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A note on the $<\kappa$ mantle

If $\tau \subset OR \times OR$, then we write $\tau''\{\xi\}$ for $\{\eta : (\xi, \eta) \in \tau\}$; we call τ <u>λ-small</u> iff $\overline{\overline{\tau''\{\xi\}}} \leq \lambda$ for all ξ, λ a cardinal.

<u>Lemma 1.</u> Let $V = W[g]$, where g is $\mathbb{P}$-generic over W, $\mathbb{P} \in W$ and $\mathbb{P}$ has the λ^+-c.c.

Every λ-small $\tau \subset \theta \times OR$ in V is covered by a λ-small $s \subset \theta \times OR$ in W.

Proof: For each $\xi < \theta$, pick $a_\xi \in [OR]^\lambda \cap W$, $a_\xi \supset \tau''\{\xi\}$. Then pick $R \subset \theta \times ([OR]^\lambda \cap W)$, $R \in W$, such that $a_\xi \in R''\{\xi\}$ and $\overline{\overline{R''\{\xi\}}} \leq \lambda$ for all ξ. Let $s \subset \theta \times OR$ be s.t. $s''\{\xi\} = \bigcup R''\{\xi\}$ for $\xi < \theta$. $\dashv$

Let κ be a cardinal. The $<\kappa$ <u>mantle</u> (of V) is the intersection of all W, where

$W[g] = V$, some g which is $\mathbb{P}$-generic over W for a $\mathbb{P} \in W \cap V_\kappa$.

<u>Theorem</u> 1. Let κ be inaccessible. The $<\kappa$ mantle contains an inner model, W, such that every $X \in V$ is $\mathbb{P}$-generic over W for some $\mathbb{P}$ which has the κ^+-c.c.

Proof: Every $<\kappa$ ground P is identified by the witnessing $\mathbb{P}$, g and a rank initial segment of P of height $<\kappa$, so that there are at most κ $<\kappa$ grounds P of V. Let

$$(P_i : i < \kappa)$$

be a list of all of them.

Let $f : \theta \longrightarrow OR$. We claim that there is some $\tau \subset \theta \times OR$ in the $<\kappa$ mantle such that $\tau \supset f$ and τ is κ-small.

In order to show this, let us construct sequences $(\tau_{i,j} : i < \kappa,\ j \leq i)$, $(\lambda_i : i < \kappa)$

as follows.

For each $i<\kappa$, let λ_i be the least regular cardinal $(<\kappa)$ such that each W_j, $j+1\leq i$, is a $<\kappa$ ground via a forcing which has the λ_i-c.c. We shall maintain that $r_{i,j}$ is λ_i-small.

Suppose $r_{i,j}$ has been chosen to be λ_i-small, $i<\kappa$, $j+1\leq i$. Using Lemma 1, we then pick $r_{i,j+1}\in W_j$, $r_{i,j+1}\supset r_{i,j+1}$, $r_{i,j+1}\subset\Theta\times\mathrm{OR}$, $r_{i,j+1}$ λ_i-small.

If $j\leq i$ is a limit ordinal, $i<\kappa$, and $r_{i,\bar{j}}$ has been chosen for all $\bar{j}<j$, then we set

$$r_{i,j}=\bigcup\{r_{i,\bar{j}}:\bar{j}<j\}.$$

If $r_{i,i}$ has been chosen, $i<\kappa$, we set $r_{i+1,0}=r_{i,i}$.

If $i<\kappa$ is a limit ordinal and $r_{\bar{i},j}$ has been chosen for all $j\leq\bar{i}<i$, then we set

$$r_{i,0}=\bigcup\{r_{\bar{i},j}:j\leq\bar{i}<i\}.$$

This finishes the construction. We finally set

$$\tau = \bigcup\{\tau_{i,j} : j \leq i < \kappa\}.$$

Of course, $\tau \subset \Theta \times OR$, and τ is κ-small. Also, $f \subset \tau$. We need to verify that τ is in the $<\kappa$ mantle.

Let $i < \kappa$. To show that $\tau \in W_i$, it suffices to show that if $A \subset \Theta \times OR$, $A \in W_i$, $\overline{\overline{A}} \leq \lambda_i$, then $\tau \cap A \in W_i$ (this follows by approximation). But $\tau \cap A = \tau_{i^*, j^*} \cap A$ for a tail end of $j^* \leq i^* < \kappa$; in particular, there is some $i^* \geq i+1$ s.t. $\tau \cap A = \tau_{i^*, i+1} \cap A$; as $\tau_{i^*, i+1} \in W_i$, $\tau \cap A \in W_i$.

Theorem 1 now follows by well-known arguments. In fact, we may finish off the proof ~~of~~ of Theorem 1 as follows.

For each λ there is some Θ_λ and some κ-small $\tau_\lambda \subset \Theta_\lambda \times OR$ such that $\tau_\lambda \in$ the $<\kappa$ mantle and if $f \in H_\lambda$ is a function from an ordinal to the ordinals, then there is some $\tau \in L[r_\lambda]$ such that τ is κ-small, $\tau \supset f$.

There is a proper class X s.t. for all $\lambda, \lambda' \in X$, $\left(H_{(2^{2^\kappa})^+}\right)^{L[r_\lambda]} = \left(H_{(2^{2^\kappa})^+}\right)^{L[r_{\lambda'}]}$.

By Bukowský's theorem and the folklore result ~~[illegible]~~ theorem 2.2 of "The long extender algebra," H_λ^V is generic over $L[r_\lambda]$ for some forcing of size 2^{2^κ} which has the κ^+-c.c., all $\lambda \in X$, and we may assume that X was chosen in a way that this forcing is always the same.

Then $W = \bigcup \{ H_\lambda^{L[r_\lambda]} : \lambda \in X \}$ is as desired. $\dashv$

<u>Theorem 2.</u> Let W be as in Theorem 1. Then H_{κ^+} is $\mathbb{P}$-generic over W for some forcing of size 2^κ which has the κ^+-c.c.,

and $W[H_{\kappa^+}] = V$.

Proof: We just need to see that $\mathcal{P}\mathcal{P}(\kappa) \subset W[H_{\kappa^+}]$. Trivially, $\mathcal{P}(\kappa) \subset W[H_{\kappa^+}]$.

$\mathcal{P}\mathcal{P}(\kappa)$ is $\mathbb{P}$-generic over $W[H_{\kappa^+}]$ for some $\mathbb{P}$ which has the κ^+-c.c., by Bukowský.

Let g be $\mathbb{P}$-gen. with $\mathcal{P}\mathcal{P}(\kappa) \subset W[H_{\kappa^+}][g]$, and let $\tau^g = \mathcal{P}\mathcal{P}(\kappa)$.

Assume there is no $p \in g$ which decides "$\check{X} \in \tau$" for all $X \subset \kappa$, $X \in W[H_{\kappa^+}]$ ($\Leftrightarrow X \in V$).

We may then produce ~~[illegible]~~ an antichain $(q_i : i < \kappa^+)$ in $\mathbb{P}$ as usual. Contradiction!

$\dashv$

<u>Theorem 3.</u> Let κ be a measurable cardinal. The $<\kappa$ mantle is a model of ZFC.

Proof: Let P be a $<\kappa$ ground of V, say $P[g] = V$, g $\mathbb{P}$-generic over P for some

$\mathbb{P} \in P$ of size $< \kappa$. Let $j : V \longrightarrow M = \text{ult}(V; \mathcal{U})$ where $\mathcal{U}$ is a fixed measure on κ witnessing κ is a measurable cardinal.

We have that $\mathcal{U} \cap P \in P$ witnesses that κ is measurable in P, and $j(P) = \text{ult}(P; \mathcal{U} \cap P)$ and $j \restriction P$ is the ultrapower embedding. [E.g., there is an elementary embedding from $\text{ult}(P; \mathcal{U} \cap P)[g]$ to $j(P)[g] = M$ given by $\left(j^P_{\mathcal{U} \cap P}(f)(\kappa)\right)^g \mapsto j(f)^g(\kappa)$ which is onto, etc.]

$j(P)$ is a $< \kappa$ ground of M. By Theorem 2, there is an inner model $W \subset \bigcap \{ j(P) : P \text{ is a } < \kappa \text{ ground of } V \}$ such that W is a κ^+ ground of M.

Let $\widetilde{W} = W[j \restriction OR]$. (We may think of $W = L[A]$, $A \subset OR$, and then $\widetilde{W} = L[A, j \restriction OR]$.)

Also write W^* for the $< \kappa$ mantle of V.

We have that $W \subset \bigcap \{ ult(P; U \cap P) :$ P is a $<\kappa$ ground of $V \} \subset$

$\bigcap \{ P : P$ is a $<\kappa$ ground of $V \} = W^*$.

Also, j is amenable to every $<\kappa$ ground P of V, so that

(1) $$\tilde{W} = W[j \restriction OR] \subset W^*.$$

Let X be a set of ordinals in W^*. Then $j(X)$ is in every $<j(\kappa)$ ground of M, in particular, $j(X) \in W$ (as W is a κ^+ ground of M). But then $X = \{ \xi : j(\xi) \in j(X) \}$ $\in W[j \restriction OR]$. This shows that

(2) $$\mathcal{P}(OR) \cap W^* \subset \tilde{W}.$$

Therefore, W^*, the $<\kappa$ mantle of V, is equal to $\tilde{W}$ *) and hence a model of ZFC.

$\dashv$

*) using a theorem of Vopěnka and Balcar