From set theoretic to inner model theoretic geology*†

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1 Introduction

In the 1970ies, Lev Bukovský proved a beautiful criterion for when V is a generic extension of a given inner model W, see [1] and [2]. Bukovský's theorem recently served as a very useful tool in set theoretic geology, see e.g. [21] and [22], and also in inner model theoretic geology, see [13] and [15].

In this paper we shall give a proof of Bukovský's theorem and a presentation of Woodin's extender algebra (see e.g. [8], [4], [3], [20, pp. 1657ff.]) in a uniform fashion – one argument and one forcing will produce both results, see Theorem 3.11.

We shall also reproduce Usuba's results on the set directedness of grounds and on the mantle of V in the presence of an extendible cardinal.

We shall then discuss the application of these techniques to recent developments in inner model theoretic geology, namely to the theory of *Varsovian models*.

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2 Basic concepts

Definition 2.1 *Let M be an inner model of V. Let δ be a regular cardinal.*

(1) M δ-covers V *if for all sets* $X \subset M$, $X \in V$, *with* $\text{Card}(X) < \delta$ *there is some* $Y \supset X$, $Y \in M$, *such that* $\text{Card}(Y) < \delta$.

(2) M uniformly δ-covers V *iff for all functions* $f \in V$ *with* $\text{dom}(f) \in M$ *and* $\text{ran}(f) \subset M$ *there is some function* $g \in M$ *with* $\text{dom}(g) = \text{dom}(f)$ *such that* $f(x) \in g(x)$ *and* $\text{Card}(g(x)) < \delta$ *for all* $x \in \text{dom}(g)$.

(3) M δ-approximates V *iff for all* $A \in V$, *if* $A \cap a \in M$ *for every* $a \in M$ *with* $\text{Card}(a) < \delta$, *then* $A \cap M \in M$.

Trivially, "M uniformly δ-covers V" implies "M δ-covers V," and if M uniformly δ-covers V and $\mu \geq \delta$ is regular, then M also uniformly μ-covers V. Also, if M δ-approximates V and $\mu \geq \delta$ is regular, then M also μ-approximates V. (1) is equivalent to the statement where X is assumed to be a set of ordinals, (2) is equivalent to the statement where f is assumed to be a function from an ordinal to the ordinals, and (3) is equivalent to the statement where A is assumed to be a (characteristic function of a) set of ordinals.

If there is some poset $\mathbb{P} \in M$ having the δ–c.c. in M and some g which is $\mathbb{P}$-generic over M such that $V = M[g]$, then M uniformly δ-covers V, see e.g. the proof of [17, Lemma 6.32]. Bukovský's Theorem 3.11 will say that the converse is true also.

Lemma 2.2 *Let M be an inner model, and let δ be a regular cardinal. Asume that M uniformly δ-covers V. The following are true.*

(a) For every $\alpha \geq \delta$, if $C \in \mathcal{P}([\alpha]^{<\delta}) \cap V$ is club in V, then there is some $D \in \mathcal{P}([\alpha]^{<\delta}) \cap M$ with $D \subset C$ and D is club in M.

(a') For every $\alpha \geq \delta$, if $C \in \mathcal{P}([\alpha]^{<\delta}) \cap V$ is club in V, then there is some $S \in \mathcal{P}([\alpha]^{<\delta}) \cap M$ with $S \subset C$ and S is stationary in M.

(b) For every cardinal $\theta \geq \delta$, if $C \in \mathcal{P}([H_\theta^V]^{<\delta}) \cap V$ is club in V, then there is some $D \in \mathcal{P}([H_\theta^M]^{<\delta}) \cap M$ which is club in M and such that for all $X \in D$ there is some $Y \in C$ with $X = Y \cap M$.

(b') For every cardinal $\theta \geq \delta$, if $C \in \mathcal{P}([H_\theta^V]^{<\delta}) \cap V$ is club in V, then there is some $S \in \mathcal{P}([H_\theta^M]^{<\delta}) \cap M$ which is stationary in M and such that for all $X \in S$ there is some $Y \in C$ with $X = Y \cap M$.

Proof. We will in fact show that our hypotheses imply (a), and that (a) $\iff$ (b), (a) $\implies$ (a'), and (a') $\iff$ (b').

(a): Let $f\colon [\alpha]^{<\omega} \to \alpha$, $f \in V$, be such that if $X \in [\alpha]^{<\delta}$ and $f''[X]^{<\omega} \subset X$, then $X \in C$. By Definition 2.1 (2), there is $g \in M$ such that $\mathrm{dom}(g) = \mathrm{dom}(f)$, and $f(x) \in g(x)$ and $\mathrm{Card}(g(x)) < \delta$ for all $x \in [\alpha]^{<\omega}$. Inside M, let D be the set of all $X \in [\alpha]^{<\delta}$ such that $\bigcup g''[X]^{<\omega} \subset X$. Then $D \subset C$ and D is club in M.

(a) $\implies$ (b): Let $f\colon [H_\theta^V]^{<\omega} \to H_\theta^V$, $f \in V$, be such that if $X \in [H_\theta^V]^{<\delta}$ and $f''[X]^{<\omega} \subset X$, then $X \in C$. Let $f^*\colon \omega \times [H_\theta^V]^{<\omega} \to H_\theta^V$, $f^* \in V$, be such that $f^*(0, \vec{x}) = f(\vec{x})$ and for all m, n_1, ..., $n_k < \omega$ there is some $n < \omega$ such that for all $\vec{x}_1$, ..., $\vec{x}_k$, $f^*(n, \vec{x}_1 \ldots \vec{x}_k) = f^*(m, f^*(n_1, \vec{x}_1) \ldots f^*(n_k, \vec{x}_k))$.

Let $e\colon \alpha \to H_\theta^M$, $e \in M$, be bijective. Let $\bar{f} \in V$ be the partial function with domain contained in $\omega \times [\alpha]^{<\omega}$ defined as the pullback of $f^* \cap H_\theta^M$ under e^{-1}, i.e., $\bar{f}(n, \vec{\xi}) \downarrow$ iff $f^*(n, e(\vec{\xi})) \in H_\theta^M$, in which case $\bar{f}(n, \vec{\xi}) = e^{-1}(f^*(n, e(\vec{\xi})))$.

We then have that $C' = \{X \in [\alpha]^{<\delta} \colon \bar{f}''\omega \times [X]^{<\omega} \subset X\}$ is club in V. By (b), let $D' \subset C'$, $D' \in M$, be club. Then $D = \{e''X \colon X \in D'\} \in M$ is club in $[H_\theta^M]^{<\delta}$.

If $X \in D'$ and if $Y = e''X \cup f^{*}{}''(\omega \times [e''X]^{<\omega})$, then

(i) $Y \cap H_\theta^M = e''X$, and

(ii) $f''[Y]^{<\omega} \subset Y$, so that $Y \in C$.

In other words, D is as desired for (b).

(b) $\implies$ (a) is easy, (a) $\implies$ (a') is trivial, and (a') $\iff$ (b') is exactly like the proof of (a) $\iff$ (b). □

Theorem 2.3 *Let M be an inner model of V, and let δ be an infinite regular cardinal. Assume that M uniformly δ-covers V Then M δ^+-approximates V.*

Proof. Let us call any set A of functions an *antichain* iff for all a, $b \in A$ with $a \neq b$ there is some $i \in \mathrm{dom}(a) \cap \mathrm{dom}(b)$ with $a(i) \neq b(i)$.

Assume that $B\colon \alpha \to 2$, for some ordinal α, is such that $B \in V \setminus M$ but $B \restriction x \in M$ for all $x \in M$ with $\mathrm{Card}(x) \leq \delta$. We aim to derive a contradiction.

Let us write $\mathcal{F}$ for the collection of all functions $a \in M$ such that there is some $x \subset \alpha$ of size $< \delta$ such that $a\colon x \to 2$. Using the fact that M uniformly δ–covers V, we may pick a function g in M such that if $A \subset \mathcal{F}$ is an antichain with $A \in M$, then

(i) $g(A) \in M$ is a subset of A of size $< \delta$, and

(ii) if there is some (unique!) $a \in A$ with $a = B \upharpoonright \text{dom}(a)$, then $a \in g(A)$.

We call $a \in \mathcal{F}$ *legal* iff for no antichain $A \in M$, $a \in A \setminus g(A)$. Notice that being legal is defined inside M (from the parameter $g \in M$).

Every $B \upharpoonright x$, where $x \in M$, $x \subset \alpha$, and x has size $< \delta$, is legal.

If $A \subset \mathcal{F}$ is an antichain with $A \in M$, and if every $a \in A$ is legal, then we must have $g(A) = A$, from which it follows that A has size $< \delta$.

Modulo breakdown, we shall now construct an antichain $A = \{a_i \colon i < \delta\}$ of legal elements of $\mathcal{F}$ of size δ as follows. Let $<_{\mathcal{F}} \in M$ be a well order of $\mathcal{F}$

Assume $(a_j \colon j < i)$ has already been chosen, where $i < \delta$. Suppose that $(a_j \colon j < i) \in M$. Otherwise we let the construction break down. Write $x = \bigcup\{\text{dom}(a_j) \colon j < i\}$, so that $x \in M$ and $B \upharpoonright x \in M$. There must then be some legal $a \in \mathcal{F}$ such that $a \supset B \upharpoonright x$, but $a \neq B \upharpoonright \text{dom}(a)$, as otherwise B would be the union of all legal $a \in \mathcal{F}$ with $a \supset B \upharpoonright x$ and thus B would be in M. Let a_i be the $<_{\mathcal{F}}$-least legal $a \in \mathcal{F}$ such that $a \supset B \upharpoonright x$ and $a \neq B \upharpoonright \text{dom}(a)$.

We claim that the construction does not break down and that $(a_i \colon i < \delta) \in M$. Otherwise let $i \leq \delta$ be least such that $(a_j \colon j < i) \notin M$. Let $x = \bigcup\{\text{dom}(a_j) \colon j < i\}$. As M certainly δ^+-covers V, we may pick some $y \in M$, $y \supset x$, $\text{Card}(y) \leq \delta$. Then $B \upharpoonright y \in M$, and $(a_j \colon j < i)$ may inside M be recursively defined as follows. For $j < i$, a_j is the $<_{\mathcal{F}}$-least legal $a \in \mathcal{F}$ such that $a \supset (B \upharpoonright y) \upharpoonright \bigcup\{\text{dom}(a_k) \colon k < j\}$ and $a \neq (B \upharpoonright y) \upharpoonright \text{dom}(a)$. But then $(a_j \colon j < i) \in M$ after all. Contradiction!

Hence $A = \{a_i \colon i < \delta\} \in M$, and A is easily seen to be an antichain consisting of legal elements of $\mathcal{F}$. This is a contradiction! □

Theorem 2.3 becomes false if in its statement "M δ^+-approximates V" is replaced by "M δ-approximates V": e.g. consider the case that V is generic over M via forcing with a δ-Souslin tree in M.

Theorem 2.4 (R. Laver, W.H. Woodin, J.D. Hamkins, see [9], [7]**)** *Let M_0 and M_1 be inner models of V. Let δ be an infinite regular cardinal. Assume that both M_0 and M_1 δ-cover V and δ-approximate V, and assume also that $[\delta^+]^{<\delta} \cap M_0 = [\delta^+]^{<\delta} \cap M_1$ (where δ^+ is being computed in V). Then $M_0 = M_1$.*

Proof. By a theorem of Vopěnka and Balcar, see [23] (see also [10, Theorem 13.28]) it suffices to prove that M_0 and M_1 have the same sets of ordinals.

We first claim that

$$[\text{OR}]^{<\delta} \cap M_0 = [\text{OR}]^{<\delta} \cap M_1. \tag{1}$$

To verify (1), let $X \in [\mathrm{OR}]^{<\delta} \cap M_0$. Because both M_0 and M_1 δ-cover V, it is straightforward to construct a sequence $\langle X_i \colon i < \delta \rangle$ such that

(a) $X \subset X_0$,

(b) $X_j \supset X_i$ for $i < j < \delta$,

(c) $X_i \in [\mathrm{OR}]^{<\delta}$ for $i < \delta$,

(d) $X_i \in M_0$ for even $i < \delta$, and

(e) $X_i \in M_1$ for odd $i < \delta$.

Write $Y = \bigcup \{X_i \colon i < \delta\}$. As both M_0 and M_1 δ-approximate V, $Y \in M_0 \cap M_1$.

Let $e \colon \gamma \cong Y$ be the inverse of transitive collapse of Y, so that $e \in M_0 \cap M_1$. Also, $\gamma < \delta^+$ (as being computed in V).

We now have $e^{-1}``X \in M_0$. By $[\gamma]^{<\delta} \cap M_0 \subset [\gamma]^{<\delta} \cap M_1$, it follows then that $e^{-1}``X \in M_1$ and thus $X = e``(e^{-1}``X) \in M_1$. By symmetry, we showed (1).

Now let X be *any* set of ordinals in M_0. Let $a \in [\mathrm{OR}]^{<\delta} \cap M_1$. By (1), $a \in M_0$, so that $X \cap a \in M_0$ and hence also $X \cap a \in M_1$ by (1) again. As M δ-approximates V, this verifies that $X \in M_1$.

By symmetry then, M_0 and M_1 have the same set of ordinals from which we may conclude that $M_0 = M_1$. □

Let us formulate a special case of Corollary 2.4 (where $M_0 = M$ and $M_1 = V$) as a separate statement.

Corollary 2.5 *Let M be an inner model of V. Let δ be an infinite regular cardinal. Assume that M both δ-covers V as well as δ-approximates V, and assume also that $[\delta]^{<\delta} \cap V \subset M$. Then $M = V$.*

Proof. If $X \in [\mathrm{OR}]^{<\delta} \cap V$, we may pick some $Y \supset X$, $Y \in [\mathrm{OR}]^{<\delta} \cap M$, using δ-covering; if $e \colon \gamma \cong Y$ denotes the (inverse of the) transitive collapse of Y, then $\gamma < \delta$ and $e^{-1}``X \in M$ by $[\gamma]^{<\delta} \cap V \subset M$, so $e \in M$ gives $X = e``(e^{-1}``X) \in M$. This shows that

$$[\mathrm{OR}]^{<\delta} \cap V \subset M. \tag{2}$$

The rest is then as in the proof of Theorem 2.4.

3 Bukovský's theorem and Woodin's extender algebra

In this section, we present Woodin's extender algebra in such a way that this also allows us to reprove Bukovský's theorem. Woodin's extender algebra is usually defined in the presence of a Woodin cardinal, see e.g. [20, pp. 1657ff.], but it turns out that the presence of a regular uncountable cardinal suffices.

The terminology used in the following definition is inspired by [18, section 4].

Definition 3.1 *Let δ and μ be cardinals, let $\mathcal{E}$ be a collection of elementary embeddings, and let X be a function with domain $\mathcal{E}$. Write $\theta = \max\{\delta, \mu\}^+$. We say that $\langle \mathcal{E}, X \rangle$ is δ-*rich *at μ iff for all $A \in {}^{\delta}(H_\theta)$ there is some $j \in \mathcal{E}$ and some $\bar{A} \in \mathrm{dom}(j)$ of size $< \delta$ such that*

(a) if $j\colon N \to M$, then both N and M are transitive models of ZFC^-,

(b) $X(j)$ is a transitive set,

(c) $A \cap X(j) \subset j(\bar{A})$,

(d) $j``\bar{A} \subset A \cap X(j)$, and

(e) $(A \cap X(j)) \setminus \mathrm{ran}(j) \neq \emptyset$.

We shall associate a partial order to each $\langle \mathcal{E}, X \rangle$ which is δ-*rich* at μ. Before doing so, let us see how to obtain rich pairs.

Definition 3.2 *Let δ and μ be cardinals. Write $\theta = \max\{\delta, \mu\}^+$. Let $\mathcal{E}(\delta, \mu)$ be the collection of all elementary embeddings $j\colon N \to H_\theta$ such that*

(a) N is transitive and of size $< \delta$, and

(b) $j(\mathrm{crit}(j)) = \delta$.

Let $\mathcal{E}^+(\delta, \mu)$ be the collection of all elementary embeddings $j\colon N \to H_\theta$ such that (a) and (b) hold and in fact also

(a^+) there is some (successor) cardinal $\bar{\theta} < \theta$ such that $N = H_{\bar{\theta}}$.

Let $X(\delta, \mu)$ be the function with domain $\mathcal{E}(\delta, \mu)$ and constant value H_θ, and let $X^+(\delta, \mu)$ be the function with domain $\mathcal{E}^+(\delta, \mu)$ and constant value H_θ.

Lemma 3.3 *Let δ and μ be cardinals. Write $\theta = \max\{\delta, \mu\}^+$.*

(1) Assume that δ is regular and uncountable. Then $\langle \mathcal{E}(\delta, \mu), X(\delta, \mu)\rangle$ is δ-rich at μ.

(2) Assume that δ is $2^{<\theta}$-supercompact. Then $\langle \mathcal{E}^+(\delta, \mu), X^+(\delta, \mu)\rangle$ is δ-rich at μ.

Proof. (1): Given $A \in {}^{\delta}(H_\mu)$, let $j\colon N \to H_\theta$ be in $\mathcal{E}(\delta, \mu)$ such that $A \in \operatorname{ran}(j)$. Then (a) through (e) of Definition 3.1 will be satisfied with $\bar{A} = j^{-1}(A)$. (Note that (e) just follows from the fact that $\operatorname{Card}(N) < \delta$.)

(2): This is by the same proof as for (1): if δ is $2^{<\theta}$-supercompact, then by the Magidor characterization of supercompactness (see e.g. [17, Problems 4.29 and 10.21]) for every $A \in {}^{\delta}(H_\theta)$ there is some $j\colon H_{\bar{\theta}} \to H_\theta$ in $\mathcal{E}^+(\delta, \mu)$ such that $A \in \operatorname{ran}(j)$. □

Definition 3.4 *Let δ be a cardinal. Let $\mathcal{E}^*(\delta)$ be the collection of all elementary embeddings $j\colon V \to M$ such that*

(a) M is transitive, and

(b) $\operatorname{crit}(j) < \delta$ and $j(\operatorname{crit}(j)) \leq \delta$.

Let $X^*(\delta)$ be the function with domain $\mathcal{E}^*(\delta)$ such that for each $j\colon V \to M$ in $\mathcal{E}^*(\delta)$, $X^*(\delta)(j) = V_{\alpha(j)}$, where $\alpha(j)$ is the strength *of j, i.e., the largest ordinal α with $V_\alpha \subset M$.*

The attentive reader will notice that $\mathcal{E}^*(\delta)$ as in Definition 3.2 will have to be a collection of proper classes. However, it is always possible to pick $\mathcal{E}^*$ in such a way that the elementary embeddings from the collection $\mathcal{E}^*$ witnessing the relevant properties may all be coded by set sized *extenders*, see e.g. [17, section 10.3], so that we may in fact think of $\mathcal{E}^*$ as being (coded by) a *set*.

Lemma 3.5 *Let δ and μ be cardinals. Write $\theta = \max\{\delta, \mu\}^+$.*

(1) Assume that δ is a Woodin cardinal. Then $\langle \mathcal{E}^*(\delta), X^*(\delta)\rangle$ is δ-rich at δ.

(2) Assume that δ is $2^{<\theta}$-supercompact. Then $\langle \mathcal{E}^*(\delta), X^*(\delta)\rangle$ is δ-rich at μ.

Proof. (1): Fix $A \in {}^{\delta}(H_\delta) \subset V_\delta$. Let $\kappa < \delta$ be A-strong up to κ, see e.g. [17, Definition 10.75]. We may pick some $j\colon V \to M$ in $\mathcal{E}^*(\delta)$ such that $\operatorname{crit}(j) = \kappa$ and if $X^*(\delta)(j) = V_{\alpha(j)}$, $\alpha(j)$ being the strength of j, then

(i) $\alpha(j) < j(\kappa)$,

(ii) $A \restriction (\kappa + 1) \in V_{\alpha(j)}$, and

(iii) $j(A) \cap V_{\alpha(j)} = A \cap V_{\alpha(j)}$.

Let $\bar{A} = A \cap V_\kappa$. Then (a) through (e) of Definition 3.1 are satisfied: (c) follows from the fact that (i) and (iii) give $j(\bar{A}) \cap X^*(\delta)(j) = j(A) \cap X^*(\delta)(j) = A \cap X^*(\delta)(j)$, (d) is trivial, and (e) is given by (ii).

(2): This is by the proof of (2) of Lemma 3.3. Recall that any elementary embedding $j : H_{\bar{\theta}} \to H_\theta$ (where $\bar{\theta}$ and θ are successor cardinals) may be extended to an elementary embedding $\hat{j} : V \to M$, where $\hat{j} \supset j$ and M is transitive (see e.g. [17, section 10.3]). □

Let us now fix δ, μ, and $\langle \mathcal{E}, X \rangle$ such that δ and μ are cardinals and $\langle \mathcal{E}, X \rangle$ is δ-rich at μ. We aim to associate a partial order $\mathbb{P} = \mathbb{P}(\delta, \mu, \mathcal{E}, X)$ to δ, μ, and $\langle \mathcal{E}, X \rangle$ which tries to make a given $a \subset \mu$, a possibly not in V, $\mathbb{P}$-generic over V.

By way of terminology, if δ is a Woodin cardinal and $\mu \leq \delta$, then the special case $\mathbb{P}(\delta, \mu, \mathcal{E}^*(\delta), X^*(\delta))$ will be an instance of Woodin's *extender algebra*. The general version of the forcing $\mathbb{P}(\delta, \mu, \mathcal{E}, X)$ which we are about to define will be used to prove Bukovský's Theorem 3.11.

In what follows, the reader may sometimes want to think of V as an inner model of the true universe of all sets so that sets outside of V actually exist. By an "outer model" W (of V) we then mean an inner model of the true universe of all sets with $W \supset V$. Most of the constructions to follow are still to be performed inside V, though, which is why we decided to use this letter to denote the ground model over which we are going to force with $\mathbb{P}$.

Let $\bar{\mathcal{L}}$ be the infinitary language with atomic fomulae "$\check{\xi} \in \dot{a}$," for $\xi < \mu$, and such that the set of formulae is closed under negation and infinite disjunctions of the form $\bigvee\!\!\!\bigvee \Gamma$ for all well–ordered sets Γ of fomulae with $\mathrm{Card}(\Gamma) < \delta$. The language $\bar{\mathcal{L}}$ has size $\mu^{<\delta}$.

For $a \subset \mu$, where possibly a is not in V but in some outer model of V, say $a \in V^{\mathrm{Col}(\omega, \mu^{<\delta})}$, and for $\varphi \in \bar{\mathcal{L}}$, we may define the meaning of "$a \vDash \varphi$" in the obvious recursive fashion: $a \vDash$ "$\check{\xi} \in \dot{a}$" iff $\xi \in a$, $a \vDash \neg \varphi$ iff $a \nvDash \varphi$, and $a \vDash \bigvee\!\!\!\bigvee \Gamma$ iff $a \vDash \varphi$ for some $\varphi \in \Gamma$. Inside $V^{\mathrm{Col}(\omega, \mu^{<\delta})}$, the relation "$a \vDash \varphi$" is Borel in the codes.

For $A \subset \bar{\mathcal{L}}$, $a \vDash A$ means $a \vDash \varphi$ for all $\varphi \in A$. For $A \cup \{\varphi\} \subset \bar{\mathcal{L}}$ ($A \cup \{\varphi\}$ being in V), we write

$$A \vdash \varphi \tag{3}$$

iff in $V^{\mathrm{Col}(\omega, \mu^{<\delta})}$, for all $a \subset \mu$, if $a \vDash A$, then $a \vDash \varphi$. (3) is thus defined over V, and inside $V^{\mathrm{Col}(\omega, \mu^{<\delta})}$, (3) is Π^1_1 in the codes. By Σ^1_1 absoluteness, for any outer model $W \supset V$ of V, (3) is thus equivalent with the fact that in $W^{\mathrm{Col}(\omega, \mu^{<\delta})}$, for all $a \subset \mu$, if $a \vDash A$, then $a \vDash \varphi$.

For $A \subset \bar{\mathcal{L}}$ (A being in V), A is called *consistent* iff there is no $\varphi \in \bar{\mathcal{L}}$ such that $A \vdash \varphi$ and $A \vdash \neg\varphi$, which in turn is easily seen to be equivalent with the fact that in $V^{\mathrm{Col}(\omega,\mu^{<\delta})}$ (equivalently, in $W^{\mathrm{Col}(\omega,\mu^{<\delta})}$ for any outer model $W \supset V$ of V) there is some $a \subset \mu$ with $a \vDash A$.

Let us define the set $T = T(\delta, \mu, \mathcal{E}, X)$ of *axioms*. For $\psi \in \bar{\mathcal{L}}$, we stipulate that $\psi \in T$ iff there are $j \in \mathcal{E}$, $\bar{A} \in \mathrm{dom}(j)$ of size $< \delta$, and $\varphi \in j(\bar{A}) \cap X(j)$ such that $j(\bar{A}) \subset \bar{\mathcal{L}}$, and ψ is equal to

$$\varphi \to \bigvee\!\!\!\bigvee j\text{"}\bar{A}.$$

We write $\mathcal{L}$ for the set of all $\varphi \in \bar{\mathcal{L}}$ such that $T \cup \{\varphi\}$ is consistent. For φ, $\varphi' \in \mathcal{L}$, we also write

$$\varphi \leq_{\mathcal{L}} \varphi' \tag{4}$$

just in case $T \cup \{\varphi\} \vdash \varphi'$. We then set

$$\mathbb{P} = \mathbb{P}(\delta, \mu, \mathcal{E}, X) = \langle \mathcal{L}, \leq_{\mathcal{L}} \rangle. \tag{5}$$

Lemma 3.6 *$\mathbb{P} = \mathbb{P}(\delta, \mu, \mathcal{E}, X)$ has the δ-chain condition.*

Proof. Let $A \in {}^{\delta}\mathcal{L}$. We may pick $j \in \mathcal{E}$ and $\bar{A} \in \mathrm{dom}(j)$ of size $< \delta$ such that (a) through (e) of Definition 3.1 hold true. By (e), we may pick $\varphi \in (A \cap X(j)) \setminus \mathrm{ran}(j)$. By (c), $\varphi \in j(\bar{A})$. By (d), $j\text{"}\bar{A} \subset A \cap X(j) \subset \mathcal{L}$, so that

$$\varphi \to \bigvee\!\!\!\bigvee j\text{"}\bar{A}$$

is an axiom in T and $j\text{"}\bar{A} \cup \{\varphi\} \subset A$ with $\varphi \notin j\text{"}\bar{A}$. We have shown that A cannot be an antichain. □

Let $a \subset \mu$, a not necessarily in V. We set

$$g_a = \{\varphi \in \mathbb{P} \colon a \vDash \varphi\}.$$

Lemma 3.7 *Let $a \subset \mu$, a not necessarily in V. Assume that $a \vDash T = T(\delta, \mu, \mathcal{E}, X)$. Then $g_a \subset \mathbb{P}$ is a $\mathbb{P}$-generic filter over V and*

$$a = \{\xi < \mu \colon \text{“}\check{\xi} \in \dot{a}\text{”} \in g_a\} \in V[g_a],$$

and hence $V[g_a] = V[a]$.[1]

[1]Here, $V[a]$ denotes the smallest transitive model W of ZFC with $V \cup \{a\} \subset W$.

Proof. If φ, $\varphi' \in \mathcal{L}$, $a \vDash \varphi$, and $\varphi \leq_{\mathcal{L}} \varphi'$, then $a \vDash \varphi'$: by definition, $a' \vDash \varphi$ implies that $a' \vDash \varphi'$ for all $a' \subset \mu$, $a' \in V^{\mathrm{Col}(\omega,\mu^{<\delta})}$; but then by Σ^1_1 absoluteness this holds true for all $a' \subset \mu$ whatsoever which exist in any outer model of V. If φ, $\varphi' \in \mathbb{P}$, $a \vDash \varphi$, and $a \vDash \varphi'$, then $a \vDash \varphi \wedge \varphi'$,[2] $\varphi \wedge \varphi' \in \mathcal{L}$ by $a \vDash T$ and Σ^1_1 absoluteness, and clearly $\varphi \wedge \varphi' \leq_{\mathcal{L}} \varphi$ and $\varphi \wedge \varphi' \leq_{\mathcal{L}} \varphi'$. Hence g_a is a filter.

Now let $A \in V$ be a maximal antichain in $\mathbb{P}$. By Claim 3.6, $A \in [\mathcal{L}]^{<\delta}$. If $g_a \cap A = \emptyset$, then $a \vDash \neg \bigvee\!\!\!\bigvee A$. By $a \vDash T$ and absoluteness, $\neg \bigvee\!\!\!\bigvee A \in \mathcal{L}$, and

$$A \cup \{\neg \bigvee\!\!\!\bigvee A\} \supsetneq A$$

is an antichain. Contradiction!

The rest is easy. □

We now aim to state a criterion for when a given $a \subset \mu$, a not necessarily in V, satisfies T.

Definition 3.8 *Let δ and μ be cardinals. Let $\langle \mathcal{E}, X\rangle$ be δ-rich at μ. Let $a \subset \mu$, a not necessarily in V. We say that $\langle \mathcal{E}, X\rangle$ admits the* a-lifting property *iff for all $j\colon N \to M$ in $\mathcal{E}$ where both N and M are transitive models of ZFC^-, there is an elementary embedding $\hat{\jmath}\colon \hat{N} \to \hat{M}$ and there is some $b \in \mathrm{dom}(\hat{\jmath})$ such that*

(a) both $\hat{N}$ and $\hat{M}$ are transitive models of ZFC^- with $\hat{N} \supset N$ and $\hat{M} \supset M$,

(b) $\hat{\jmath} \supset j$, and

(c) $\hat{\jmath}(b) \cap X(j) = a \cap X(j)$.

Lemma 3.9 *Let δ and μ be cardinals. Let $a \subset \mu$, a not necessarily in V. Let $\langle \mathcal{E}, X\rangle$ be δ-rich at μ, and assume $\langle \mathcal{E}, X\rangle$ to admit the a-lifting property. Then $a \vDash T = T(\delta, \mu, \mathcal{E}, X)$.*

Proof. Let $j\colon N \to M$ be in $\mathcal{E}$, where both N and M are transitive models of ZFC^-, let $\bar{A} \in N = \mathrm{dom}(j)$ be of size $< \delta$, and let $\varphi \in j(\bar{A}) \cap X(j)$. Assume that $j(\bar{A}) \subset \bar{\mathcal{L}}$. We need to see that

$$a \vDash \varphi \to \bigvee\!\!\!\bigvee j\text{"}\bar{A}. \tag{6}$$

Let $\hat{\jmath}\colon \hat{N} \to \hat{M}$ and $b \in \hat{N} = \mathrm{dom}(\hat{\jmath})$ be as in (a) through (c) of Definition 3.8.

[2] $\varphi \wedge \varphi'$ is short for $\neg \bigvee\!\!\!\bigvee\{\neg\varphi, \neg\varphi'\}$.

Suppose that $a \vDash \varphi$. By $\varphi \in j(\bar{A}) \cap X(j)$ and (c) (and (a)) of Definition 3.8, we then get that

$$\hat{M} \vDash \text{“}\exists\bar{\varphi} \in j(\bar{A})\ \hat{\jmath}(b) \vDash \bar{\varphi},\text{”}$$

so that by (b) of Definition 3.8,

$$\hat{N} \vDash \text{“}\exists\bar{\varphi} \in \bar{A}\ b \vDash \bar{\varphi},\text{”}$$

and hence we may choose some $\bar{\varphi} \in \bar{A}$ such that

$$\hat{M} \vDash \text{“}\hat{\jmath}(b) \vDash j(\bar{\varphi}).\text{”}$$

But then

$$\hat{\jmath}(b) \vDash \bigvee\!\!\!\bigvee j\text{''}\bar{A},$$

which implies that

$$a \vDash \bigvee\!\!\!\bigvee j\text{''}\bar{A}$$

by (d) of Definition 3.1 and (c) of Definition 3.8. □

The proof of the following is straightforward.

Lemma 3.10 *Let δ and μ be cardinals, and write $\theta = \max\{\delta, \mu\}^+$. Let $a \subset \mu$, a not necessarily in V. Suppose that (in V) there is a stationary set $S \subset [H_\theta]^{<\delta}$ such that for all $X \in S$,*

(a) there is some $j \colon N \to H_\theta$ in $\mathcal{E}(\delta, \mu)$ with $X = \mathrm{ran}(j)$, and

(b) there is some $\hat{\jmath} \colon \hat{N} \to H_\theta^{V[a]}$ such that $\hat{\jmath} \supset j$ and $a \in \mathrm{ran}(\hat{\jmath})$.

Write $\bar{\mathcal{E}} = \{j \in \mathcal{E}(\delta, \mu) \colon \mathrm{ran}(j) \in S\}$ and $X = X(\delta, \mu) \restriction \bar{\mathcal{E}}$. Then $\langle \bar{\mathcal{E}}, X \rangle$ is δ-rich at μ and admits the a-lifting property.

The same is true if $\mathcal{E}(\delta, \mu)$ is replaced by $\mathcal{E}^+(\delta, \mu)$ and $X(\delta, \mu)$ is replaced by $X^+(\delta, \mu)$.

The following is an attempt to summarize what we have been doing in this section.

Theorem 3.11 (Bukovský, see [1]**)** *Let $W \supset V$ be an outer model, and let δ be an infinite regular cardinal in V. The following are equivalent.*

(a) V uniformly δ-covers W.

(b) *For every $\alpha \geq \delta$, if $C \in \mathcal{P}([\alpha]^{<\delta}) \cap W$ is club in W, then there is some $D \in \mathcal{P}([\alpha]^{<\delta}) \cap V$ with $D \subset C$ and D is club in V.*

(b') *For every $\alpha \geq \delta$, if $C \in \mathcal{P}([\alpha]^{<\delta}) \cap W$ is club in W, then there is some $S \in \mathcal{P}([\alpha]^{<\delta}) \cap V$ with $S \subset C$ and S is stationary in V.*

(c) *For every cardinal $\theta \geq \delta$, if $C \in \mathcal{P}([H_\theta^W]^{<\delta}) \cap W$ is club in W, then there is some $D \in \mathcal{P}([H_\theta^V]^{<\delta}) \cap V$ which is club in V and such that for all $X \in D$ there is some $Y \in C$ with $X = Y \cap V$.*

(c') *For every cardinal $\theta \geq \delta$, if $C \in \mathcal{P}([H_\theta^W]^{<\delta}) \cap W$ is club in W, then there is some $S \in \mathcal{P}([H_\theta^V]^{<\delta}) \cap V$ which is stationary in V and such that for all $X \in S$ there is some $Y \in C$ with $X = Y \cap V$.*

(d) *There is some poset $\mathbb{P} \in V$ such that $\mathbb{P}$ has the δ-c.c. in V, $\mathbb{P}$ has size 2^δ in W, and $W = V[g]$ for some g which is $\mathbb{P}$–generic over V.*

Proof. (d) $\Longrightarrow$ (a): This is a standard fact, see the remark in the second paragraph after Definition 2.1.

(a) $\Longrightarrow$ (b) $\Longleftrightarrow$ (b') $\Longrightarrow$ (c) $\Longleftrightarrow$ (c') is given by Lemma 2.2.

(c') $\Longrightarrow$ (d): Let $a \in \mathcal{P}(\mu) \cap W$ for some cardinal μ, and let $\theta = \max\{\delta, \mu\}^+$. In W, the set C of all $Y \prec H_\theta^W$ such that $\mathrm{Card}(Y) < \delta$ and $a \in Y$ is club. Let S be as given by (c'). In V, let $\mathcal{E} = \{j \in \mathcal{E}(\delta, \mu)\colon \mathrm{ran}(j) \in S\}$, and let $X = X(\delta, \mu) \restriction \mathcal{E}$. Obviously, $\langle \mathcal{E}, X \rangle$ is δ-rich at μ and admits the a-lifting property, cf. Lemma 3.10, so that by Lemmas 3.6, 3.7, and 3.9, a is $\mathbb{P}$-generic over V, where $\mathbb{P} = \mathbb{P}(\mu, \delta, \mathcal{E}, X)$ has the δ-c.c. In particular, V uniformly δ-covers W by (d) $\Longrightarrow$ (a).

In W, let $e\colon 2^\delta \to \mathcal{P}(\delta)$ be a bijection, and let

$$a = \{\delta \cdot \eta + \xi \colon \eta < 2^\delta \wedge \xi \in e(\eta)\}.$$

Then, by what we showed so far, a is $\mathbb{P}(\delta, 2^\delta, \mathcal{E}, X)$-generic over V for the appropriate $\langle \mathcal{E}, X \rangle$. But clearly $V[a]$ uniformly δ-covers W and $\mathcal{P}(\delta) \cap W \subset V[a]$ (equivalently, $[\delta^+]^\delta \cap W \subset V[a]$), so that $W = V[a]$ by Theorem 2.3 and Corollary 2.5. □

Let us end this section by stating a consequence of what has been worked out for the special case of the extender algebra. The following is a prototype result by W.H. Woodin about the extender algebra, the papers [8], [4], [3], and [20] contain more general material also due to W.H. Woodin on the extender algebra.

Theorem 3.12 (Woodin) *Let $W \supset V$ be an outer model, and let δ be a Woodin cardinal in V. Suppose that every $j\colon V \to M$ from $\tilde{E}^*(\delta)$ lifts to some $\hat{j}\colon W \to \hat{W}$. Let μ be smaller than the least measurable cardinal of V. Then every $a \in \mathbb{P}(\mu) \cap W$ is $\mathbb{P}(\delta, \mu, \mathcal{E}^*(\delta), X^*(\delta))$-generic over V.*

4 Usuba's theorems

Definition 4.1 *Let κ be a cardinal. An inner model M is called a* κ-ground of V *iff there is some forcing $\mathbb{P} \in M$ of size $< \kappa$ and some g which is $\mathbb{P}$-generic over M such that $V = M[g]$. M is called a* ground *iff M is a κ ground for some cardinal κ.*

We write

$$\mathbb{M}_{<\kappa} = \bigcap \{M \colon M \text{ is a } \kappa\text{-ground of } V\},$$

and call it the κ-mantle *of V. Also,*

$$\mathbb{M} = \bigcap \{\mathbb{M}_{<\kappa} \colon \kappa \text{ is a cardinal }\}$$

is the mantle of V.

A. Lietz has shown that $\mathbb{M}_{<\kappa}$ need not be a model of ZFC, even if κ is inaccessible. This is dual to Theorem 4.7 below.

Theorem 4.2 (Usuba, see [21]**)** Let κ be a regular cardinal. Let $\{W_i \colon i < \kappa\}$ be a collection of inner models[3] of V such that each W_i, $i < \kappa$, uniformly κ-covers V and κ-approximates V. There is then an inner model $M \subset \bigcap\{W_i \colon i < \kappa\}$ which uniformly κ^+-covers V.

Proof. Write $\tilde{M} = \bigcap\{W_i \colon i < \kappa\}$. We first claim that for all functions $f \in V$ with $\mathrm{dom}(f) \in \mathrm{OR}$ and $\mathrm{ran}(f) \subset \mathrm{OR}$ there is some function $g \in \tilde{M}$ with $\mathrm{dom}(g) = \mathrm{dom}(f)$ such that $f(x) \in g(x)$ and $\mathrm{Card}(g(x)) \leq \kappa$ for all $x \in \mathrm{dom}(g)$.

To see this, fix such a fuction f, say $f \colon \theta \to \alpha$. Let $\langle M_i \colon i < \kappa\rangle$ be a list such that

(a) for each $i < \kappa$ there is some $j < \kappa$ with $M_i = W_j$, and

(b) for each $j < \kappa$, the set $\{i < \kappa \colon W_j = M_i\}$ is cofinal in κ.

Using the fact that every W_i uniformly κ-covers V it is then easy to construct $\langle r_i \colon i < \kappa\rangle$ such that for all $i < \kappa$,

(a) $r_i \subset \theta \times \alpha$,

(b) $r_i \in W_i$,

[3] Of course the language of BGC doesn't let us talk about collections of proper classes, so instead of $\{W_i \colon i < \kappa\}$ we should refer to a proper class $\mathcal{W}$ from which each W_i, $i < \kappa$, may be read off as $\{x \colon (i, x) \in \mathcal{W}\}$. Similar remarks apply to all our future quantification about collections of proper classes.

and for all $\xi < \theta$,

(c) $\mathrm{Card}(r_i"\{\xi\}) < \kappa$ and

(d) $r_i"\{\xi\} \supset \{f(\xi)\} \cup \bigcup_{j<i} r_j"\{\xi\}$.

Write

$$r = \bigcup_{i<\kappa} r_i.$$

Let $j < \kappa$. If $a \in W_j$ has size $< \kappa$, then $a \cap r = a \cap r_i$ for all sufficiently big $i < \kappa$, so that if $i < \kappa$ is sufficiently big with $W_j = M_i$, then $a \cap r = a \cap r_i \in M_i = W_j$. As W_j κ-approximates V, we then have that $r \in W_j$. Hence, as j was arbitrary, $r \in \tilde{M}$.

We may then let g with $\mathrm{dom}(g) = \theta$ be defined by $g(\xi) = r"\{\xi\}$ for $\xi < \theta$. Then $g \in \tilde{M}$ and g is as desired.

By replacing a single function by a vector of functions, the very same proof shows that for every α and for every collection $\vec{f} = (f_i : i < \alpha)$ of functions with $\mathrm{dom}(f_i) \in \mathrm{OR}$ and $\mathrm{ran}(f_i) \subset \mathrm{OR}$ for all $i < \alpha$ there is some collection $\vec{g} = (g_i : i < \alpha) \in \tilde{M}$ such that for each $i < \alpha$, $\mathrm{dom}(g_i) \supset \mathrm{dom}(f_i)$, and $f_i(\xi) \in g_i(\xi)$ and $\mathrm{Card}(g_i(\xi)) \leq \kappa$ for all $\xi \in \mathrm{dom}(f_i)$. To verify this, we may first assume that all f_i, $i < \alpha$, have a common domain, δ; we may then apply the above argument to the function f with domain $\alpha \times \delta$, where $f(i, \xi) = f_i(\xi)$ for $i < \alpha$ and $\xi < \delta$.

In the situation of the preceeding paragraph, let us *ad hoc* say that $\vec{g}$ κ^+-covers $\vec{f}$. We may let $\langle \vec{g}_\theta : \theta \in \mathrm{Card} \rangle$ be such that for each cardinal θ, $\vec{g}$ κ^+-covers some list of all functions from ordinals to ordinals which exist H_θ^V.

There is a proper class X of cardinals such that $[\kappa^{+3}]^{\kappa^+} \cap L[\vec{g}_\theta] = [\kappa^{+3}]^{\kappa^+} \cap L[\vec{g}_{\theta'}]$ for all θ, $\theta' \in X$. By Theorem 2.3 and a localized version of Theorem 2.4 we then get that $H_\theta^{L[\vec{g}_\theta]} = H_\theta^{L[\vec{g}_{\theta'}]}$ for all $\theta \leq \theta'$, both being in X. But then

$$M = \bigcup \{ H_\theta^{L[\vec{g}_\theta]} : \theta \in X \}$$

is as desired. □

Corollary 4.3 *Let κ be a regular cardinal such that $2^{<\kappa} = \kappa$. There is some ground $M \subset \mathbb{M}_{<\kappa}$ of V for which there is some forcing $\mathbb{P} \in M$ and some g which is $\mathbb{P}$-generic over M such that*

(a) $\mathbb{P}$ has the κ^+-c.c. in M,

(b) $\mathrm{Card}(\mathbb{P}) = 2^{\kappa^+}$, as being computed in V, and

(c) $V = M[g]$.

Definition 4.1 may also performed inside any model of ZFC^-. The second part of the following Lemma follows from (a localized version of) Theorem 2.4 using a simple pigeonhole argument.

Lemma 4.4 *Let κ be a cardinal. For all cardinals $\theta \geq \kappa$, $\mathbb{M}_{<\kappa}^{H_\theta} \subset \mathbb{M}_{<\kappa} \cap H_\theta$, and for all but set many cardinals $\theta \geq \kappa$, $\mathbb{M}_{<\kappa} \cap H_\theta = \mathbb{M}_{<\kappa}^{H_\theta}$.*

Lemma 4.5 (Hamkins, Reitz) *Let $\kappa \leq \lambda$ both be cardinals with $\mathrm{cf}(\lambda) \geq \kappa$. Let $W \subset V$ be an inner model such that W κ-covers and κ-approximates V. Let $E = \langle E_a \colon a \in H_\lambda \rangle$ be a V-extender with critical point κ.[4] Then $E \cap W \in W$.*

Proof. Let

$$j \colon V \to_E M$$

be the ultrapower of V by E, where M is transitive. As E has support H_λ, $H_\lambda \subset M$. Both $W \cap H_\lambda$ and $j(W) \cap H_\lambda$ κ-cover and κ-approximate H_λ and they have the same intersection with $[\kappa^+]^{<\kappa}$, so that (a localized version of) Theorem 2.4 then implies that

$$W \cap H_\lambda \in M \text{ and } j(W) \cap H_\lambda = W \cap H_\lambda. \tag{7}$$

By κ-approximation, it suffices to prove that $E \cap Z \in W$ for every $Z \in W$ of cardinality $< \kappa$. So let us fix such a Z. By κ-covering, we may cover $E \cap Z$ by a set $\{(a_i, X_i) \colon i < \theta\} \in W$, where $\theta < \kappa$.

Write $\vec{a} = \langle a_i \colon i < \theta \rangle$ and $\vec{X} = \langle X_i \colon i < \theta \rangle$. By $\mathrm{cf}(\lambda) \geq \kappa$, we may assume that $\vec{a}$ was picked in a way that $\vec{a} \in H_\lambda$. We have that $\mathrm{TC}(\{\vec{a}\}) \in W \cap H_\lambda = j(W) \cap H_\lambda$, so that $j(\vec{X}) \cap \mathrm{TC}(\{\vec{a}\}) \in j(W) \cap H_\lambda = W \cap H_\lambda$. Hence

$$\{\vec{a}, \vec{X}, j(\vec{X}) \cap \mathrm{TC}(\{\vec{a}\})\} \subset W. \tag{8}$$

But then $(a, X) \in E \cap Z$ iff $(a, X) \in Z$ and there is some $i < \theta$ such that $a = \vec{a}(i)$, $X = \vec{X}(i)$, and $a \in j(\vec{X}) \cap \mathrm{TC}(\{\vec{a}\})$. Hence $E \cap Z$ may be computed in W from the objects (8). □

We aim to show that if κ is a measurable cardinal, then $\mathbb{M}_{<\kappa}$ is always a model of ZFC, and that if κ is extendible, then $\mathbb{M}_{<\kappa} = \mathbb{M}$ (the latter being a theorem by T. Usuba). Both facts may be derived as corollaries of the following.

[4]We explicitly allow E to be *long*.

Lemma 4.6 *Let $\kappa \le \lambda$ be cardinals. Let $W \subset \mathbb{M}_{<\kappa}$ be a λ-ground of V. Let θ be a sufficiently big cardinal (so as to satisfy the conclusion of Lemma 4.4). Let*

$$j \colon V \to M$$

be an elementary embedding with critical point κ, where M is transitive. Assume that

(a) $W \cap H^M_{j(\theta)} \in M$, and

(b) $W \cap H^M_{j(\theta)}$ is a $< j(\kappa)$-ground of $H^M_{j(\theta)}$.

Then[5]

$$\bigcup_{\alpha<\theta} \mathcal{P}(\alpha) \cap \mathbb{M}_{<\kappa} \subset \bigcup_{\alpha<\theta} \mathcal{P}(\alpha) \cap L[W, j \restriction \alpha]. \tag{9}$$

If in addition j is the ultrapower embedding given by the ultrapower of V by a V-extender $E = \langle E_a \colon a \in H_{\lambda'} \rangle$ for some cardinal $\lambda' \ge \kappa$ with $\mathrm{cf}(\lambda') \ge \kappa$, then

$$L[W, j \restriction \alpha] \subset \mathbb{M}_{<\kappa}, \tag{10}$$

so that in particular

$$\bigcup_{\alpha<\theta} \mathcal{P}(\alpha) \cap \mathbb{M}_{<\kappa} = \bigcup_{\alpha<\theta} \mathcal{P}(\alpha) \cap L[W, j \restriction \alpha] \tag{11}$$

Proof. (9): Let $X \in \mathbb{M}_{<\kappa}$ be a set of ordinals with $\sup(X) < \theta$. Then $X \in (\mathbb{M}_{<\kappa})^{H_\theta}$, so that using (b), $j(X) \in (\mathbb{M}_{<j(\kappa)})^{H^M_{j(\theta)}} \subset W$. But then $X = j^{-1}{}''j(X) \in L[W, j \restriction \sup(X)]$.

"$\supset$" of (11): We have $W \subset \mathbb{M}_{<\kappa}$ by hypothesis. Let P be any $< \kappa$-ground of V. For any ordinal α, $j \restriction \alpha \in P$ follows from Lemma 4.5. Hence $j \restriction \alpha \in \mathbb{M}_{<\kappa}$ for all ordinals α. □

Theorem 4.7 *Let κ be a measurable cardinal. Then $\mathbb{M}_{<\kappa}$ is a model of* ZFC*.*

Proof. Let U be a measure on κ witnessing that κ is a measurable cardinal, and let $j \colon V \to_U M$ be the ultrapower embedding. Inside M, let W be a $< (2^{\kappa^+})^+$-ground

[5] Here and in what follows, $L[W, j \restriction \alpha]$ is the least transitive model N of ZFC with $W \cup \{j \restriction \alpha\} \subset N$.

of M ("$(2^{\kappa^+})^+$" being computed in M) with $W \subset (\mathbb{M}_{<\kappa})^M$ (see Corollary 4.3). Then $W \subset (\mathbb{M}_{<\kappa})^M$ and W is a $< j(\kappa)$-ground of M.

If P is a $< \kappa$-ground of M, then there is a $< \kappa$-ground Q of V such that $P = \mathrm{ult}(Q; U \cap P) \subset Q$. This gives that

$$\mathbb{M}^M_{<\kappa} \subset \mathbb{M}_{<\kappa}.$$

It then easily follows from Lemma 4.6 that $\mathbb{M}_{<\kappa}$ and $L[W, j \restriction \mathrm{OR}]$[6] have the same sets of ordinals. As $L[W, j \restriction \mathrm{OR}]$ is a model of ZFC and $\mathbb{M}_{<\kappa}$ is a model of ZF, the theorem of Vopěnka and Balcar, see [23] (see also [10, Theorem 13.28]) implies that $\mathbb{M}_{<\kappa} = L[W, j \restriction \mathrm{OR}]$, so that in particular $\mathbb{M}_{<\kappa}$ is a model of ZFC. $\square$

Definition 4.8 *Let $A \subset V$. Let us call a cardinal κ A-long iff for all cardinals $\theta \geq \kappa$ there is some elementary embedding*

$$j \colon V \to M$$

such that

(a) M is transitive,

(b) κ is the critical point of j,

(c) $j(\kappa) > \theta$,

(d) $j(\mu)$ is a cardinal (in V) for every V-cardinal $\mu \leq \theta$, and

(e) $A \cap H_{j(\theta)} \in M$.

Recall that a cardinal κ is *extendible* iff for every $\theta > \kappa$ there is some ρ and some elementary embedding $j \colon V_\theta \to V_\rho$ with critical κ such that $j(\kappa) > \theta$. If κ is extendible, then κ is A-long for every $A \subset V$.

Theorem 4.9 (Usuba, see [22]**)** Let κ be $\mathbb{M}_{<\kappa}$-long. Then $\mathbb{M}_{<\kappa} = \mathbb{M}$.

Proof. Let $W \subset \mathbb{M}_{<\kappa}$ be a ground of V, say W is a λ-ground. Let $\theta > \lambda$ be an arbitrary limit cardinal, and let

$$j \colon V \to M$$

be such that

[6] $L[W, \restriction \mathrm{OR}]$ is the least transitive model N of ZFC with $W \cup \{j \restriction \alpha \colon \alpha \in \mathrm{OR}\} \subset N$, i.e., $L[W, j \restriction \mathrm{OR}] = \bigcup \{L[W, j \restriction \alpha] \colon \alpha \in \mathrm{OR}\}$.

(a) M is transitive,

(b) κ is the critical point of j,

(c) $j(\kappa) > \lambda$,

(d) $j(\theta)$ is a cardinal (in V), and

(e) $W \cap H_{j(\theta)} \in M$.

By (e), (a) of Lemma 4.6 holds true. Also, W is a λ-ground of V, so that W uniformly λ^+-covers V. But then $W \cap H_{j(\theta)} \in M$ uniformly λ^+ covers $H^M_{j(\theta)}$, so that

$$W \cap H_{j(\theta)} = W \cap H^M_{j(\theta)} \text{ is a } < j(\kappa)\text{-ground of } H^M_{j(\theta)} \tag{12}$$

and (b) of Lemma 4.6 holds true.

We claim that

$$j \restriction \alpha \in W \text{ for every } \alpha < \theta. \tag{13}$$

(13) implies that $H_\theta^{\mathbb{M}_{<\kappa}} \subset H_\theta^W$ by (9), so that $H_\theta^{\mathbb{M}_{<\kappa}} = H_\theta^W$. As θ was arbitrarily large, this will have shown that $\mathbb{M}_{<\kappa} = W$, so that $\mathbb{M}_{<\kappa} = \mathbb{M}$.

To show (13), let us assume without loss of generality that $\alpha \geq \lambda$ is a regular cardinal, and let $\langle S_\beta \colon \beta < \alpha \rangle \in W$ be a partition of $\alpha \cap \mathrm{cf}^W(\omega)$ into stationary sets. (As W is a λ-ground of V, stationarity of subsets of λ is absolute between W and V.) Write $\langle T_\beta \colon \beta < j(\alpha) \rangle = j(\langle S_\beta \colon \beta < \alpha \rangle)$. Write $\tilde{\alpha} = \sup(j\text{"}\alpha)$. By a result of R. Solovay,

$$j\text{"}\alpha = \{\beta < \tilde{\alpha} \colon T_\beta \cap \tilde{\alpha} \text{ is stationary in } \tilde{\alpha}\}. \tag{14}$$

Let us verify (14). Notice that j is continuous at every ordinal of cofinality ω. Hence $j\text{"}\alpha$ contains an ω-club. Then if $C \subset \tilde{\alpha}$ is club, $C \cap j\text{"}\alpha$ contains an ω-club, i.e., $j^{-1}\text{"}C$ contains an ω-club. Hence if $\beta < \alpha$ and $C \subset \tilde{\alpha}$ is club, then there is some $\xi \in S_\beta \cap j^{-1}\text{"}C$, so $j(\xi) \in T_{j(\beta)} \cap C$ and $T_{j(\beta)}$ is shown to be stationary. On the other hand, if $\beta < \tilde{\alpha}$ is not in the range of j, then T_β is disjoint from $j\text{"}\alpha$, where the latter contains an ω-club. We have shown (14).

We have that $\langle S_\beta \colon \beta < \alpha \rangle \in W \cap H_\theta \subset (\mathbb{M}_{<\kappa})^{H_\theta}$, so that $\langle T_\beta \colon \beta < j(\alpha) \rangle \in (\mathbb{M}_{<j(\kappa)})^{H^M_{j(\theta)}}$, which is contained in $W \cap H_{j(\theta)}$ by (12). But then (14) tells us that $j \restriction \alpha$ may be computed inside W from $\langle T_\beta \colon \beta < j(\alpha) \rangle$. $\square$

Corollary 4.10 *Let κ be a cardinal. The following are equivalent.*

(1) κ is $\mathbb{M}_{<\kappa}$-long.

(2) κ is extendible.

Proof. "(2) $\Longrightarrow$ (1)" is easy, see the remark before Theorem 4.9.

"(1) $\Longrightarrow$ (2)": Let $W = \mathbb{M}_{<\kappa} = \mathbb{M}$. By Corollary 4.3 and Theorem 4.7, there is some $\mathbb{P} \in \mathbb{M}$ of size 2^{κ^+} and some g which is $\mathbb{P}$-generic over $\mathbb{M}$ such that $V = \mathbb{M}[g]$. Let $\lambda = (2^{\kappa^+})^+$, let $\theta > \lambda$ be an arbitrary limit cardinal, and let $j \colon V \to M$ have propertes (a) through (e) from the proof of Theorem 4.9.

We may assume without loss of generality that $\mathbb{P} = (2^{\kappa^+}; \leq)$, so that $g \subset 2^{\kappa^+}$. By the proof of Theorem 4.9, $j \restriction (2^{\kappa^+}) \in \mathbb{M} \cap H_{j(\theta)} \in M$. Hence

$$g = (j \restriction (2^{\kappa^+}))^{-1} {}'' j(g) \in M.$$

But then $H_{j(\theta)} = H^{\mathbb{M}}_{j(\theta)}[g] \in M$, in other words, $j \restriction H_\theta \colon H_\theta \to H_{j(\theta)}$. As θ was arbitrarily large, κ is extendible. □

If κ is $\mathbb{M}_{<\kappa}$-long, then $\mathbb{M} = \mathbb{M}_{<\kappa}$ by Theorem 4.9, so that κ is then also $\mathbb{M}$-long. However, κ can be $\mathbb{M}$-long without being $\mathbb{M}_{<\kappa}$-long: in M_1, the least iterable inner model with one Woodin cardinal, every measurable cardinal is $\mathbb{M}$-long (see e.g. [6, Theorem 3.18]), but as every $\mathbb{M}_{<\kappa}$-long cardinal is extendible by Corollary 4.10, M_1 doesn't have any $\mathbb{M}_{<\kappa}$-long cardinal.

5 Varsovian models

Set theoretic geology studies the collection of grounds and the mantle of V or of any inner model of V, see [5]. It has turned out to be a fruitful program to restrict the focus to extender models: *inner model theoretic geology* analyzes the grounds and the mantle of given extender models.

An *extender model* is a proper class sized premouse of the form $L[E]$ where E codes a coherent sequence of (partial and total) extenders, see e.g. [20, Definition 2.19]. We have to warn the reader that this last section will have not many explanations and proofs and that it will be a difficult read for people with no appropriate background in inner model theory. The hope is that it may make some people curious.

Definition 5.1 *Let $W \subset V$ be an inner model. W is called a* bedrock *iff there is no ground P of W with $P \subsetneq W$.*

If W is a bedrock, then W is its own mantle. In the light of Theorem 4.2, the following is true for every inner model W:

(1) Either the mantle $\mathbb{M}^W$ of W is the $\subset$-smallest ground of W in which case there are only set many grounds[7] of W and $\mathbb{M}^W$ is a bedrock,

(2) or else the mantle $\mathbb{M}^W$ of W is not a ground of W in which case there are proper class many grounds[8] of W.

It is part of the folklore that if the extender model $L[E]$ doesn't have an inner model with a Woodin cardinal, then $L[E]$ is a bedrock. This is true because in this situation, $L[E]$ will think "I'm the core model" and the core model is absolute to forcing extensions (these are both theorems of J. Steel), so that every ground of $L[E]$ must contain all of $L[E]$.

On the other hand (see e.g. the first paragraph of [13, Introduction]):

Theorem 5.2 (W.H. Woodin) *Let $L[E]$ be an extender model such that $L[E] \vDash$ "There is a Woodin cardinal." Then $L[E]$ is* not *a bedrock.*

Proof sketch. Let us first suppose that δ is least such that δ is Woodin in an inner model (equivalently, in an extender model $L[E]$, where $E \subset V_\delta^{L[E]}$). Let $L[E]$ be an extender model with $E \subset V_\delta^{L[E]}$ and $L[E] \vDash$ "δ is a Woodin cardinal." Let K^c be the result of performing a (1-small) K^c construction inside $L[E]$. By a slight variant of Theorem 3.12, K^c will be a ground of $L[E]$. As we may "delay" adding total measures on the sequence of K^c (e.g. by demanding that the critical point of a new extender added during the construction is strictly bigger than the least measurable cardinal), we may easily make sure that $K^c \subsetneq L[E]$.

Now if $L[E]$ is an arbitrary extender model with a Woodin cardinal, let δ be the least Woodin of $L[E]$. Instead of doing a K^c construction, we have to perform a "fully backgrounded construction" as in [11, Chap. 11] but with any smallness hypothesis being dropped. Let M denote the H_δ of the result of this construction, and let $P = \mathcal{P}(M)$ be the result of performing a "$\mathcal{P}$ construction" above M inside $L[E]$, see [19]. It may then be verified that P is a nontrivial ground of $L[E]$. □

The paper [6] partially generalized this result and analyzed the mantle of tame extender models which do not have a strong cardinal.

The *minimal core* of a given inner model W is the result of working inside W and stacking collapsing mice with no total extenders and which are seen to be fully iterable inside W, see [6, Definitions 3.28 and 3.30, and Lemma 3.31]. In particular,

[7]i.e., there is some ordinal κ such that $\{W_i : i < \kappa\}$ is the collection of grounds of W; cf. the footnote on p. 13

[8]i.e., not set many

the minimal core is a *lower-part premouse*, i.e., a premouse with no total measures and hence no measurable cardinals.

Theorem 5.3 (Fuchs-Schindler, see [6, Theorem 3.33]**)** *Let $L[E]$ be an extender model. Assume that*

(1) $L[E]$ is tame,

(2) $L[E]$ does not have a strong cardinal,

(3) inside $L[E]$, $L[E]$ is not fully iterable,

(4) $L[E]$ is fully iterable in V, and

(5) E is $\mathrm{OD}^{L[E]}$, and for arbitrarily large strong cutpoints θ of $L[E]$, if g is $\mathrm{Col}(\omega,\theta)$-generic over $L[E]$, and if E_g is the natural extension of $E \upharpoonright (\theta,\infty)$ to $L[E][g]$, then E_g is $\mathrm{OR}^{L[E][g]}$.

Then the mantle of $L[E]$ is equal to the minimal core of $L[E]$.

For a while, the role of hypothesis (2) in Theorem 5.3 was not clear, and it looked like more sophisticated arguments might allow us to drop this hypothesis. Another possibility – before Usuba proved Theorem 4.2 – was that models which satisfy (2) and (3) of Theorem 5.3 provide a counterexample to the set directedness of grounds, i.e., the conclusion of Theorem 4.2.

Both scenarios were wrong, the latter one by Theorem 4.2, and the former one by [13]. The least extender model which satisfies (3), (4), and the *negation* of (2) of Theorem 5.3 is called M_{sw} and it is the least fully iterable $L[E]$ which has a strong cardinal above a Woodin cardinal. The paper [13] showed that if κ is the strong cardinal of M_{sw}, then the mantle of M_{sw} is equal to the κ-mantle $\mathbb{M}_{<\kappa}^{M_{\mathrm{sw}}}$ of M_{sw}, so that $\mathbb{M}_{<\kappa}^{M_{\mathrm{sw}}}$ is the $\subset$-least ground of M_{sw} and is hence a bedrock. In the light of Theorem 4.9, this means that the strong cardinal of M_{sw} plays the same role for M_{sw} as an extendible cardinal plays for V as far as geology goes:

$$\frac{\text{strong cardinal}}{M_{\mathrm{sw}}} = \frac{\text{extendible cardinal}}{V}.$$

In fact, [13] gave a fairly complete analysis of the mantle of M_{sw}. In (3) of Theorem 5.4 below, $\mathcal{M}_\infty$ is the transitive direct limit of all iterates P of M_{sw} via trees $\mathcal{T}$ of length $\lambda+1$ such that $\mathcal{T} \upharpoonright \lambda \in M_{\mathrm{sw}}$, $[0,\lambda]_{\mathcal{T}}$ does not drop, and $\mathcal{T}$ lives on M_{sw} up to its Woodin cardinal. It can be shown that this direct limit system may be covered by a system which is inside $M_{\rm sw}$ and has the same direct limit; in particular, $\mathcal{M}_\infty$ is a definable inner model of $M_{\rm sw}$. For an ordinal ρ, ρ^* is the common value of the image of ρ under the map from P into $\mathcal{M}_\infty$ for any P which is sufficiently far out in the system which gives rise to $\mathcal{M}_\infty$; $\rho \mapsto \rho^*$ is also definable in $M_{\rm sw}$. In (4) of Theorem 5.4, Σ is supposed to be the canonical iteration strategy for $\mathcal{M}_\infty$, restricted to trees which exist in $M_{\rm sw}$'s mantle and which live on $M_{\rm sw}$ up to its Woodin cardinal.

Theorem 5.4 (Sargsyan-Schindler, see [13]) *Let κ denote the strong cardinal of $M_{\rm sw}$. The mantle $\mathbb{M}^{M_{\rm sw}}$ of $M_{\rm sw}$ is equal to each of the following inner models.*

(1) The κ-mantle $\mathbb{M}^{M_{\rm sw}}_{<\kappa}$ of $M_{\rm sw}$.

(2) $\mathsf{HOD}^{(M_{\rm sw})^{\mathrm{Col}(\omega,<\kappa)}}$.

(3) $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$.

(4) $L[\mathcal{M}_\infty | \delta^{\mathcal{M}_\infty}, \Sigma]$.

Also,

$$(H_{\delta^{\mathcal{M}_\infty}})^{\mathbb{M}^{M_{\rm sw}}} = (H_{\delta^{\mathcal{M}_\infty}})^{\mathcal{M}_\infty}$$

and $\delta^{\mathcal{M}_\infty}$ is a Woodin cardinal in $\mathbb{M}^{M_{\rm sw}}$.

Hence the mantle of $M_{\rm sw}$ has a Woodin cardinal, call it δ, and by (4) of Theorem 5.4 this mantle is the closure of its H_δ under the iteration strategy for its H_δ. It can be shown the existence of a strong cardinal above a Woodin cardinal is the least large cardinal hypothesis from which a ZFC-model may be constructed which has a Woodin cardinal and knows how to iterate itself. Cf. [14]. The mantle of $M_{\rm sw}$ is a ground of $M_{\rm sw}$ as being witnessed by a natural forcing which can explicitly written down in an elegant fashion, see [15].

The mantle of $M_{\rm sw}$ may also represented as a premouse $\mathcal{P}$ which has short and long extenders on its sequence: Let η denote the $\mathcal{M}_\infty$-cardinal successor of $\mathcal{M}_\infty$'s strong cardinal; if $E^{\mathcal{P}}_\nu$ is on $\mathcal{P}$'s sequence, then

(a) If $\nu < \eta$, then $E^{\mathcal{P}}_\nu = E^{\mathcal{M}_\infty}_\nu$.

(b) If $\nu = \eta$, then $E^{\mathcal{P}}_\nu \restriction \delta^{\mathcal{M}_\infty} = (\rho \mapsto \rho^*) \restriction \delta^{\mathcal{M}_\infty}$.

(c) If $\nu > \eta$, then $E^{\mathcal{P}}_\nu = E^{M_{\rm sw}}_\nu \cap \mathcal{P}|\nu$.

The mantle of $M_{\rm sw}$ thus appears to be a natural object; [13] coined the term *Varsovian model* for it and denoted it by $\mathcal{V}$. Being construed as a premouse with short and long extenders (as in the previous paragraph), $\mathcal{V}$ belongs to a new category of *strategic premice*, and – starting out with extender models which satisfy stronger large cardinal hypotheses – we may iterate generalizations of the process which leads from $M_{\rm sw}$ to its Varsovian model $\mathcal{V}$ finitely or infinitely many times.

If M is an extender model or more generally a strategic premouse which inductively was obtained via applications of the Varsovian model operator and which has a strong cardinal κ above a distinguished Woodin cardinal δ, then we may define the Varsovian model of M as follows. Let $\mathcal{M}_\infty^M$ be the transitive direct limit of all iterates P of M via trees $\mathcal{T}$ of length $\lambda+1$ such that $\mathcal{T} \upharpoonright \lambda \in M$, $[0,\lambda]_\mathcal{T}$ does not drop, and $\mathcal{T}$ lives on M up to δ. For an ordinal ρ, $(\rho^*)^M$ is the comon value of the image of ρ under the map from P into $\mathcal{M}_\infty^M$ for any P which is sufficiently far out in the system which gives rise to $\mathcal{M}_\infty^M$. Under favourable circumstances, $L[\mathcal{M}_\infty^M, \rho \mapsto (\rho^*)^M]$ is a definable inner model of M, and the analysis which leads to Theorem 5.4 may be applied to represent $L[\mathcal{M}_\infty^M, \rho \mapsto (\rho^*)^M]$ again as a strategy premouse. $M \mapsto L[\mathcal{M}_\infty^M, \rho \mapsto (\rho^*)^M]$ is the *Varsovian model operator*, and the output $\mathcal{V}^M = L[\mathcal{M}_\infty^M, \rho \mapsto (\rho^*)^M]$ is the Varsovian model associated with M.

The strategic premice which arise via applications of the Varsovian model operator resemble the hod mice which come out of the analysis of HOD of natural models of determinacy, see [12].

The next natural model beyond $M_{\rm sw}$ to apply this new machinery to is $M_{\rm swsw}$, the least fully iterabe $L[E]$ which has cardinals $\delta_0 < \kappa_0 < \delta_1 < \kappa_1$ such that each δ_i is Woodin and each κ_i is strong, $i \in \{0,1\}$. This is done in [15]. It is shown there that the mantle of $M_{\rm swsw}$ is equal to the κ_1-mantle of $M_{\rm swsw}$ which in turn may be represented as a strategic premouse – a premouse with *two* Woodin cardinals which knows how to iterate itself.

There is also to be limit stages in the construction of Varsovian models. Suppose that for each $n < \omega$,

$$\pi_n \colon \mathcal{V}_n \to (\mathcal{M}_\infty)^{\mathcal{V}_n} \subset \mathcal{V}_{n+1} = L[(\mathcal{M}_\infty)^{\mathcal{V}_n}, \rho \mapsto (\rho^*)^{\mathcal{V}_n}]$$

is (in V) the natural embedding from the $n^{\rm th}$ Varsovian model into its direct limit which is a subclass of the $n{+}1^{\rm st}$ Varsovian model. There is then an obvious procedure for how to define a direct limit of $\langle \mathcal{V}_n, \pi_n \colon < \omega \rangle$. For $n \leq m < \omega$ we may let

$$\pi_{n,m} = \pi_{m-1} \circ \ldots \circ \pi_n.$$

Then $\pi_{n,m}$ is an embedding from $\mathcal{V}_n$ into $\pi_{n,m}(\mathcal{V}_n) = \pi_{n+1,m}((\mathcal{M}_\infty)^{\mathcal{V}_n})$. For $n < \omega$, we may let

$$(\mathcal{V}_n^\omega, \pi_{n,\omega}) = \text{dir lim}\langle \pi_{n,m}(\mathcal{V}_n), \pi_{m,m'} \restriction \pi_{n,m}(\mathcal{V}_n) \colon n \leq m \leq m' < \omega\rangle,$$

and we may let the direct limit model $\mathcal{V}_\omega$, the ω^{th} Varsovian model of $\mathcal{V}_0$, be defined as $\bigcup_{n<\omega} \mathcal{V}_n^\omega$.

It turns out that under favourable circumstances, $\mathcal{V}_\omega$ is a ground of $\mathcal{V}_0$ again as being witnessed by a natural forcing which can explicitly written down in an elegant fashion. Also, $\mathcal{V}_\omega$ is (unlike in the successor case) *properly* contained in the λ-mantle of $\mathcal{V}_0$, where λ is the supremum of the strongs and Woodins of $\mathcal{V}_0$ which were made use of in this process. $\mathcal{V}_\omega$ will be a strategic premouse with (at least) ω Woodin cardinals which knows how to iterate itself. This and more general limit cases will be explored in [16].

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