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Bernstein sets don't give Vitali sets

<u>Defn.</u> $B \subset {}^\omega\omega$ is called <u>Bernstein</u> iff

$B \cap P \neq \emptyset \neq P \setminus B$ for all perfect $P \subset {}^\omega\omega$.

Letting $E_0 \subset ({}^\omega\omega)^2$ be the Vitali equivalence relation defined by $x \, E_0 \, y$ iff $\exists n_0 \, \forall n \geq n_0 \; x(n) = y(n)$, $V \subset {}^\omega\omega$ is called <u>Vitali</u> iff V picks exactly one element from each E_0-equivalence class, i.e.,

~~[illegible]~~ $\forall x \, \exists y \, \forall z \, ((z \, E_0 \, x \wedge z \in V) \leftrightarrow z = y)$.

Building upon [WWY], [N, p.48] proves that ZF + "there is a Bernstein set" does not yield a "Vitali set" V^{rec} in the redefined sense that V^{rec} picks exactly one element from each Turing degree. We here show that a slight variation of the arguments of [WWY] and [N] show that ZF + "there is a Bernstein set" does not yield a Vitali set in the original sense as defined above.

<u>Thm.</u> ZF + "there is a Bernstein set" does not prove "there is a Vitali set"

Proof: Let G be $\mathbb{C}(\omega_1)$-generic over L, where $\omega_1 = \omega_1^L$ and $\mathbb{C}(\omega_1)$ is the finite support product of ω_1 Cohen forcings, cf. [Sch, p. 105].

For $\alpha < \omega_1$ let $\beta(\alpha)$ be the least $\beta > \omega$, $\beta > \alpha$, such that $L_\beta \models$ "$\overline{\overline{\alpha}} \leq \aleph_0$," and let $e_\alpha : \omega \leftrightarrow L_\alpha$ be the $L_{\beta(\alpha)}$-least bijection. Let $E_\alpha \subset \omega \times \omega$ be such that $(\omega; E_\alpha) \overset{e_\alpha}{\cong} (L_\alpha; \in)$, and let g_α be the set of all $n < \omega$ such that there are $\xi < \alpha$, $k, m < \omega$ with $e_\alpha(n) = (\xi, k, m)$ and $p(\xi)(k) = m$ for some $p \in G$. I.e., E_α is a canonical code for L_α, and g_α codes $G \restriction \alpha$ relative to E_α.

We may define $z_\alpha : \omega \longrightarrow \omega$ by

$$z_\alpha(2\ell) = \begin{cases} 1 & \text{iff } (\ell)_0 \; E_\alpha \; (\ell)_1 \\ 0 & \text{iff } (\ell)_0 \; \not{E}_\alpha \; (\ell)_1 \end{cases} \quad \text{and}$$

$$z_\alpha(2\ell+1) = \begin{cases} 1 & \text{iff } \ell \in g_\alpha \\ 0 & \text{iff } \ell \notin g_\alpha \end{cases} .$$

Here, $(\ell)_0$ and $(\ell)_1$ is the first and second component, resp., of ℓ when construed as a pair of natural numbers, i.e., fixing a canonical $e : \omega \leftrightarrow \omega\times\omega$, $e(\ell) = ((\ell)_0, (\ell)_1)$.

<u>Claim 1.</u> For every $x \in {}^\omega\omega \cap L[G]$ there is some $\alpha < \omega_1$ such that $x \in L[z_\alpha]$.

Pf.: Given x, there is some α with $x \in L_\alpha[G \restriction \alpha]$. But $L_\alpha[G\restriction\alpha] \in L[z_\alpha]$. $\dashv$

<u>Claim 2.</u> Let $s \in \mathbb{C}(\omega_1)$ be any condition. Let G^s be the collection of all $p \in \mathbb{C}(\omega_1)$ for which there is a $q \in G$ with $\mathrm{dom}(p(\xi)) = \mathrm{dom}(q(\xi))$ for all $\xi < \omega_1$ and

$$p(\xi)(k) = \begin{cases} s(\xi)(k) & \text{if } k \in \mathrm{dom}(s(\xi)) \\ q(\xi)(k) & \text{o.w.} \end{cases}$$

then G^s is $\mathbb{C}(\omega_1)$-generic over L, and $L[G^s] = L[G]$. Also, $s \in G^s$.

Pf.: Cf. [Sch, p. 123, Problem 6.7]. $\dashv$

<u>Claim 3.</u> Let $s \in \mathbb{C}(\omega_1)$ be any condition.

Let $\alpha < \omega_1$, let $g_\alpha^s = \{n : \exists \xi < \alpha \, \exists k, m < \omega \, [$
$e_\alpha(n) = (\xi, k, m) \wedge p(\xi)(k) = m$ for some $p \in G^s]\}$,
and let $z_\alpha^s : \omega \to \omega$ be defined by

$$z_\alpha^s(2\ell) = \begin{cases} 1 & \text{iff } (\ell)_0 \; E_\alpha \; (\ell)_1 \\ 0 & \text{iff } (\ell)_0 \; \not E_\alpha \; (\ell)_1 \end{cases}$$

$$z_\alpha^s(2\ell+1) = \begin{cases} 1 & \text{iff } \ell \in g_\alpha^s \\ 0 & \text{iff } \ell \notin g_\alpha^s \end{cases} .$$

(I.e., z_α^s is defined as z_α above except for using g_α^s instead of g_α.)

Then $$z_\alpha^s \; E_0 \; z_\alpha .$$

Pf.: Immediate, as there are only finitely many pairs (ξ, k) s.t. there are $p \in G^s$ and $q \in G$ with $p(\xi)(k) \neq q(\xi)(k)$. $\dashv$

<u>Definition.</u> Let us write $d_\alpha = \{z : z \, E_0 \, z_\alpha\}$ for the E_0-equivalence class of z_α.

By Claim 3, $\{z_\alpha^s : s \in \mathbb{C}(\omega_1)\} \subset d_\alpha$, and by Claim 1:

<u>Claim 1'.</u> For every $x \in {}^\omega\omega \cap L[G]$ there is some $\alpha < \omega_1$ s.t. $x \in L[z]$ for all $z \in d_\alpha$.

Let us now consider the model

$$N = \mathrm{HOD}^{L[G]}_{({}^\omega\omega \cap L[G]) \cup \{(d_\alpha : \alpha < \omega_1)\}},$$

i.e. the class of all $X \in L[G]$ which inside $L[G]$ are hereditarily ordinal definable from parameters in $({}^\omega\omega \cap L[G]) \cup \{(d_\alpha : \alpha<\omega_1)\}$, cf. [Sch, p. 86].

$N \models ZF$.

<u>Claim 4.</u> $N \models \neg AC$, i.e., the axiom of choice fails in N, in fact: there is no well-ordering of the reals in N.

Prf.: Suppose $L[G]$ has a well-ordering of its reals which is definable from $\vec{\alpha} \in OR$, $\vec{y} \in {}^\omega\omega \cap L[G]$, and $(d_\alpha : \alpha<\omega_1)$. Let $\alpha<\omega_1$ be such that $\vec{y} \in L_\alpha[G \restriction \alpha]$. Let $\tilde{\alpha} > \alpha$,

$\tilde\alpha < \omega_1$. Then $z_{\tilde\alpha}$ must be definable in $L[G]$ from $\vec\alpha, \gamma, \vec y$, and $(d_\alpha : \alpha < \omega_1)$ for some ordinal γ.

Let us assume w.l.o.g. that $\vec y \in {}^\omega\omega \cap L$; the argument in the general case is just a simple variant of the argument that is to come.

There is a formula φ such that for all $k, m < \omega$, $z_{\tilde\alpha}(k) = m$ iff

$L[G] \models \varphi(k, m, \vec\alpha, \gamma, \vec y, (d_\alpha : \alpha < \omega_1))$. Let the formula ψ define $(d_\alpha : \alpha < \omega_1)$ from G over $L[G]$, i.e., $L[G] \models \forall \vec d\, (\vec d = (d_\alpha : \alpha < \omega_1) \leftrightarrow \psi(\vec d, G))$.

Hence $z_{\tilde\alpha}(k)(m)$ iff

$L[G] \models$ "$\varphi(k, m, \vec\alpha, \gamma, \vec y, \vec d)$, where $\psi(\vec d, G)$," iff

$\exists p \in G\ p \Vdash^{\mathbb{C}(\omega_1)}_{L}$ "$\varphi(\check k, \check m, \check{\vec\alpha}, \check\gamma, \check{\vec y}, \vec d)$, where $\psi(\vec d, \dot G)$."

Suppose $s \in \mathbb{C}(\omega_1)$ is any condition such that

$s \Vdash^{\mathbb{C}(\omega_1)}_{L}$ "$\neg\varphi(\check k, \check m, \check{\vec\alpha}, \check\gamma, \check{\vec y}, \vec d)$, where $\psi(\vec d, \dot G)$."

Using Claim 2, G^s is $\mathbb{C}(\omega_1)$-generic over L, $s \in G^s$, and $L[G^S] = L[G]$. This gives that

$L[G^S] = L[G] \models$ "$\neg\varphi(k, m, \vec{\alpha}, \gamma, \vec{y}, \vec{d})$, where $\psi(\vec{d}, G^S)$."

However, Claim 3 tells us that if $\psi(\vec{d}, G^S)$ holds true in $L[G]$, then in fact $\vec{d} = (d_\alpha : \alpha < \omega_1)$, and therefore $L[G] \models \neg\varphi(k, m, \vec{\alpha}, \gamma, \vec{y}, (d_\alpha : \alpha < \omega_1))$.

Contradiction!

We have shown that $z_{\tilde{\alpha}}(k) = m$ iff

$\exists p \in G \quad p \Vdash^{\mathbb{C}(\omega_1)}_{L}$ "$\varphi(\check{k}, \check{m}, \check{\vec{\alpha}}, \check{\gamma}, \check{\vec{y}}, \vec{d})$, where $\psi(\vec{d}, \dot{G})$"

iff $\quad \mathbb{1}_{\mathbb{C}(\omega_1)} \Vdash^{\mathbb{C}(\omega_1)}_{L}$ "$\varphi(\check{k}, \check{m}, \check{\vec{\alpha}}, \check{\gamma}, \check{\vec{y}}, \vec{d})$, where $\psi(\vec{d}, \dot{G})$."

But then $z_{\tilde{\alpha}} \in L$, cf. [Sch, p. 118]. However, $L[z_{\tilde{\alpha}}]$ contains a Cohen real over L.

Contradiction!

We have verified Claim 4. $\dashv$

<u>Claim 5.</u> $N \models$ "There is no Vitali set."

Prf.: Suppose there is some $V \in N$ such that $V \cap d_\alpha$ is a singleton for each $\alpha < \omega_1$.

There is then a sequence $(z^*_\alpha : \alpha < \omega_1)$ in N s.t. $z^*_\alpha \in d_\alpha$ for every $\alpha < \omega_1$. Let $<_\alpha$ be the canonical well-ordering of $L[z^*_\alpha]$ as being defined inside $L[z^*_\alpha]$. For $x \in {}^\omega\omega \cap N = {}^\omega\omega \cap L[G]$ let $\alpha(x)$ be the least $\alpha < \omega_1$ such that $x \in L[z^*_\alpha]$. By Claim 1′, $\alpha(x)$ is always well-defined. We may then define a well-order $<$ of ${}^\omega\omega \cap N$ inside N as follows. $x < y$ iff $\alpha(x) < \alpha(y)$ or $\alpha(x) = \alpha(y)$ and $x <_\alpha y$.
Contradiction with Claim 4. $\dashv$

<u>Claim 6.</u> $N \models$ "there is a Bernstein set."

Prf.: Let

$$B = \{ b \in {}^\omega\omega : \exists \text{ even } \alpha \; [\, b \in L[z] \text{ for all/some } z \in d_{\alpha+1} \wedge b \notin L[z] \text{ for all/some } z \in d_\alpha \,] \},$$

and $$B' = \{ b \in {}^\omega\omega : \exists \text{ odd } \alpha \; [\, b \in L[z] \text{ for all/some } z \in d_{\alpha+1} \wedge b \notin L[z] \text{ for all/some } z \in d_\alpha \} ,$$

as being defined in N.

Obviously, $B \cap B' = \emptyset$.

Let $P \subset {}^\omega\omega$ be a perfect set in N, say $P = [T]$ for some perfect tree T, $T \in L[z]$, $z \in d_\alpha$, α even. We work in N.

Pick $z^* \in d_{\alpha+1}$. We may easily find some $b \in {}^\omega\omega$ such that $L[T, b] = L[z^*]$. In particular, $b \in L[z^*]$. If $b \in L[z]$, then $L[z^*] = L[T, b] \subset L[z]$, which contradicts $z^* \in d_{\alpha+1}$ and $z \in d_\alpha$. Hence $b \notin L[z']$ for any $z' \in d_\alpha$.

We have shown that $B \cap P \neq \emptyset$.

Virtually the same argument shows that $B' \cap P \neq \emptyset$.

But then B is Bernstein. ⊣ ⊣ (Thm.)

[N] A. Nies, dl./dropboxusercontent.com/u/370127/Blog/Blog2012.pdf, p. 48.

[Sch] R. Schindler, Set theory. Exploring independence and truth, Springer-Verlag 2014.

[WWY] W. Wang, L. Wu, L. Yu, Cofinal maximal chains in the Turing degrees.