A note on the $\lt \kappa$ mantle

If $\tau \subset \mathcal{O} \times \mathcal{O}$, then we write $\tau^{\{\xi\}}$ for $\{\xi : (\xi, \eta) \in \tau\}$; we call $\tau$ $\lambda$-small if $\tau^{\{\xi\}} \leq \lambda$ for all $\xi$, $\lambda$ a cardinal.

**Lemma 1.** Let $V = W[G]$, where $g$ is $\mathcal{P}$-generic over $W$, $\mathcal{P} \in W$ and $\mathcal{P}$ has the $\lambda^+-$c.c.

Every $\lambda$-small $\tau \subset \theta \times \mathcal{O}$ in $V$ is covered by a $\lambda$-small $\tau \subset \theta \times \mathcal{O}$ in $W$.

**Proof:** For each $\xi \in \theta$, pick $a_\xi \in [\mathcal{O}]^\lambda \cap N$, $a_\xi \supset \tau^{\{\xi\}}$. Then pick $R \subset \theta \times ([\mathcal{O}]^\lambda \cap N)$, $R \in W$, such that $a_\xi \in R^{\{\xi\}}$ and $R^{\{\xi\}} \leq \lambda$ for all $\xi$. Let $s \subset \theta \times \mathcal{O}$ be s.t. $s^{\{\xi\}} = \bigcup R^{\{\xi\}}$ for $\xi \in \theta$.

Let $\kappa$ be a cardinal, the $\lt \kappa$ mantle (of $V$) is the intersection of all $W$, where...
$W[g] = V$, some $g$ which is $P$-generic over $W$ for a $P \in W \cap V$.

**Theorem 1.** Let $\kappa$ be inaccessible. The $\kappa$ mantle contains a inner model $W$, such that every $X \in V$ is $P$-generic over $W$ for some $P$ which has the $\kappa^+$-c.c.

**Proof:** Every $\kappa$ ground $P$ is identified by the witnessing $TP, g$ and a rank initial segment of $P$ of height $\kappa$, so that there are at most $\kappa \cdot \kappa^+$ grounds $P$ of $V$. Let

$$\left( P_i : i < \kappa \right)$$

be a list of all of them.

Let $f : \Theta \rightarrow OR$. We claim that there is some $r \in \Theta \times OR$ in the $\kappa$ mantle such that $r \supset f$ and $r$ is $\kappa$-small.

In order to show this, let us construct sequences $\left( r_{ij} : i < \kappa, j \leq i \right)$, $\left( \lambda_i : i < \kappa \right)$,
as follows.

For each \( i \leq \lambda \), let \( \tau_i \) be the least regular cardinal \( (\leq \lambda) \) such that each \( W_j \), \( j+1 \leq i \), is a \( \leq \lambda \) ground via a forcing which has the \( \lambda_i \)-c.c. We shall maintain that \( \tau_{ij} \) is \( \lambda_i \)-small.

Suppose \( \tau_{ij} \) has been chosen to be \( \lambda_i \)-small, \( i \leq \lambda \), \( j+1 \leq i \). Using Lemma 1, we then pick \( \tau_{ij+1} \in W_j \), \( \tau_{ij+1} \supset \tau_{ij} \), \( \tau_{ij+1} \cap \Theta \times \Theta \).

If \( j \leq i \) is a limit ordinal, \( i \leq \lambda \), and \( \tau_{ij} \) has been chosen for all \( j < i \), then we set \( \tau_{ij} = U \{ \tau_{ij} : j < i \} \).

If \( \tau_i \) has been chosen, \( i \leq \lambda \), we set \( \tau_{i+1,0} = \tau_i \).

If \( i \leq \lambda \) is a limit ordinal and \( \tau_{ij} \) has been chosen for all \( j \leq i < \lambda \), then we set \( \tau_{i,0} = U \{ \tau_{ij} : j \leq i < \lambda \} \).
This finishes the construction. We finally set 
\[ \tau = \bigcup \{ \tau_{i,j} : j \leq i < \kappa \} \, . \]

Of course, \( \tau \subset \Theta \times OR, \) and \( \tau \) is \( \kappa \)-small.

Also, \( f \in \tau. \) We need to verify that \( \tau \) is in the \( \kappa \)-mantle.

Let \( i < \kappa. \) To show that \( \tau \in W_i, \) it suffices to show that if \( A \subset \Theta \times OR, \ A \in W_i, \)

\( \bar{A} \leq \gamma_i, \) then \( \tau \cap A \in W_i. \) (This follows by approximation.) But \( \tau \cap A = \tau_{i*, j*} \cap A \) for a tail end of \( j* \leq i* < \kappa; \) in particular, there is some \( i* \geq i+1 \) s.t. \( \tau \cap A = \tau_{i*, i+1} \cap A; \)

as \( \tau_{i*, i+1} \in W_i, \) \( \tau \cap A \in W_i. \)

Theorem 1 now follows by well-known arguments.

In fact, we may finish off the proof of Theorem 1 as follows.

For each \( \lambda, \) there is some \( \Theta_{\lambda} \) and some \( \kappa\)-small \( \tau_{\lambda} \subset \Theta_{\lambda} \times OR \) such that \( \tau_{\lambda} \in \) the \( \kappa \)-mantle and if \( f \in H_{\lambda} \) is a function from a
ordinal to the ordinals, then there is some \( r \in L[\mathcal{E}] \) such that \( r \) is \( \kappa \)-small, \( r \supset f \).

There is a proper class \( X \) s.t. for all \( \lambda, \lambda' \in X \),
\[
\left( \frac{H}{(2^{2^{\lambda'}})^+} \right)^{L[\mathcal{E}]} = \left( \frac{H}{(2^{2^{\lambda}})^+} \right)^{L[\mathcal{E}]}.
\]

By Bukovsky's theorem and the folklore result, Theorem 2.2 of "The long extendible algebra," \( H^\lambda \) is generic over \( L[\mathcal{E}] \) for some forcing of size \( 2^{2^\kappa} \) which has the \( \kappa^+-\text{c.c.} \), all \( \lambda \in X \), and we may assume that \( X \) was chosen in a way that this forcing is always the same.

Then \( W = \bigcup \{ H^\lambda_{\mathcal{E}} : \lambda \in X \} \) is as desired.

**Theorem 2.** Let \( W \) be as in Theorem 1.

Then \( H^{\kappa^+} \) is \( P \)-generic on \( W \) for some forcing of size \( 2^{\kappa} \) which has the \( \kappa^+-\text{c.c.} \),
and \( W[H_{k+}^+] = V \).

Proof: We just need to see that \( PP(\kappa) \subset W[H_{k+}^+] \). Trivially, \( P(\kappa) \subset W[H_{k+}^+] \).

\( PP(\kappa) \) is \( \mathcal{P} \)-generic over \( W[H_{k+}^+] \) for some \( \mathcal{P} \) which has the \( \kappa^+ \)-c.c., by Bukowsky.

Let \( g \) be \( \mathcal{P} \)-gen. with \( PP(\kappa) \subset W[H_{k+}^+]g \), and let \( \tau^g = PP(\kappa) \).

Assume there is no \( p \in g \) which decides "\( \exists \tau \in T \) for all \( X \subset \kappa \), \( X \in W[H_{k+}^+] \) (\( \exists \tau \in V \)).

We may then produce \( \mathcal{Q} \) an antichain \( \{q_i : i < \kappa^+\} \) in \( \mathcal{P} \) as usual. Contradiction!

Theorem 3. Let \( \kappa \) be a measurable cardinal.

The \( < \kappa \) mantle is a model of \( ZFC \).

Proof: Let \( P \) be a \( < \kappa \) ground of \( V \), say \( P[g] = V \), \( g \) \( \mathcal{P} \)-generic over \( P \) for some
There exists a cardinal \( \kappa \) such that \( \mathcal{P} \subseteq \kappa \). Let \( j : V \rightarrow M = \text{ul}(V; \mathcal{P}) \) where \( \mathcal{P} \) is a fixed measure on \( \kappa \) witnessing \( \kappa \) is a measurable cardinal.

We have that \( \mathcal{U} \cap \mathcal{P} \subseteq \mathcal{P} \) whenever \( \kappa \) is a measurable cardinal in \( \mathcal{P} \), and \( j(\mathcal{P}) = \text{ul}(\mathcal{P}; \mathcal{U} \cap \mathcal{P}) \) and \( j^\mathcal{P} \mathcal{P} \) is the ultrapower embedding. [E.g., this is an elementary embedding from \( \text{ul}(\mathcal{P}; \mathcal{U} \cap \mathcal{P}) \mathcal{E}_0 \) to \( j(\mathcal{P})[\mathcal{E}_0] = M \) given by \( (j^\mathcal{P} \mathcal{P} \mathcal{E}_0 (x)) \mapsto j^\mathcal{P} \mathcal{P} \mathcal{E}_0 (x) \) which is onto, etc.]

\( j(\mathcal{P}) \) is a \( \kappa \)-ground of \( M \). By Theorem 2, there is an inner model \( W \subseteq \bigcap \{ j(\mathcal{P}) : \mathcal{P} \text{ is a } \kappa \text{-ground of } V \} \) such that \( W \) is a \( \kappa^+ \)-ground of \( M \).

Let \( \widehat{W} = W[j^\mathcal{P} \mathcal{P} \mathcal{R}] \), (we may think of \( W = L[A], A \subseteq \kappa^+, \text{ and then } \widehat{W} = L[A, j^\mathcal{P} \mathcal{P} \mathcal{R}] \).)

Also write \( W^* \) for the \( \kappa \)-mantle of \( V \).
We have that \( W \subset \cap \{ \text{ult}(p; \text{uup}) : P \text{ is a } <k \text{ ground of } V \} \subset \cap \{ P : P \text{ is a } <k \text{ ground of } V \} = W^* \).

Also, \( j \) is amenable to every \( <k \) ground \( P \) of \( V \), so that

\[
(1) \quad \tilde{W} = W[\mathcal{J}^{\forall} \mathcal{O}^R] \subset W^*.
\]

Let \( X \) be a set of ordinals in \( W^* \). Then \( j(X) \) is in every \( <j(x) \) ground of \( M \), in particular, \( j(X) \in W \) (as \( W \) is a \( k^+ \) ground of \( M \)). But then \( X = \{ \exists \tilde{y} : j(\tilde{y}) \in j(X) \} \in W[\mathcal{J}^{\forall} \mathcal{O}^R] \). This shows that

\[
(2) \quad \mathcal{P}(\mathcal{O}^R) \cap W^* \subset \tilde{W}.
\]

Therefore, \( W^* \), the \( <k \) mantle of \( V \), is equal to \( \tilde{W} \) and hence a model of \( ZFC \). 

*) using a theorem of Vopěnka and Balcar