The long extender algebra*†

Ralf Schindler

August 17, 2016

Abstract

Generalizing Woodin's extender algebra, cf. e.g. [8], we isolate the long extender algebra as a general version of Bukowský's forcing, cf. [1], in the presence of a supercompact cardinal.

1 Introduction.

Recently, a wonderful theorem of Lev Bukowský, cf. [1], found some interesting applications in set theoretic geology, cf. [9], which proves the set–directedness of the collection of all grounds of a given model of set theory, [5], which analyzes the mantle of the least inner model with a strong cardinal above a Woodin cardinal, and [6].[1]

Said theorem of Bukowský gives a necessary and sufficient criterion for when V is a λ–c.c. generic extension of a given inner model W. Inspired by [3], this paper explores the relationship between Bukowský's result and W. Hugh Woodin's extender algebra, cf. e.g. [8, pp. 1657ff.]. A special case of Bukowský's forcing may be construed as a version of the extender algebra.

The current paper proposes a generalization of the extender algebra to long extenders, cf. the forcing $\mathbb{P}^{\mathcal{E}}$ defined in section 4 below. The *long extender algebra* $\mathbb{P}^{\mathcal{E}}$ then corresponds to a general version of Bukowský's forcing in the presence of a supercompact cardinal.

---

*2000 Mathematics Subject Classifications: 03E15, 03E45, 03E60.

†Keywords: Forcing, extender algebra.

[1]The terms "ground," "bedrock," and "mantle" are taken from [4]. If $\bar{W} \subset W$ are both inner models, then $\bar{W}$ is a ground of W iff W is a generic extension of $\bar{W}$. The mantle of W is the intersection of all grounds of W. That the collection of all grounds be set–directed means that the intersection of any collection of grounds which may be indexed by a set contains a ground.

The author would like to thank Grigor Sargsyan for his great hospitality and generous support during the author's visits to Warsaw (Poland) in November, 2015, and to Brookylyn (USA) in February through April, 2016. Some of the insights which are documented by the current paper were obtained during those visits and would not have been achieved without the many discussions with Sargsyan during those visits.

The author would also like to thank his hosts at the School of Mathematics of the IPM, Tehran (Iran), for their exceptional hospitality during his visit in Oct 2015.

2 A criterion for an inner model to be V.

Definition 2.1 *Let W be an inner model of V. Let λ be an infinite cardinal. We say that W* uniformly λ–covers *V iff for all functions $f \in V$ with $\mathrm{dom}(f) \in W$ and $\mathrm{ran}(f) \subset W$ there is some function $g \in W$ with $\mathrm{dom}(g) = \mathrm{dom}(f)$ such that $f(x) \in g(x)$ and $\mathrm{Card}(g(x)) < \lambda$ for all $x \in \mathrm{dom}(g)$.*

If there is some poset $\mathbb{P} \in W$ having the λ–c.c. in W and some g which is $\mathbb{P}$–generic over W such that $V = W[g]$, then W uniformly λ–covers V. Bukowský's Theorem 3.3 will say that the converse is true also.

The following is probably part of the folklore.

Theorem 2.2 *Let W be an inner model of V, and let λ be an infinite regular cardinal. Assume that W uniformly λ–covers V, and assume also that $\mathcal{P}(2^{<\lambda}) \cap V \subset W$. Then $W = V$.*

Proof. Let us call any set Γ of functions an *antichain* iff for all a, $b \in \Gamma$ with $a \neq b$ there is some $i \in \mathrm{dom}(a) \cap \mathrm{dom}(b)$ with $a(i) \neq b(i)$.

It is easily seen that the hypotheses on W give that

$$^{2^{<\lambda}}W \subset W. \tag{1}$$

To verify (1), notice first that by $\mathcal{P}(2^{<\lambda}) \cap V \subset W$, W computes the cardinal successor of $2^{<\lambda}$ correctly and for every $\gamma < (2^{<\lambda})^+$, $\mathcal{P}(\gamma) \cap V \subset W$.

Now let $f \colon 2^{<\lambda} \to \mathrm{OR}$, $f \in V$. Using the fact that W uniformly λ–covers V, let $g \in W$ be a function with $\mathrm{dom}(g) = 2^{<\lambda}$ such that $g(\xi)$ is a set of ordinals, $f(\xi) \in g(\xi)$, and $\mathrm{Card}(g(\xi)) < \lambda$ for all $\xi < 2^{<\lambda}$. Let $e \colon \gamma \cong \bigcup \mathrm{ran}(g)$ be the (inverse of the) transitive collapse of $\bigcup \mathrm{ran}(g)$, so that $e \in W$ and $\gamma < (2^{<\lambda})^+$. As $\mathcal{P}(\gamma) \cap V \subset W$, the function $e^{-1} \circ f \colon 2^{<\lambda} \to \gamma$ is in W, which gives that $f = e \circ (e^{-1} \circ f) \in W$. We showed (1).

Assume that $A\colon \alpha \to 2$, for some ordinal α, is such that $A \in V \setminus W$. Let us write $\mathcal{F}$ for the collection of all functions a such that there is some $x \subset \alpha$ of size $< \lambda$ such that $a\colon x \to 2$. Using again the fact that W uniformly λ–covers V,[2] we may pick a function g in W such that if $\Gamma \subset \mathcal{F}$ is an antichain with $\Gamma \in W$, then

(i) $g(\Gamma) \in W$ is a subset of Γ of size $< \lambda$, and

(ii) if there is some (unique!) $a \in \Gamma$ with $a = A \restriction \mathrm{dom}(a)$, then $a \in g(\Gamma)$.

We call $a \in \mathcal{F}$ *legal* iff for no antichain $\Gamma \in W$, $a \in \Gamma \setminus g(\Gamma)$. Notice that being legal is defined inside W (from the parameter $g \in W$).

Every $A \restriction x$, where $x \subset \alpha$ has size $< \lambda$, is legal.

If $\Gamma \subset \mathcal{F}$ is an antichain with $\Gamma \in W$, and if every $a \in \Gamma$ is legal, then we must have $g(\Gamma) = \Gamma$, from which it follows that Γ has size $< \lambda$.

Let $\theta \gg \alpha$ be such that $\theta^{<\lambda} = \theta$. Let

$$X \prec (H_\theta; \in, \{A\}, \mathcal{F}, g, H_\theta \cap W)$$

be such that ${}^{<\lambda}X \subset X$ and $\mathrm{Card}(X) = 2^{<\lambda}$. By (1), $X \cap W \in W$, and of course

$$X \cap W \prec (H_\theta \cap W; \in, \mathcal{F}, g) \in W. \tag{2}$$

Write $\sigma\colon \bar{W} \cong X \cap W$ for the (inverse of the) transitive collapse of $X \cap W$, so that $\sigma \in W$. σ extends to $\tilde{\sigma}\colon H \cong X$, the (inverse of the) transitive collapse of X.

Notice that $\mathcal{P}(2^{<\lambda}) \cap V \subset W$ gives that $\bar{A} = \tilde{\sigma}^{-1}(A) \in W$, which in turn yields that

$$A \restriction (X \cap \alpha) = \sigma\text{"}\bar{A} \in W. \tag{3}$$

We are now going to derive a contradiction from (3).

Using (3), we may work inside W and define a sequence $(a_i\colon i < \lambda)$ of elements of $\mathcal{F}$ such that $a_i \in X$ and $\mathrm{dom}(a_i) \supset \mathrm{dom}(a_j)$ for all $j < i < \lambda$ as follows. Assume $(a_j\colon j < i)$ has already been chosen. Notice that $(a_j\colon j < i) \in X$ by ${}^{<\lambda}X \subset X$. Write $x = \bigcup_{j<i} \mathrm{dom}(a_j)$, so that $x \in X$. Clearly, for every $\xi < \alpha$ there is some legal $a \in \mathcal{F}$ such that $x \cup \{\xi\} \subset \mathrm{dom}(a)$ and $a = A \restriction \mathrm{dom}(a)$ (just pick $A \restriction (x \cup \{\xi\})$). There must then be some $\xi < \alpha$ such that there are legal a and b in $\mathcal{F}$ with $x \cup \{\xi\} \subset \mathrm{dom}(a) \cap \mathrm{dom}(b)$ and $a(\xi) \neq b(\xi)$, as otherwise A would be the union of all legal $a \in \mathcal{F}$ with $a \supset A \restriction x$ and thus A would be in W.

By (2) we must then have inside X some $\xi < \alpha$ and some legal a and b in $\mathcal{F}$ with $x \cup \{\xi\} \subset \mathrm{dom}(a) \cap \mathrm{dom}(b)$ and $a(\xi) \neq b(\xi)$. By (3), we may then choose in W some

[2]This use is now substantial, in contrast to the previous one.

$\xi \in \alpha \cap X$ and some $a \in \mathcal{F} \cap X$ such that $x \cup \{\xi\} \subset \mathrm{dom}(a)$, $a \restriction x = (A \restriction (X \cap \alpha)) \restriction x$ ($= A \restriction x$), and $a(\xi) \neq (A \restriction (X \cap \alpha))(\xi)$ ($= A(\xi)$). Let $a_i = a$.

Writing $\Gamma = \{a_i \colon i < \lambda\}$, $\Gamma \in W$, and Γ is an antichain consisting of legal functions. But this is a contradiction! □

3 Bukowsky's theorem.

Let us fix $W \subset V$, an inner model, and let λ and μ be infinite cardinals, $\lambda \leq \mu$. We aim to define a poset in W which will be a candidate for generically adding a given subset of μ.

Working in W, let $\mathcal{L}$ be the infinitary language with atomic fomulae "$\check{\xi} \in \dot{a}$," for $\xi < \mu$, and such that the set of formulae is closed under negation and infinite disjunctions of the form $\bigvee\!\!\!\bigvee \Gamma$ for all well–ordered sets Γ of fomulae with $\mathrm{Card}(\Gamma) < \lambda$. Writing $\mu^{<\lambda} = (\mu^{<\lambda})^W$, $\mathcal{L}$ has size $\mu^{<\lambda}$.

For $A \subset \mu$, $A \in V^{\mathrm{Col}(\omega,\mu^{<\lambda})}$, and $\varphi \in \mathcal{L}$, we may define the meaning of "$A \vDash \varphi$" in the obvious recursive fashion: $A \vDash$ "$\check{\xi} \in \dot{a}$" iff $\xi \in A$, $A \vDash \neg\varphi$ iff $A \nvDash \varphi$, and $A \vDash \bigvee\!\!\!\bigvee \Gamma$ iff $A \vDash \varphi$ for some $\varphi \in \Gamma$. Inside $V^{\mathrm{Col}(\omega,\mu^{<\lambda})}$, the relation "$A \vDash \varphi$" is Borel in the codes. For $\Gamma \subset \mathcal{L}$, $A \vDash \Gamma$ means $A \vDash \varphi$ for all $\varphi \in \Gamma$. For $\Gamma \cup \{\varphi\} \in \mathcal{P}(\mathcal{L}) \cap W$, we write

$$\Gamma \vdash \varphi \tag{4}$$

iff in $W^{\mathrm{Col}(\omega,\mu^{<\lambda})}$, for all $A \subset \mu$, if $A \vDash \Gamma$, then $A \vDash \varphi$. (4) is thus defined over W, and inside $W^{\mathrm{Col}(\omega,\mu^{<\lambda})}$, (4) is Π^1_1 in the codes. By absoluteness, (4) is thus equivalent with the fact that in $V^{\mathrm{Col}(\omega,\mu^{<\lambda})}$, for all $A \subset \mu$, if $A \vDash \Gamma$, then $A \vDash \varphi$. For $\Gamma \in \mathcal{P}(\mathcal{L}) \cap W$, Γ is called *consistent* iff there is no $\varphi \in \mathcal{L}$ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$, which in turn is easily seen to be equivalent with the fact that in $W^{\mathrm{Col}(\omega,\mu^{<\lambda})}$ (equivalently, in $V^{\mathrm{Col}(\omega,\mu^{<\lambda})}$) there is some $A \subset \mu$ with $A \vDash \Gamma$.

Now let

$$g \colon [\mathcal{L}]^\lambda \cap W \to [\mathcal{L}]^{<\lambda} \cap W,\ g \in W$$

be a function such that

(i) $g(\Gamma) \subset \Gamma$, and

(ii) $\mathrm{Card}(g(\Gamma)) < \lambda$

for all $\Gamma \in [\mathcal{L}]^\lambda \cap W$. Let us call $\varphi \in \mathcal{L}$ *illegal* iff there is some $\Gamma \in [\mathcal{L}]^\lambda \cap W$ such that $\varphi \in \Gamma \setminus g(\Gamma)$, and let us write T^g for the set of all formulae of the form[3]

$$\varphi \to \bigvee\!\!\!\bigvee g(\Gamma), \tag{5}$$

[3] $\varphi \to \varphi'$ is short for $\bigvee\!\!\!\bigvee\{\neg\varphi, \varphi'\}$.

where φ is illegal, $\Gamma \in [\mathcal{L}]^\lambda \cap W$, and $\varphi \in \Gamma \setminus g(\Gamma)$.

Let us write $\mathbb{P}^g$ for the set of all $\varphi \in \mathcal{L}$ such that $T^g \cup \{\varphi\}$ is consistent. We also write

$$\varphi \leq_{\mathbb{P}^g} \varphi' \tag{6}$$

for $T^g \cup \{\varphi\} \vdash \varphi'$.

Claim 3.1 *$\mathbb{P}^g$ has the λ–c.c. inside W.*

Proof. Let $\Gamma \in [\mathbb{P}^g]^\lambda \cap W$. Let $\varphi \in \Gamma \setminus g(\Gamma)$. By (5), $\varphi \leq_{\mathbb{P}^g} \bigvee\!\!\!\bigvee g(\Gamma)$, so that Γ cannot be an antichain. $\square$

For an arbitrary choice of g, we might have that $\mathbb{P}^g$ is quite trivial, or even $\mathbb{P}^g = \emptyset$.

Let $A \subset \mu$, $A \in V$. We set

$$G_A = \{\varphi \in \mathbb{P}^g \colon A \vDash \varphi\}.$$

Claim 3.2 *Assume that $A \vDash T^g$. Then $G_A \subset \mathbb{P}^g$ is a $\mathbb{P}^g$–generic filter over W and*

$$A = \{\xi < \mu\colon \text{ “}\check{\xi} \in \dot{a}\text{”} \in G_A\} \in W[G_A].$$

Proof. If φ, $\varphi' \in \mathbb{P}^g$, $A \vDash \varphi$, and $\varphi \leq_{\mathbb{P}^g} \varphi'$, then $A \vDash \varphi'$ using absoluteness. If φ, $\varphi' \in \mathbb{P}^g$, $A \vDash \varphi$, and $A \vDash \varphi'$, then $A \vDash \varphi \wedge \varphi'$,[4] $\varphi \wedge \varphi' \in \mathbb{P}^g$ by $A \vDash T^g$, and clearly $\varphi \wedge \varphi' \leq_{\mathbb{P}^g} \varphi$ and $\varphi \wedge \varphi' \leq_{\mathbb{P}^g} \varphi'$. Hence G_A is a filter.

Now let $\Gamma \in W$ be a maximal antichain in $\mathbb{P}^g$. By Claim 3.1, $\Gamma \in [\mathbb{P}^g]^{<\lambda}$. If $G_A \cap \Gamma = \emptyset$, then $A \vDash \neg \bigvee\!\!\!\bigvee \Gamma$. By $A \vDash T^g$, $\neg \bigvee\!\!\!\bigvee \Gamma \in \mathbb{P}^g$, and

$$\Gamma \cup \{\neg \bigvee\!\!\!\bigvee \Gamma\} \supsetneq \Gamma$$

is an antichain. Contradiction!

The rest is easy. $\square$

Theorem 3.3 (Lev Bukowský) *Let $W \subset V$ be an inner model, and let λ be an infinite regular cardinal such that W uniformly λ–covers V. Let $e\colon 2^{2^{<\lambda}} \to \mathcal{P}(2^{<\lambda})$ be a bijection, and let*

$$A = \{2^{<\lambda} \cdot \eta + \xi \colon \eta < 2^{2^{<\lambda}} \wedge \xi \in e(\eta)\}.$$

There is then some poset $\mathbb{P} \in W$ such that

[4] $\varphi \wedge \varphi'$ is short for $\neg \bigvee\!\!\!\bigvee\{\neg\varphi, \neg\varphi'\}$.

(a) $\mathbb{P}$ *has the* λ*–c.c. in* W*,*

(b) $\mathbb{P}$ *has size* $2^{2^{<\lambda}}$ *in* W*,*

(c) A *is* $\mathbb{P}$*–generic over* W*, and*

(d) $V = W[A]$*.*

Proof. Let us write
$$\mu = 2^{2^{<\lambda}},$$
as being computed in V.

By the fact that W uniformly λ–covers V, we may find a function
$$g\colon [\mathcal{L}]^{\lambda} \to [\mathcal{L}]^{<\lambda},\ g \in W$$
such that for all $\Gamma \in [\mathcal{L}]^{\lambda} \cap W$,

(i) $g(\Gamma) \subset \Gamma$,

(ii) $\mathrm{Card}(g(\Gamma)) < \lambda$, and

(iii) if $A \vDash \varphi$ for *some* $\varphi \in \Gamma$, then $A \vDash \bigvee\!\!\!\bigvee g(\Gamma)$.

For this choice of g, $A \vDash T^g$. Hence by Claim 3.2, G_A is $\mathbb{P}^g$–generic over W, and $A \in W[G_A]$. This gives (a), (b), and (c). Clearly, $W[G_A]$ inherits from W the fact that it uniformly λ–covers V, so that (d) is given by Theorem 2.2. □

Recall that for a regular cardinal λ and an ordinal $\alpha \geq \lambda$ a set $C \subset [\alpha]^{<\lambda}$ is called *club* iff

(a) for all $\gamma < \lambda$ and all $\{X_i \colon i < \gamma\} \subset C$ we have $\bigcup\{X_i \colon i < \gamma\} \in C$, and

(b) for all $x \in [\alpha]^{<\lambda}$ there is some $X \in C$ with $x \subset X$.

Theorem 3.3 immediately leads to the following characterization.

Corollary 3.4 *Let* $W \subset V$ *be an inner model, and let* λ *be an infinite regular cardinal. The following are equivalent.*

(a) W *uniformly* λ*–covers* V*.*

(b) For every $\alpha \geq \lambda$*, if* $C \in \mathcal{P}([\alpha]^{<\lambda}) \cap V$ *is club in* V*, then there is some* $D \in \mathcal{P}([\alpha]^{<\lambda}) \cap W$ *with* $D \subset C$ *and* D *is club in* W*.*

(c) *There is some poset* $\mathbb{P} \in W$ *such that* $\mathbb{P}$ *has the* λ*–c.c. in* W*,* $\mathbb{P}$ *has size* $2^{2^{<\lambda}}$ *in* W*, and* $V = W[g]$ *for some* g *which is* $\mathbb{P}$*–generic over* W*.*

Proof. (a) $\Longrightarrow$ (c) is given by Theorem 3.3.

To show (c) $\Longrightarrow$ (b), fix $\alpha \geq \lambda$ and $C \in \mathcal{P}([\alpha]^{<\lambda}) \cap V$ which is club in V. Let $\mathbb{P}$ and g be as in (c), let $\theta \gg \max(\alpha, 2^{2^{<\lambda}})$, and let $\tau \in W^{\mathbb{P}} \cap H_\theta^W$ be a name of a function $f \colon {}^{<\omega}\alpha \to \alpha$, $f \in V$, such that if $Z \in [\alpha]^{<\lambda}$, $f\text{"}{}^{<\omega}Z \subset Z$, then $Z \in C$. Inside W, there is some $D^* \in \mathcal{P}([H_\theta^W]^{<\lambda}) \cap W$ which is club in W, such that if $X \in D^*$, then

(i) $X \in [H_\theta^W]^{<\lambda}$,

(ii) $X \prec (H_\theta^W; \in, \mathbb{P}, \tau)$, and

(iii) if $A \in \mathcal{P}(\mathbb{P}) \cap X$ is an antichain in $\mathbb{P}$, then $A \subset X$.

and $D = \{X \cap \alpha \colon X \in D^*\} \subset [\alpha]^{<\lambda}$ is club in W. If $X \in D^*$, then $f\text{"}{}^{<\omega}(X \cap \alpha) \subset X[g] \cap \alpha = X \cap \alpha$, i.e., $X \cap \alpha \in C$.

To show (b) $\Rightarrow$ (a), let $f \colon \theta \to \alpha$, $f \in V$. Let $C \in \mathcal{P}([\alpha]^{<\lambda}) \cap V$ be is club in V such that if $X \in C$ and $\xi \in X \cap \theta$, then $f(\xi) \in X$. Let $D \in \mathcal{P}([\alpha]^{<\lambda}) \cap W$ be such that $D \subset C$ and D is club in W. Working inside W, pick for each $\xi < \theta$ some $X \in D$ such that $\xi \in X$, and call it X_ξ. Define g with $\mathrm{dom}(g) = \theta$ inside W by $g(\xi) = X \cap \alpha$. □

4 From Bukowský to Woodin and beyond.

As in the previous section, let us fix $W \subset V$, an inner model, and let λ and μ be infinite cardinals, $\lambda \leq \mu$. We are going to use some of the terminology of [7, Definitions 10.45, 10.55, 10.57]. If E is a (κ, ν)–extender over V (cf. [7, Definition 10.45]), then we shall write

(i) $\mathrm{crit}(E)$ for the critical point κ of E,

(ii) $\mathrm{lh}(E)$ for the length ν of E,

(iii) $\sigma(E)$ for the space[5] $\sup\{\mu_a + 1 \colon a \in [\mathrm{lh}(E)]^{<\omega}\}$ of E, and

[5] μ_a is the ordinal $\bar{\mu}$ such that the measure E_a of E lives on $[\bar{\mu}]^{\mathrm{Card}(a)}$, cf. [7, Definition 10.45 (1)].

(iv) $\rho(E)$ for the strength of E, i.e., for the largest β such that[6] $V_\beta \subset \mathrm{ult}(V;E)$.

We shall write $\pi_E\colon V \to \mathrm{ult}(V;E)$ for the ultrapower embedding. If E is an extender over W, then we write π_E^W for the ultrapower map induced by forming the ultrapower of W by E inside W.

Let us work entirely inside W until further notice. As before, we let $\mathcal{L}$ be the infinitary language with atomic fomulae "$\check{\xi} \in \dot{a}$," for $\xi < \mu$, and such that the set of formulae is closed under negation and infinite disjunctions of the form $\bigvee\!\!\!\bigvee \Gamma$ for all well–ordered sets Γ of fomulae with $\mathrm{Card}(\Gamma) < \lambda$.

Let $\mathcal{E}$ be a class of (short or long) extenders such that $\mathrm{crit}(E) < \sigma(E) < \lambda$. We let $T^{\mathcal{E}}$ be the collection of all sentences of $\mathcal{L}$ of the form

$$\varphi \to \bigvee\!\!\!\bigvee \pi_E{}''\Gamma, \tag{7}$$

where $E \in \mathcal{E}$, $\Gamma \in [\mathcal{L}]^{\mathrm{crit}(E)} \cap V_{\sigma(E)}$, and $\varphi \in \pi_E(\Gamma) \cap V_{\rho(E)}$.

We may define $\mathbb{P}^{\mathcal{E}}$ in much the same way as $\mathbb{P}^g$ was defined above. To be explicit, we write $\mathbb{P}^{\mathcal{E}}$ for the set of all $\varphi \in \mathcal{L}$ such that $T^{\mathcal{E}} \cup \{\varphi\}$ is consistent. We also write

$$\varphi \leq_{\mathbb{P}^{\mathcal{E}}} \varphi' \tag{8}$$

for $T^{\mathcal{E}} \cup \{\varphi\} \vdash \varphi'$.

It is easy to see that if $\mu = \lambda$ and $\mathcal{E} \subset V_\lambda$ is a class of short extenders, then $\mathbb{P}^{\mathcal{E}}$ is exactly W. Hugh Woodin's extender algebra associated with $\mathcal{E}$, cf. e.g. [8, pp. 1657ff.] or [2].

We say that $\mathcal{E}$ is *rich* iff for every $\Gamma \in [\mathcal{L}]^\lambda$ there is some (κ, ν)–extender $E \in \mathcal{E}$ such that

(i) $\kappa < \sigma(E) < \lambda$,

(ii) $\pi_E(\kappa) = \lambda \leq \mu \leq \rho(E)$,

(iii) $\Gamma \in \mathrm{ran}(\pi_E) \cap V_{\rho(E)}$, and

(iv) $(\pi_E)^{-1}(\Gamma) \in V_{\sigma(E)}$.

For future references, cf. Theorem 4.5, let us also refer to $\mathcal{E}$ as "(λ, μ)*–rich*."

If λ is a supercompact cardinal, then by exploiting Magidor's characterization of "supercompactness," cf. e.g. [7, Problems 4.29 and 10.21], there is some rich $\mathcal{E}$. This follows immediately from the argument for [7, Problem 4.29].

[6] We assume here and in what follows that $\mathrm{ult}(V;E)$ is always well–founded, so that we may identify it with its own transitive collapse.

If $\mathcal{E}$ is rich, then we may define a function $g\colon [\mathcal{L}]^\lambda \to [\mathcal{L}]^{<\lambda}$ as follows. For $\Gamma \in [\mathcal{L}]^\lambda$ let us pick some $E \in \mathcal{E}$ with properties (i) through (iv) above, and then define

$$g(\Gamma) = \Gamma \cap \operatorname{ran}(\pi_E).$$

This gives a function g with properties (i) and (ii) as on p. 4.

The following is immediate.

Claim 4.1 *Suppose that $\mathcal{E}$ is rich. Then $T^{\mathcal{E}} \vdash \psi$ for every $\psi \in T^g$.*

Proof. Let φ be illegal and $\varphi \in \Gamma \setminus g(\Gamma)$ for some $\Gamma \in [\mathcal{L}]^\lambda$. Let $E \in \mathcal{E}$ be the extender which was used to define $g(\Gamma)$.

Write $\bar{\Gamma} = (\pi_E)^{-1}(\Gamma)$. We have that $\bar{\Gamma} \in [\mathcal{L}]^{\operatorname{crit}(E)}$ and

$$\bigvee\!\!\!\bigvee \pi_E{}''\bar{\Gamma} = \bigvee\!\!\!\bigvee \operatorname{ran}(\pi_E) \cap \Gamma = \bigvee\!\!\!\bigvee g(\Gamma),$$

and therefore

$$\varphi \to \bigvee\!\!\!\bigvee g(\Gamma)$$

by an instance of (7). We showed the relevant instance of T^g. $\square$

We immediately get from Claims 3.1 and 4.1:

Claim 4.2 *Suppose that $\mathcal{E}$ is rich. Then $\mathbb{P}^{\mathcal{E}}$ has the λ–c.c.*

Let us now step out of W. Let $A \subset \mu$, $A \in V$. We set

$$G'_A = \{\varphi \in \mathbb{P}^{\mathcal{E}}\colon A \vDash \varphi\}.$$

Virtually the same proof as the one of Claim 3.2 combined with Claim 4.1 shows:

Claim 4.3 *Suppose that $\mathcal{E} \in W$ is rich inside W. Assume also that $A \vDash T^{\mathcal{E}}$. Then $G'_A \subset \mathbb{P}^{\mathcal{E}}$ is a $\mathbb{P}^{\mathcal{E}}$–generic filter over W and*

$$A = \{\xi < \mu\colon \text{ ``}\check{\xi} \in \dot{a}\text{''} \in G'_A\} \in W[G'_A],$$

and G_A as defined on p. 5 is a $\mathbb{P}^g$–generic filter over W and

$$A = \{\xi < \mu\colon \text{ ``}\check{\xi} \in \dot{a}\text{''} \in G_A\} \in W[G_A].$$

Our next theorem produces a sufficient criterion for $A \subset \mu$, $A \in V$, to be generic over W which seems more useful than Claim 4.3.

This theorem suggests a way of making sets of ordinals generic over iterates of models with supercompact cardinals. However, the existence of iterable models with supercompact cardinals is a key problem of contemporary set theory.

Theorem 4.4 *Let $W \subset V$ be an inner model, let λ be a regular cardinal, and let $\mu \geq \mu$ be a cardinal. Let $\mathcal{E} \in W$ be a class of W–extenders which is rich inside W.*

Let $A \subset \mu$, $A \in V$, and suppose that for every $E \in \mathcal{E}$ there is some elementary embedding $\tilde{\pi} \supset \pi_E^W$ such that

$$\tilde{\pi} \colon V \to M$$

and $A \in \operatorname{ran}(\tilde{\pi})$.

Then G'_A is $\mathbb{P}^{\mathcal{E}}$–generic over W, G_A is $\mathbb{P}^g$–generic over W, and

$$A \in W[G_A] \cap W[G'_A].$$

Proof. By Claim 4.3, we only need to verify that $A \vDash T^{\mathcal{E}}$. Let $E \in \mathcal{E}$, and let

$$\pi_E^W : W \to \operatorname{ult}(W; E)$$

be the associated embedding as being formed inside W. By our hypotheses, there is some elementary embedding $\tilde{\pi} \supset \pi_E^W$ such that

$$\tilde{\pi} \colon V \to M$$

and $A \in \operatorname{ran}(\tilde{\pi})$. Write $\bar{A} = \tilde{\pi}^{-1}(A)$. Let $\kappa = \operatorname{crit}(E)$.

Let $\Gamma \in [\mathcal{L}]^\kappa \cap V^W_{\sigma(E)}$ and $\varphi \in \pi_E^W(\Gamma) \cap V^W_{\rho(E)}$, where $\sigma(E)$ and $\rho(E)$ are defined inside W.

Let us assume that $A \vDash \varphi$. Then

$$M \vDash \text{“}\exists \varphi' \in \tilde{\pi}(\Gamma)\, A \vDash \varphi',\text{”}$$

so that

$$V \vDash \text{“}\exists \varphi' \in \Gamma\, \bar{A} \vDash \varphi'.\text{”} \tag{9}$$

Let φ' be a witness to (9). Then

$$V \vDash \text{“}\bar{A} \vDash \varphi',\text{”}$$

hence

$$M^* \vDash \text{“}A \vDash \tilde{\pi}(\varphi'),\text{”}$$

which implies that

$$A \models \bigvee\!\!\!\bigvee \pi_E^W \text{"}\Gamma.$$

We have verified that A satisfies an arbitrary instance of $T^{\mathcal{E}}$. □

The attentive reader might now be tempted to use Theorem 4.4 to formulate a criterion for when V is generic over a given weak extender model W for λ being supercompact via some forcing with the λ–c.c. (cf. [10]), but any such criterion we were able to come up with gives its conclusion from its hypothesis already via Bukowský's Theorem 3.3, cf. Corollary 3.4. By a theorem of Woodin, cf. [11], it is consistent that $V = \mathsf{HOD}$ and if W is *any* weak extender model for the supercompactness of some λ which is Σ_2–definable from some $\alpha < \lambda$, then necessarily $W = V$. (But cf. [10, Theorem 174].)

We do have:

Theorem 4.5 *Let $W \subset V$ be an inner model, and let λ be a cardinal such that λ is supercompact inside W. The following are equivalent.*

(a) For every $a \in V$ there is some poset $\mathbb{P} \in W$ such that $\mathbb{P}$ has the λ–c.c. in W and there is some g which is $\mathbb{P}$–generic over W such that $a \in W[g]$.

(b) $V = W[g]$, where g is generic over W for the long extender algebra $\mathbb{P}^{\mathcal{E}}$, where $\mathcal{E} \in W$ is $(\lambda, 2^\lambda)$–rich inside W.

Proof. (b) $\Rightarrow$ (a) is trivial.

Let us show (a) $\Rightarrow$ (b). Write $\mu = 2^\lambda$, and let $A \subset \mu$ code $\mathcal{P}(\lambda) \cap V$ as in the statement of Theorem 3.3. Inside V, let $C \in [H_{\mu^+}]^{<\lambda}$ be club such that if $X \in C$, then

$$X \prec (H_{\mu^+}; \in, A).$$

By (a) and Corollary 3.4, there is some $D \in [H_{\mu^+}^W]^{<\lambda} \cap W$ such that $D \subset \{X \cap H_{\mu^+}^W \colon X \in C\}$ and D is club in W. Inside W, let $\mathcal{E}$ be the set of all extenders E with $\rho(E) = \mu + 1$ and $\mathrm{ran}(\pi_E^W) \cap H_{\mu^+}^W \in D$.

As λ is supercompact in W, $\mathcal{E}$ is rich inside W, via Magidor's characterization of "supercompactness," cf. e.g. [7, Problems 4.29 and 10.21]. It is easy to see that the hypotheses of Theorem 4.4 are satisfied. We then get (b) by the conclusion of Theorem 4.4. □

References

[1] Lev Bukovský, *Characterization of generic extensions of models of set theory*, Fundamenta Mathematica **83** (1973), pp. 35–46.

[2] Philipp Doebler and Ralf Schindler, *The extender algebra and vagaries of Σ^2_1 absoluteness*, Münster Journal of Mathematics **6** (2013), pp. 117–166.

[3] Sy Friedman, Sakaé Fuchino, and Hiroshi Sakai, *On the set-theoretic multiverse*, unpublished, available at
`http://kurt.scitec.kobe-u.ac.jp/~fuchino/papers/FrFuRoSa-e.pdf`

[4] Gunter Fuchs, Joel Hamkins, and Jonas Reitz, *Set-theoretic geology*, Annals of Pure and Applied Logic **166** (2015), pp. 464–501.

[5] Grigor Sargsyan and Ralf Schindler, *Varsovian models I*, in preparation.

[6] Grigor Sargsyan and Ralf Schindler, *Varsovian models II*, in preparation.

[7] Ralf Schindler, *Set theory. Exploring independence and truth*, Universitext, Springer–Verlag 2014.

[8] John Steel, *An outline of inner model theory*, in: Handbook of set theory (Foreman, Kanamori, eds.), Springer–Verlag 2010.

[9] Toshimichi Usuba, in preparation. (Draft sent to the present author by email on Oct 26, 2015.)

[10] W. Hugh Woodin, *Suitable extender models I*, Journal of Mathematical Logic **10** (2010), pp. 101–339.

[11] W. Hugh Woodin, private communication, June 01, 2016.