Dilemmas and truths in set theory

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Set Theory and Cantor's Continuum Hypothesis

- Set theory started with the following theorem of Georg Cantor.
- Cantor (Nov 11, 1873, in a letter to R. Dedekind): $\mathbb{R}$ is uncountable. I.e., there are uncountably many real numbers.
- Cantor's first proof of this used nested intervals.
- But how many real numbers are there?
- Continuum Hypothesis (CH): For every uncountable $A \subset \mathbb{R}$ there is a bijection $f : \mathbb{R} \to A$.
- Cantor's Program: Show CH by "induction on the complexity" of $A \subset \mathbb{R}$.

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- Young (1906): Every uncountable G_δ– oder F_σ–set $A \subset \mathbb{R}$ contains a perfect subset.
- Aleksandrov/Hausdorff (1916): Every uncountable *Borel* set $A \subset \mathbb{R}$ contains a perfect subset.
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Cantor's Generalized Continuum Hypothesis

- In addition to sets of natural numbers, of reals, of sets of reals, etc., Cantor started considering sets *in general*.
- "By a 'set' we understand any gathering-together M of determined well-distinguished objects m of our intuition or of our thought, into a whole." (Cantor, 1995)
- This idea leads to the cumulative hierarchy of sets.
- For every set x whatsoever, the *power set* $\mathcal{P}(x)$ exists.

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- Cantor's Theorem (1892): Let x be any set. There is no surjection $f: x \to \mathcal{P}(x)$.
- This time, Cantor's proof uses a diagonal argument.
- *How big* is $\mathcal{P}(x)$ in comparison to x?
- Generalized Continuum Hypothesis (GCH): For every infinite set x and every $A \subset \mathcal{P}(x)$, there is either a surjection $f: x \to A$ or else a bijection $f: \mathcal{P}(x) \to A$.
- We need to talk about axiomatizations of set theory in order to discuss CH and GCH.

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The axiom system ZFC (Zermelo–Fraenkel with choice)

- Any two sets with the same elements are equal.
- For all x and y, $\{x, y\}$, $\bigcup x$, and $\mathcal{P}(x)$ exist.
- There is an infinite set.
- Separation. For all x and for all formulae $\varphi(y)$, $\{y \in x \colon \varphi(y)\}$ exists.
- Replacement. For all x and for all formulae $\varphi(y, z)$ such that for all $y \in x$ there is a unique z with $\varphi(y, z)$, $\{z \colon \exists y \in x \varphi(y, z)\}$ exists.
- Every x with $\emptyset \notin x$ admits a choice function.
- Every nonempty set has an $\in$–least element.

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- It may be shown, though, that these parameters are not needed:
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- ZFC formalizes the idea (albeit somewhat indirectly) that the universe of set theory arises from nothing ($\emptyset$) through the operations $x \mapsto \mathcal{P}(x)$ and $x \mapsto \bigcup x$ in a cumulative fashion:
- If we define $V_\alpha = \bigcup\{\mathcal{P}(V_\beta) \colon \beta < \alpha\}$ for ordinals α, then ZFC proves that every x is an element of some V_α. The V_α's are called *ranks*.
- Provably, there is no set of all sets. (By Cantor's Theorem: if v were such a set, then there would be a surjection from v onto $\mathcal{P}(v)$.)
- However, we may introduce a new category of objects, *classes* ("inconsistent multiplicities" in the language of Cantor), and there will be a class of all sets.

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Classes and Truth

- The introduction of classes is tantamount to adding a *truth predicate* to the language of set theory.
- BGC (Bernays–Gödel with choice) results from ZFC by adding a new sort of variables, class variables X, Y, ..., and demanding that the universe of all classes is closed under the logical oprations; instead of talking about formulae in Separation and Replacement we now talk about classes.
- A philosophical credo. In contrast to sets, classes do not exist *de re*, they just exist *de dicto*. Otherwise the collection of all classes would just be another rank of the set theoretical universe, and what appeared to be classes are in fact sets.

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- Tarski (1936)/Mostowski (1950): Whereas the truth predicate for set theory cannot be defined in the language of ZFC, it may be defined in the language of BGC in a Δ^1_1 fashion. All instances of the Tarski schema

$$\varphi \longleftrightarrow \ulcorner\varphi\urcorner \text{ is true}$$

  for set theoretical φ may be proven in BGC.
- Sch (2002): The Tarski *sentence* of negation,

$$\forall\ulcorner\varphi\urcorner\,(\ulcorner\neg\,\varphi\urcorner \text{ is true } \longleftrightarrow \neg\,\ulcorner\varphi\urcorner \text{ is true })$$

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- Sch (1993): We may in fact define a Tarskian hierarchy of meta–languages for the language of set theory.
- Each language $\mathcal{L}_\alpha$ comes with a new sort of variables for classes of type α. We demand that if $\varphi(x)$ is from $\mathcal{L}_\beta$, some $\beta < \alpha$, then

$$\{x \colon \varphi(x)\}$$

exists as a class of type α.
- The truth predicate for $\bigcup_{\beta<\alpha} \mathcal{L}_\beta$ may then be defined in $\mathcal{L}_\alpha$, and we may formulate natural theories BGC^α which prove the appropriate Tarski schemas.

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Large cardinals

- Replacement may be construed as a "large cardinal axiom." It says that for every formula φ there is a rank V_α which refects φ, i.e.,

$$\varphi(x_1, ...x_k) \longleftrightarrow V_\alpha \models \varphi(x_1, ..., x_k)$$

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- Here is a list of some of the large cardinal concepts which are on the market nowadays.
- Inaccessible $<$ Mahlo $<$ weakly compact $<$ measurable $<$ strong $<$ Woodin $<$ subcompact $<$ supercompact $<$ I_0 $<$ ...
- Shelah/Woodin (1990): If there are infinitely many Woodin cardinals, then CH holds for all projective sets.
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- Bernays has formulated a system of class theory which proves the existence of inaccessible and Mahlo cardinals via reflection principles.
- Bernays' System $B_{\texttt{refl}}$ is BGC together with the following schema of reflection. For every formula φ in the language of BGC with no class quantifiers,

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- On the other hand, there are many statements which imply the *consistency* of the existence of large cardinals with ZFC, in fact the existence of canonical inner models with such large cardinals.
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