GCH and $\diamond$ in $L[U]$.

A coarse Dodd-Jensen premouse is a model $M = (L_\alpha[U]; \in, U)$ s.t. $M \models ZFC^-$ (ZFC without the power set axiom) plus $M \models$ "U is a normal $<\kappa$-complete nonprincipal ultrafilter on κ," some $\kappa \in M$.

If M is a coarse DJ premouse, we may define the ultrapower $i^M_U : M \longrightarrow \mathrm{ult}(M; U)$ using functions in M. $\mathrm{ult}(M; U)$ will always be well-founded, so that we may identify it with a transitive structure; i^M_U will be fully elementary, so that $\mathrm{ult}(M; U)$ will again be a coarse DJ premouse.

<u>Lemma 1.</u> Let $M = (J_\alpha[U]; \in, U)$ be a coarse DJ premouse, let U be on κ, and let $i^M_U : M \longrightarrow \mathrm{ult}(M; U)$ be the ultrapower map. Then $\mathcal{P}(\kappa) \cap \mathrm{ult}(M; U) = \mathcal{P}(\kappa) \cap M$.

Proof: "$\supset$": If $A \in \mathcal{P}(\kappa) \cap M$, then $A = i^M_U(A) \cap \kappa \in \mathrm{ult}(M; U)$.

"$\subset$": Let $A \in \mathcal{P}(\kappa) \cap \mathrm{ult}(M; U)$. Say $A = i^M_U(f)(\kappa)$, so that $\xi \in A$ iff $\{\eta < \kappa : \xi \in f(\eta)\} \in U$. Write $A_\xi = \{\eta < \kappa : \xi \in f(\eta)\}$ for $\xi < \kappa$. By the proof of "$\supset$", $(A_\xi : \xi < \kappa) \in \mathrm{ult}(M; U)$, in fact $(A_\xi : \xi < \kappa) = (A'_\xi \cap \kappa : \xi < \kappa)$, where $(A'_\xi : \xi < i^M_U(\kappa)) = i^M_U((A_\xi : \xi < \kappa))$. Then $\xi \in A$ iff $\{\eta < \kappa : \xi \in f(\eta)\} \in U$ iff $\kappa \in i^M_U(A_\xi)$ iff $\kappa \in$ the ξ^{th} element of $i^M_U((A_{\bar\xi} : \bar\xi < \kappa))$, so $A \in \mathrm{ult}(M; U)$. $\dashv$

Given some coarse JD premouse, M, we may define a putative iteration

$$(M_i, \pi_{ij} : i \leq j \leq \alpha)$$

of length $\alpha + 1$ as usual. M is called <u>iterable</u> iff M_α is well-founded (transitive) for all putative iterations $(M_i, \pi_{ij} : i \le j \le \alpha)$ of M of length α.

$L[u]$ itself is iterable, so that if $\eta > \text{crit}(u)$ is such that $L_\eta[u] \models ZFC^-$ and if

$$\sigma : L_\alpha[\bar{u}] \longrightarrow L_\eta[u]$$

is fully iterable, then $L_\alpha[\bar{u}] = (L_\alpha[\bar{u}]; \in, \bar{u})$ is an iterable coarse DJ premouse.

<u>Lemma 2.</u> (Comparison) Let $M = (L_\alpha[u]; \in, u)$ be an iterable coarse DJ premouse. Let $\mu > \alpha$ be a regular cardinal (in V), and let $\mathcal{F}_\mu$ denote the club filter on μ (as being defined in V). Let $(M_i, \pi_{ij} : i \le j \le \mu)$ be the iteration of M of length μ. There is then α' s.t.

$$(*) \qquad M_\mu = (L_{\alpha'}[\mathcal{F}_\mu]; \in, \mathcal{F}_\mu \cap L_{\alpha'}[\mathcal{F}_\mu]).$$

Proof: Write $\kappa_i = \text{crit}(u_i)$, when $M_i = (L_{\alpha_i}[u_i]; \in, u_i)$, $i \le \mu$.

If $i < \mu$, then $\pi_{i\,i+1}(A) \in U_{i+1}$ iff $A \in U_i$ iff $\kappa_i \in \pi_{i\,i+1}(A)$, which implies that for all $i < j \leq \mu$, if $B \in \operatorname{ran}(\pi_{ij})$, then

$$B \in U_j \quad \text{iff} \quad \kappa_i \in B .$$

Then if $j \leq \mu$ is a limit ordinal, for all $B \in \mathcal{P}(\kappa_j) \cap M_j$,

$$B \in U_j \quad \text{iff} \quad \kappa_i \in B \text{ for a tail end of } i < j$$

This implies (*). $\dashv$

<u>Corollary 3.</u> Let M, M' be iterable coarse DJ premice. There is some μ s.t. if

$$(M_i, \pi_{ij} : i \leq j \leq \mu)$$
$$(M_i', \pi_{ij}' : i \leq j \leq \mu)$$

are the iterations of M, M' resp. of length $\mu+1$, then if $M_\mu = (L_\alpha[U_\mu], \in, U_\mu)$, $M_\mu' =$

$(L_{\alpha'}[U'_\mu]; \in, U'_\mu)$,

$\alpha \leq \alpha'$ and $U_\mu = U'_\mu \cap L_\alpha[U_\mu]$ or

$\alpha' \leq \alpha$ and $U'_\mu = U_\mu \cap L_{\alpha'}[U'_\mu]$.

Lemma 1 implies that if $M = (J_\alpha[U]; \in, U)$ is an iterable coarse DJ premouse, if

$$(M_i, \pi_{ij} : i \leq j < \mu)$$

is an iteration of M of length μ, and if $\gamma < \mathrm{crit}(U)$, then

$$\mathcal{P}(\gamma) \cap M_\mu = \mathcal{P}(\gamma) \cap M.$$

Corollary 3 then yields:

<u>Corollary 4.</u> Let M, M' be iterable coarse DJ premice, and let $\gamma < \min(\mathrm{crit}(U), \mathrm{crit}(U'))$, where U, U' is the measure of M, M', resp.; then $\mathcal{P}(\gamma) \cap M \subset M'$ or $\mathcal{P}(\gamma) \cap M' \subset M$.

For any coarse DJ premouse M, we may let $<_M$

denote the order of constructibility of M.

If $A \in M$, we let $\mathrm{pred}_{<_M}(A)$ denote $\{x \in M : x <_M A\}$.

Lemma 5. Let $M = (L_\alpha[U]; \in, U)$ be an iterable coarse DJ premouse, and let $\gamma < \alpha$. Let $A \in \mathcal{P}(\gamma) \cap M$. Then

$$\mathrm{Card}\left(\mathrm{pred}_{<_M}(A) \cap \mathcal{P}(\gamma)\right) \leq \gamma$$

inside any transitive ZFC^- model P with $M \in P$.

Proof: Let

$$\sigma : \bar{M} \longrightarrow M$$

be fully elementary such that $(\gamma+1) \cup \{A\} \subset \mathrm{ran}(\sigma)$, $\bar{M}$ transitive, and $\mathrm{Card}(\bar{M}) = \gamma$.

By Cor. 4,

$$\mathrm{pred}_{<_M}(A) \cap \mathcal{P}(\gamma) \subset \bar{M}. \qquad \dashv$$

Corollary 6. $L[U] \models GCH$.

Lemma ~~6~~7. $L[U] \models$ there is a Δ^1_3 well-ordering of $\mathbb{R}$.

Proof: For $x, y \subset \omega$, $x <_{L[U]} y$ iff there is a (ctble.) iterable DJ premouse M with $x <_M y$. Being a ctble. iterable DJ premouse is Π^1_2 in the codes. $\dashv$

Lemma 8. $L[U] \models$ " $\diamondsuit_\mu$ for all regular μ. "

Proof: We work inside $L[U] = M$.

Let us define $(C_\xi, A_\xi : \xi < \mu)$ recursively as follows.

$(C_\xi, A_\xi) =$ the $<_M$ - least (C, A) s.t.

$C \subset \xi$ is club, $A \subset \xi$, and

$\{ \bar{\xi} < \xi : A \cap \bar{\xi} = A_{\bar{\xi}} \} \cap C = \emptyset$.

We claim that $(A_\xi : \xi < \mu)$ witnesses $\diamondsuit_\mu$.

Otherwise let (C, A) be the $<_M$ - least counterexample.

Let $\sigma : \bar{M} \longrightarrow H^M_\theta$ be fully elementary, where θ is regular and large enough, $\mathrm{Card}(\bar{M}) \leq \mu$, $\mathrm{ran}(\sigma) \cap \mu \in \mu$, $\{\mu, C, A\} \subset \mathrm{ran}(\sigma)$. Set $\bar{\mu} = \sigma^{-1}(\mu)$.

Then $(C \cap \bar{\mu}, A \cap \bar{\mu}) = (C_{\bar{\mu}}, A_{\bar{\mu}})$, so

$\bar{\mu} \in \{\bar{\xi} < \mu : A \cap \bar{\xi} = A_{\bar{\xi}}\} \cap C$, and then

$$\{\bar{\xi} < \bar{\mu} : A_{\bar{\mu}} \cap \bar{\xi} = A_{\bar{\xi}}\} \cap C_{\bar{\mu}} \neq \emptyset \quad \text{↯} . \quad \dashv$$

<u>Lemma 9</u>. $L[U] \models$ "$\diamondsuit^*_{\mu^+}$ for all infinite μ"

Proof: We work inside $L[U] = M$.

For $\xi < \mu^+$, let $\mathcal{A}_\xi = \bigcup \{ \mathcal{P}(\xi) \cap \bar{M}$: $\bar{M}$ is an iterable coarse DJ premouse with $\xi = \mu^{+\bar{M}} \}$.

$\dashv$

It can be shown that in $L[U]$, $\diamondsuit^*_\mu$ holds iff μ is not ineffable.