Paul Larson, Ralf Schindler, Linzhen Wu

NS_{ω_1} is not Π_1 definable.

Let $\mathbb{C}_{\omega_1}$ denote the forcing to add a Cohen subset of ω_1 using countable approximations.

Lemma 1 (folklore) Let g be $\mathbb{C}_{\omega_1}$-generic over V, and write $D = \bigcup g \subset \omega_1$. If $S \in \mathcal{P}(\omega_1) \cap V$ is stationary in V, then both $S \setminus D$, $S \cap D$ are stationary in $V[g]$.

Proof: Let $p \in \mathbb{C}_{\omega_1}$ be any condition, and let $\tau \in V^{\mathbb{C}_{\omega_1}}$ be s.t. $p \Vdash \tau \subset \check{\omega}_1$ is club. Pick $X \prec V_\theta$, θ suff. large, $\overline{\overline{X}} = \aleph_0$, $p, \tau \in X$, $X \cap \omega_1 = \alpha \in S$. It is easy to construct $q \leq p$, $lh(q) = \alpha$, s.t. for every dense $D \subset \mathbb{C}_{\omega_1}$, $D \in X$, there is some $\xi < \alpha$ with $q \upharpoonright \xi \in D \cap X$. In particular, $q \Vdash \check{\alpha} \in \tau \cap \check{S}$. We may then extend q to q^* s.t. $q^* \Vdash \check{\alpha} \in \dot{D}$ or $q^* \Vdash \check{\alpha} \notin \dot{D}$, as we please. (Here, $\dot{D}$ is the name for D.) $\dashv$

In other words, if g is $\mathbb{C}_{\omega_1}$-generic over V, then

$$V[g] \models \text{"} \exists D \; \exists (S_i : i < \lambda) \; [$$
$$\forall S \in H^V_{\omega_2} \; (H^V_{\omega_2} \models S \subset \omega_1 \text{ is stationary} \to \exists i < \lambda \;\; S_i = S \setminus D) \wedge$$
$$D \notin NS_{\omega_1} \wedge$$
$$\forall i < \lambda \;\; S_i \notin NS_{\omega_1} \;\;].$$

This statement is Σ_1 in the parameter $H^V_{\omega_2}$ and in the language $\mathcal{L}_{\dot{\in}, \dot{NS}_{\omega_1}}$ of set theory with an additional predicate for the non-stationary ideal.

Let us define

$$S = \{ X \prec H_{\omega_2} : \overline{\overline{X}} = \aleph_1 , \; X \text{ is transitive},$$
$$\exists D \; (D \text{ is stationary} \wedge$$
$$S \setminus D \text{ is stationary for all}$$
$$S \in \mathcal{P}(\omega_1) \cap X \text{ which are stationary}) \}$$

By the above and by the proof of theorem 1.3 of [ClaSch 12], we get the following.

Lemma 2 ($MA_{\omega_1}(\sigma\text{-closed})^{++}$) S is stationary.

We're now going to use this Lemma to produce the following.

Theorem 1 (BMM^{++} + $\exists$ a Woodin cardinal) there is no $A \subset \omega_1$ and no Σ_1 formula φ in the language $\mathcal{L}_{\dot{\in}}$ of set theory s.t. for all $S \in \mathcal{P}(\omega_1)$, S is stationary iff $\varphi(S, A)$.

Proof: Assume S is stationary iff $\varphi(S, A)$, A, φ as above.

Let g be $\mathbb{P}_{<\delta}$ - generic over V, δ being a Woodin cardinal, $\mathbb{P}_{<\delta}$ being the associated full stationary tower. Let us pick g s.t. $S \in g$ which is possible by Lemma 2. Let

$$\hat{j}: V \longrightarrow M \subset V[g]$$

be the generic elementary embedding given by g, where M is transitive.

We have that $j'' H^V_{\omega_2} \in j(S)$. In particular, $j'' H^V_{\omega_2}$ is transitive, hence $j'' H^V_{\omega_2} = H^V_{\omega_2}$ and $\mathrm{crit}(j) \geq \omega_2^V$. As $\overline{\overline{H^V_{\omega_2}}} = \aleph_1$ in M, $\mathrm{crit}(j) = \omega_2$.

Via a more substantial use of $H^V_{\omega_2} \in j(S)$,

$$\begin{aligned} M \models \exists D\, (& D \text{ is stationary } \wedge \\ & S \setminus D \text{ is stationary for all} \\ & S \in \mathcal{P}(\omega_1) \cap H^V_{\omega_2} \text{ which are stationary} \\ & (\text{in } M \Leftrightarrow \text{in } V)\,). \end{aligned}$$

By Theorem 2.5.8 of [Lar 04], ${}^{<\delta}M \cap V[g] \subset M$. This buys us that

$$\begin{aligned} V[g] \models \exists D\, (& D \text{ is stationary } \wedge \\ & S \setminus D \text{ is stationary for all} \\ & S \in \mathcal{P}(\omega_1) \cap V \text{ which are stationary in } V). \end{aligned}$$

Notice that in particular $V[g]$ is a stationary set preserving extension of V.

Now let $\mathbb{P}$ denote the forcing in $V[g]$ to shoot a club thru $\omega_1 \setminus D$, i.e., $p \in \mathbb{P}$ iff $\exists \alpha < \omega_1 \, (p : \alpha+1 \longrightarrow \omega_1 \setminus D$ is continuous$)$, ordered by end-extension.

<u>Lemma 3</u> (folklore) Let h be $\mathbb{P}$-generic over $V[g]$. If $S \in \mathcal{P}(\omega_1) \cap V$ is stationary in V, then S is stationary in $V[g,h]$.

Proof: Fix S. Let $p \in \mathbb{P}$ be any condition, and let $\tau \in V[g]^{\mathbb{P}}$ be s.t. $p \Vdash \tau \subset \omega_1^V$ is club. As $S \setminus D$ is stationary in $V[g]$, we may pick $X \prec V_\theta$, θ suff. large, $\bar{\bar{X}} = \aleph_0$, $p, \tau \in X$, $X \cap \omega_1 = \alpha \in S \setminus D$. It is easy to construct a function $q : \alpha \rightarrow \omega_1 \setminus D$, q continuous and end-extending p, s.t. for every dense $D \subset \mathbb{P}$, $D \in X$, there is ~~some~~ $\xi < \alpha$ with $q \restriction \xi \in D \cap X$. Then $q^* = q \cup \{(\alpha, \alpha)\} <_{\mathbb{P}} p$, and $q^* \Vdash \check{\alpha} \in \tau \cap \check{S}$. $\dashv$

In other words, $V[g,h]$ is a stationary set preserving extension of V.

But now by our hypothesis and by the elementarity of $j: V \to M$ we get that

$$M \models \varphi(D, A),$$

and hence

$$V[g] \models \varphi(D, A),$$

as φ is Σ_1. Hence in $V[g,h]$, $\varphi(D, A)$ holds true (again as φ is Σ_1), but D is non-stationary in $V[g,h]$. Therefore,

$$V[g,h] \models \exists D \,(\, D \text{ is nonstationary} \wedge \varphi(D,A)\,).$$

This statement is Σ_1 in the parameter A and in the language $\mathcal{L}_{\in}$ of set theory, so that we may use BMM to conclude that

$$V \models \exists D \,(\, D \text{ is nonstationary} \wedge \varphi(D, A)\,).$$

Contradiction!

The conclusion of theorem 1 was thus in fact proven from the hypothesis $MA_{\omega_1}(\sigma\text{-closed})^{++}$ + BMM + $\exists$ a Woodin cardinal.

$\dashv$

We are now going to derive the same conclusion from $(*) \equiv \exists g$ $\mathbb{P}_{max}$-generic over $L(\mathbb{R})$ with $H_{\omega_2}^V \subset L(\mathbb{R})[g]$.

Theorem 2. ($(*)$) There is no $A \subset \omega_1$ and no Σ_1 formula φ in the language $\mathcal{L}_\in$ of set theory s.t. for all $S \in \mathcal{P}(\omega_1)$, S is stationary iff $\varphi(S, A)$.

Proof: Suppose that $p \in \mathbb{P}_{max}$, $p \Vdash_{L(\mathbb{R})}$ "$\forall S$ (S is stationary iff $\varphi(S, \dot{A})$)."

Here, $\dot{A}$ is the canonical name for the distinguished subset of ω_1 which gives rise to the $\mathbb{P}_{max}$-generic filter. By [LueSchSch?] we know that the conclusion of Theorem 2 is true for $A \in L(\mathbb{R})$, so that we may assume w.l.o.g. that $A \in V \setminus L(\mathbb{R})$ and hence gives rise to a $\mathbb{P}_{max}$-generic filter.

Claim 1. There is $q <_{\mathbb{P}_{max}} p$, $q = (q; NS_{\omega_1}^q, b)$, s.t. if $i : p \longrightarrow p^* = (p^*; J, b)$ witnesses

$q <_{\mathbb{P}_{max}} p$, then

$$q \Vdash \text{“} \exists T \in \mathcal{P}(\omega_1) \setminus NS^q_{\omega_1} \ (\forall S \in (\mathcal{J}^+)^{p^*} \ S \setminus T \notin NS^q_{\omega_1}) \text{”.}$$

Proof of claim 1: Let $p \in q$, p countable in q, q being such that $\exists c [(q; NS^q_{\omega_1}, c)$ is a $\mathbb{P}_{max}$-condition]. Inside q, write $\omega_1 = \dot{\bigcup}_{i<\omega_1} T_i$, where each T_i is stationary in q. Working inside q, we may produce an iteration $i : p \to p^*$ of p of length ω_1^q s.t. if $p^* = (p^*; \mathcal{J}, b)$, $S \in (\mathcal{J}^+)^{p^*}$, then $T_i \setminus \xi \subset S$ for some i, $0 < i < \omega_1^q$, some $\xi < \omega_1^q$. We may then let $T = T_0$ and $q = (q; NS^q_{\omega_1}, b)$. ⊣

Now let q be as in Claim 1, and let g be $\mathbb{P}_{max}$-generic over $L(\mathbb{R})$ with $q \in g$. Write $V = L(\mathbb{R})[g] = L(\mathbb{R})[A]$, where $A \subset \omega_1$ is the distinguished set giving rise to g. By the choice of p and $q < p$, $q \in g$, in V: $S \subset \omega_1$ is stationary iff $\varphi(S, A)$.

Let $j : q \to q^* = (q^*; K, A)$ be the iteration of q of length ω_1 given by g. By standard $\mathbb{P}_{max}$ arguments, $K = NS^V_{\omega_1} \cap q^*$.

By elementarity,

$$V \models \text{“} j(T) \in \mathcal{P}(\omega_1) \setminus K \ (\forall S \in j(J^{+p^*}) \ S \setminus j(T) \notin K),$$

in other words,

$j(T)$ is stationary in V and for all $S \in j(J^{+p^*})$, $S \setminus j(T)$ is stationary in V. Hence

$$(H^V_{\omega_2}; NS^V_{\omega_1}, A) \models \text{“} j(T) \text{ is stationary}, \ \varphi(j(T), A), \text{ and } S \setminus j(T) \text{ is stationary for all } S \in j(J^{+p^*}).\text{”}$$

Now let L be the ideal generated from $NS^V_{\omega_1} \cup \{j(T)\}$, i.e., $X \in L$ iff $(X \setminus j(T)) \cap C = \emptyset$ for some club C. It is easy to see that L is a normal ideal, and $j(J^{+p^*}) \subseteq L^+$, so

$J = L \cap j(p^*)$.

By taking a countable hull in V, we have verified the following.

<u>Claim 2.</u> There is $r = (r; \bar{I}, a) <_{\mathbb{P}_{max}} q$ such that if $j : q \to q^*$ witnesses $r <_{\mathbb{P}_{max}} q$, then

$$r \models \exists T \in \bar{I} \; (T \text{ stationary} \wedge \varphi(T, a))$$

But now if h is $\mathbb{P}_{max}$-generic over $L(\mathbb{R})$ with $r \in h$ and $k : r \to r^*$ is the iteration of length ω_1 given by h, then $\varphi(k(T), k(a))$ and $k(T)$ is nonstationary in $L(\mathbb{R})[h]$. But $r <_{\mathbb{P}_{max}} p$, contradiction. $\dashv$

[ClaSch12] Claverie, Schindler, Woodin's axiom (*), etc., JSL <u>77</u> (2012), pp. 475–498.

[Lar04] Larson, The stationary tower, AMS 2004.

[LueSchSch17] Lücke, Schindler, Schlicht, Lightface definable subsets of H_{ω_2}, JSL 83 (2017), pp. 1106–1131.