# Hamel bases and well–ordering the continuum

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**Abstract**

In ZF, the existence of a Hamel basis does not yield a well–ordering of $\mathbb{R}$.

Throughout this paper, by a *Hamel basis* we always mean a basis for $\mathbb{R}$, construed as a vector space over $\mathbb{Q}$. We denote by $E$ the *Vitali equivalence relation*, $xEy$ iff $x - y \in \mathbb{Q}$ for $x, y \in \mathbb{R}$. We also write $[x]_E = \{y \colon yEx\}$ for the $E$–equivalence class of $x$. A transversal for the set of all $E$–equivalence classes picks exactly one member from each $[x]_E$. The range of any such transversal is also called a *Vitali set*.

A set $\Lambda \subset \mathbb{R}$ is a *Luzin set* iff $\Lambda$ is uncountable but $\Lambda \cap M$ is at most countable for every meager set $M \subset \mathbb{R}$. A set $S \subset \mathbb{R}$ is a *Sierpiński set* iff $S$ is uncountable but $S \cap N$ is at most countable for every null set $N \subset \mathbb{R}$ ("null" in the sense of Lebesgue measure). A set $B \subset \mathbb{R}$ is a *Bernstein set* iff $B \cap P \neq \emptyset \neq P \setminus B$ for every perfect set $P \subset \mathbb{R}$.

It has been well–known for more than a century that the existence of a well–ordering of the reals implies the existence of all these "pathological" sets of reals: Hamel bases, Vitali sets, Luzin sets, Sierpiński sets,[5] and Bernstein sets; see e.g. the thorough discussion in [5].[6]

[1]The first author gratefully acknowledges support from the Marianne und Dr. Horst Kiesow-Stiftung, Frankfurt a.M.

[2]The second author was partially supported by the SFB 878 "Groups, geometry, and actions" from the DFG (Deutsche Forschungsgemeinschaft). This paper was typed while the second author visited Grigor Sargsyan in Warsaw (Poland) in Nov 2016, whom he would like to thank for his hospitality and support. He would also like to thank William Chan for his comments on an earlier version of this paper.

[3]The third author would like to acknowledge the support through the funding Projects NSFC 11321101 and 11401567.

[4]The fourth author gratefully acknowledges support from the National Natural Science Fund of China, No. 11322112 and 11671196 and from a Humboldt Research Fellowship for Experienced Researchers.

[5]To get Luzin and Sierpiński sets, one needs to make the additional hypothesis that CH holds true, unless e.g. one works with the concept of *generalized* Luzin and Sierpiński sets which arises from the concept of Luzin and Sierpiński sets by replacing "at most countable" with "smaller than the continuum" and works under Martin's Axiom.

[6]A discussion of "paradoxical" decompositions of the unit ball à la Hausdorff and Banach–Tarski is beyond the scope of this paper, cf. also [5].

D. Pincus and K. Prikry study the Cohen-Halpern-Lévy model $H$ in [9] and show that there is a Luzin set in $H$, thereby establishing that in ZF, the existence of a Luzin set does not imply the existence of a well–ordering of the reals. We will recall their proof below, cf. Theorem 1.5.

In ZF, the existence of a Hamel basis implies the existence of a Vitali set of reals, cf. Lemma 1.1 below. Feferman had observed that $H$ has a Vitali set, cf. [9, p. 433]. Pincus and Prikry ask:

"We would be interested in knowing whether a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ (the rationals) exists in $H$ or in any other model in which $\mathbb{R}$ cannot be well ordered." ([9, p. 433])

In [2], A. Blass shows that in ZF, if every vector space has a basis, then the axiom of choice holds true.

In the current paper we answer the question by Pincus and Prikry and show that $H$ does have a Hamel basis. This will also give Feferman's result as a corollary, cf. Corollary 2.4 below.

We shall also show that $H$ has a Bernstein sets, cf. Theorem 1.7. There is no Sierpiński set in $N$, though, cf. Lemma 1.6. Therefore, in ZF not even the conjunction of the following statements (1), (3), (4), and (5) implies the existence of a well–ordering of the reals.

(1) There is a Luzin set.

(2) There is a Sierpiński set.

(3) There is a Bernstein set.

(4) There is a Vitali set.

(5) There is a Hamel basis.

It remains open whether in ZF plus DC (the principle of dependent choice), (5) yields a well–ordering of the reals. The paper [1] shows that in ZF plus DC, the conjunction of (1), (2), and (3) does not yield a well–ordering of the reals.

# 1 Warm ups.

In what follows, we shall sometimes think of reals as elements of the Baire space ${}^{\omega}\omega$, sometimes as elements of the Cantor space ${}^{\omega}2$, and at other times think of them as actual reals. The attentive reader will have no problem sorting this out.

Let us first show that (4) implies (5). If $X$ is a set of reals, then we write $\text{span}(X)$ for the set of all $\sum_{n=1}^{m} q_n \cdot x_n$, where $m \in \mathbb{N}$, $m \geq 1$, $q_n \in \mathbb{Q}$, and $x_n \in X$ for all $n$, $1 \leq n \leq m$. By convention, we also declare $\text{span}(\emptyset) = \{0\}$.

**Lemma 1.1** *(Folklore) In* ZF*, if there is a Hamel basis, then there is a Vitali set.*

*Proof.* Let $B$ be a Hamel basis. For each real $x$, there is some finite $b \subset B$ such that $[x]_E \subset \text{span}(b)$. There is hence also a unique finite $b \subset B$ with $[x]_E \subset \text{span}(b)$ such that $c \supset b$ for every finite $c \subset B$ with $[x]_E \subset \text{span}(c)$. Let us write $b_x$ for the unique such $b \subset B$. Let $n_x = \mathrm{Card}(b_x)$, and write $b_x = \{z_x^1, \dots, z_x^{n_x}\}$, where $z_x^1 < \dots < z_x^{n_x}$ in the natural ordering of $\mathbb{R}$. Notice that $b_x = \{z_x^1, \dots, z_x^{n_x}\}$ only depends on the class $[x]_E$, not on the choice of a representative.

The rationals may be well–ordered, and any such well–ordering induces a well–ordering of the finite sequences of rationals. Let us fix some such well–ordering.

We may then define a transversal $v$ for the collection of all $E$–equivalence classes as follows. Let $x \in \mathbb{R}$, and let $v([x]_E) = y$ iff $y \in [x]_E$ and for all $y' \in [x]_E$, if $y = \sum_{k=1}^{n_x} q_k \cdot z_x^k$ and $y' = \sum_{k=1}^{n_x} q'_k \cdot z_x^k$, where $q_1, \dots, q_{n_x}, q'_1, \dots, q'_{n_x} \in \mathbb{Q}$, then $(q_1, \dots, q_{n_x}) \leq (q'_1, \dots, q'_{n_x})$ in the well–ordering of finite sequences of rationals.

We have shown that there is a Vitali set. □ (Lemma 1.1)

Let us now recall the Cohen-Halpern-Lévy model. We let $\mathbb{C}$ denote Cohen forcing, i.e., the collection of all finite sequences of natural numbers, ordered by end–extension. If $I$ is any index set, then $\mathbb{C}(I)$ denotes the finite support product of $I$ many copies of $\mathbb{C}$, i.e., $p \in \mathbb{C}(I)$ iff $p(\ell) \in \mathbb{C}$ for $\ell \in I$ and

$$\mathrm{supp}(p) = \{\ell \in I \colon p(\ell) \neq \emptyset\}$$

is finite. In what follows, $I \subseteq \omega$. If $I \cap J = \emptyset$, then $\mathbb{C}(I \cup J) \cong \mathbb{C}(I) \times \mathbb{C}(J)$.

Let us force with $\mathbb{C}(\omega)$ over $L$,[7] and let $g$ be a generic filter. Let $c_n$, $n < \omega$, denote the Cohen reals which $g$ adds. Let us write $A = \{c_n \colon n < \omega\}$ for the set of those Cohen rals. The model

$$H = H(L) = \mathsf{HOD}^{L[g]}_{A \cup \{A\}}$$

of all sets which inside $L[g]$ are hereditarily definable from parameters in $\mathrm{OR} \cup A \cup \{A\}$ is the Cohen–Halpern–Lévy model (over $L$), cf. [3, pp. 136–141], [4], and [9, p. 429]. As $L \subset H \subset L[g]$ and $\mathbb{C}(\omega)$ is countable, and hence trivially has the c.c.c., $L$, $H$, and $L[g]$ all have the same cardinals, and in particular $\omega_1^H = \omega_1^L$. It is well–known that in $H$, the reals cannot be well–ordered and in fact $A$ has no countable subset, cf. e.g. [3, pp. 136–141] and Lemma 1.2 below. Here and in what follows, a set $X$ is called *countable* iff there is some bijection $f \colon \omega \to X$, and $X$ is called *at most countable* iff $X$ is finite or countable.

For any finite $a \subset A$, we write $L[a]$ for the model constructed from the finitely many reals in $a$. Fixing some Gödelization of formulae (or some enumeration of all the rud functions, resp.) at the outset, each $L[a]$ comes with a unique canonical global well–ordering $<_a$ of $L[a]$ by which we mean the one which is induced by the *natural* order of the elements of $a$ and the fixed Gödelization device in the usual fashion. The assignment $a \mapsto <_a$, $a \in [A]^{<\omega}$, is hence in $H$.[8] This is a crucial fact.

Let us fix a bijection

$$e \colon \omega \to \omega \times \omega, \tag{1}$$

and let us write $((n)_0, (n)_1) = e(n)$.

We shall also make use the following.

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[7] We might as well force over $V$ rather than $L$, but forcing over $L$ will simplify the notation a bit.

[8] More precisely, the ternary relation consisting of all $(a, x, y)$ such that $x <_a y$ is definable over $H$.

**Lemma 1.2** *(1) Let* $a \in [A]^{<\omega}$ *and* $X \subset L[a]$, $X \in H$, *say* $X \in \mathsf{HOD}^{L[g]}_{b\cup\{A\}}$, *where* $b \supseteq a$, $b \in [A]^{<\omega}$. *Then* $X \in L[b]$.

*(2) There is no well–ordering of the reals in* $H$.

*(3)* $A$ *has no countable subset in* $H$.

*(4)* $[A]^{<\omega}$ *has no countable subset in* $H$.

*Proof sketch.* (1) Every permutation $\pi\colon \omega \to \omega$ induces an automorphism $e_\pi$ of $\mathbb{C}(\omega)$ by sending $p$ to $q$, where $q(\pi(n)) = p(n)$ for all $n < \omega$. It is clear that no $e_\pi$ moves the canonical name for $A$, call it $\dot{A}$. Let us also write $\dot{c}_n$ for the canonical name for $c_n$, $n < \omega$. Now if $a$, and $b$ are as in the statement of (1), say $b = \{c_{n_1}, \dots , c_{n_k}\}$, if $p$, $q \in \mathbb{C}(\omega)$, if $\pi \restriction \{n_1, \dots , n_k\} = \mathrm{id}$, $p \restriction \{n_1, \dots , n_k\}$ is compatible with $q \restriction \{n_1, \dots , n_k\}$, and $\mathrm{supp}(\pi(p)) \cap \mathrm{supp}(q) \subseteq \{n_1, \dots , n_k\}$, if $x \in L$, if $\alpha_1, \dots, \alpha_m$ are ordinals, and if $\varphi$ is a formula, then

$$\begin{aligned} p \Vdash^{\mathbb{C}(\omega)}_L \ \varphi(\check{x}, \check{\alpha}_1, \dots \check{\alpha}_m, \dot{c}_{n_1}, \dots \dot{c}_{n_k}, \dot{A}) \iff \\ \pi(p) \Vdash^{\mathbb{C}(\omega)}_L \ \varphi(\check{x}, \check{\alpha}_1, \dots \check{\alpha}_m, \dot{c}_{n_1}, \dots \dot{c}_{n_k}, \dot{A}) \end{aligned}$$

and $\pi(p)$ is compatible with $q$, so that the statement $\varphi(\check{x}, \check{\alpha}_1, \dots \check{\alpha}_m, \dot{c}_{n_1}, \dots \dot{c}_{n_k}, \dot{A})$ will be decided by conditions $p \in \mathbb{C}(\omega)$ with $\mathrm{supp}(p) \subseteq \{n_1, \dots , n_k\}$. But every set in $L[b]$ is coded by a set of ordinals, so if $X$ is as in (1), this shows that $X \in L[b]$.

(2) Every real is a subset of $L$. Hence by (1), if $L[g]$ had a well–ordering of the reals in $\mathsf{HOD}^{L[g]}_{a\cup\{A\}}$, some $a \in [A]^{<\omega}$, then every real of $H$ would be in $L[a]$, which is nonsense.

(3) Assume that $f\colon \omega \to A$ is injective, $f \in H$. Let $x \in {}^\omega\omega$ be defined by $x(n) = f((n)_0)((n)_1)$, so that $x \in H$. By (1), $x \in L[a]$ for some $a \in [A]^{<\omega}$. But then $\mathrm{ran}(f) \subset L[a]$, which is nonsense, as there is some $n < \omega$ such that $c_n \in \mathrm{ran}(f) \setminus a$.

(4) This readily follows from (3). □ (Lemma 1.2)

Let us recall another standard fact.

$$\text{If } a,\, b \in [A]^{<\omega}, \text{ then } L[a] \cap L[b] = L[a \cap b]. \tag{2}$$

To see this, let us assume without loss of generality that $a \setminus b \neq \emptyset \neq b \setminus a$, and say $a \setminus b = \{c_n \colon n \in I\}$ and $b \setminus a = \{c_n \colon n \in J\}$, where $I$ and $J$ are non–empty disjoint finite subsets of $\omega$. Then $\mathbb{C}(I) \cong \mathbb{C} \cong \mathbb{C}(J)$, and $a \setminus b$ and $b \setminus a$ are mutually $\mathbb{C}$–generic over $L[a\cap b]$. But then $L[a]\cap L[b] = L[a\cap b][a\setminus b]\cap L[a\cap b][b\setminus a] = L[a\cap b]$, cf. [10, Problem 6.12].

For any $a \in [A]^{<\omega}$, we write $\mathbb{R}_a = \mathbb{R} \cap L[a]$ and $\mathbb{R}^+_a = \mathbb{R}_a \setminus \bigcup\{\mathbb{R}_b \colon b \subsetneq a\}$. By [3, pp. 136–141], $(\mathbb{R}^+_a \colon a \in [A]^{<\omega})$ is a partition of $\mathbb{R}$: By Lemma 1.2 (1),

$$\mathbb{R} \cap H = \bigcup\{\mathbb{R}^+_a \colon a \in [A]^{<\omega}\}, \tag{3}$$

and $\mathbb{R}_a \cap \mathbb{R}_b = \mathbb{R}_{a\cap b}$ by (2), so that

$$\mathbb{R}^+_a \cap \mathbb{R}^+_b = \emptyset \text{ for } a, b \in [A]^{<\omega}, a \neq b. \tag{4}$$

For $x \in \mathbb{R}$, we shall also write $a(x)$ for the unique $a \in [A]^{<\omega}$ such that $x \in \mathbb{R}^+_a$, and we shall write $\#(x) = \mathrm{Card}(a(x))$.

Adrian Mathias showed that there is an $H$–definable function which assigns to each $x \in H$ an ordering $<_x$ such that $<_x$ is a well–ordering iff $x$ can be well–ordered in $H$, cf. [7, p. 182]. This gives the following as a special simple case.

**Lemma 1.3** *(A. Mathias) In $H$, the union of countably many countable sets of reals is countable.*

*Proof.* Let us work inside $H$. Let $(A_n : n < \omega)$ be such that for each $n < \omega$, $A_n \subset \mathbb{R}$ and there exists some surjection $f : \omega \to A_n$. For each such pair $n$, $f$ let $y_{n,f} \in {}^{\omega}\omega$ be such that $y_{n,f}(m) = f((m)_0)((m)_1)$. If $a \in [A]^{<\omega}$ and $y_{n,f} \in \mathbb{R}_a$, then $A_n \in L[a]$. By (2), for each $n$ there is a unique $a_n \in [A]^{<\omega}$ such that $A_n \in L[a_n]$ and $b \supset a_n$ for each $b \in [A]^{<\omega}$ such that $A_n \in L[b]$. Notice that $A_n$ is also countable in $L[a_n]$.

Using the function $n \mapsto a_n$, an easy recursion yields a surjection $g : \omega \to \bigcup\{a_n : n < \omega\}$: first enumerate the finitely many elements of $a_0$ according to their natural order, then enumerate the finitely many elements of $a_1$ according to their natural order, etc. As $A$ has no countable subset, $\bigcup\{a_n : n < \omega\}$ must be finite, say $a = \bigcup\{a_n : n < \omega\} \in [A]^{<\omega}$. But then $\{A_n : n < \omega\} \subset L[a]$. (We don't claim $(A_n : n < \omega) \in L[a]$.)

For each $n < \omega$, we may now let $f_n$ the $<_a$–least surjection $f : \omega \to A_n$. Then $f(n) = f_{(n)_0}((n)_1)$ for $n < \omega$ defines a surjection from $\omega$ onto $\bigcup\{A_n : n < \omega\}$, as desired. □ (Lemma 1.3)

**Lemma 1.4** *(1) ([6, Theorem 3.20]) Let $a \in [A]^{<\omega}$. Then $\mathbb{R}_a$ is a null set in $H$.*

*(2) If $B \subset \mathbb{R} \cap H$, $B \in H$, and $B$ is countable in $L[g]$, then $B$ is a null set in $H$.*

*Proof sketch.* (1) Let $\mathbb{R} = {}^{\omega}2$ in this argument, with the addition $+$ being the componentwise addition in $\mathbb{Z}/2\mathbb{Z}$. Let $n < \omega$ be such that $c_n \notin a$. It suffices to prove that $\mathbb{R}_a$ is null in $L[a \cup \{c_n\}]$.

In $L[a]$, let $\mathbb{R}_a = N \cup M$, where $N$ is $G_\delta$ and null set, and $M$ is $F_\sigma$ and meager, cf. e.g. [8]. Inside $L[a \cup \{c_n\}]$, let us consider $N^* + c_n = \{x + c_n : x \in N^*\}$, where $N^*$ is $L[a \cup \{c_n\}]$'s version of $N$.

Let $x \in \mathbb{R}_a$. As $N$ is comeager in $L[a]$, $N + x$ is also comeager in $L[a]$, so that $c_n \in (N + x)^* = N^* + x$, see [10, Lemma 8.9 (2)], and hence $x \in N^* + c_n$. So $\mathbb{R}_a \subseteq N^* + c_n$. But $N$ is null in $L[a]$, and hence $N^*$ and $N^* + c_n$ are null in $L[a \cup \{c_n\}]$. $\mathbb{R}_a$ is therefore contained in a null set of $L[a \cup \{c_n\}]$ and is hence itself null.

(2) Say $f : \omega \to B$, $f \in L[g]$, is an enumeration of $B$, and let $\tau \in L^{\mathbb{C}(\omega)}$ be such that $\tau^g = f$. Let us write $\tau(n)$ for the canonical name for $f(n)$ induced by $\tau$. We aim to find $N \in H$, a $G_\delta$ null set in $H$ with a code in $L$ such that $B \subset N$. Let $g : \mathbb{C}(\omega) \times \omega \to \omega$ be bijective.

Let $m < \omega$. Set $\epsilon^m = \frac{1}{m+1}$ and $\epsilon^m_n = \frac{1}{2^{n+1}} \cdot \epsilon^m$ for $n < \omega$, so that $\sum_{n=0}^{\infty} \epsilon^m_n = \epsilon^m$.

Working in $L$, for each pair $(p, k) \in \mathbb{C}(\omega) \times \omega$, write $n = g((p, k))$, and let us pick some $q \in \mathbb{C}(\omega)$, $q \leq p$, and some $s \in {}^{<\omega}\omega$ such that $q \Vdash^{\mathbb{C}(\omega)}_{L} \check{s} \subset \tau(k)$, and $\mu(U_s) \leq \epsilon^m_n$, and write $\mathcal{O}^m_n = U_s$. (Here, $U_s$ is the basis clopen set $\{x : x \supset s\}$.)

Set $\mathcal{O}^m = \bigcup\{\mathcal{O}^m_n \colon n < \omega\}$. For a given $k < \omega$, the set $\{q \in \mathbb{C}(\omega) \colon \exists n\, q \Vdash^{\mathbb{C}(\omega)}_L \tau(k) \in \mathcal{O}^m_n\}$ is dense, so that $f(n) = (\tau(n))^g \in \mathcal{O}^m_n$ for some $n$. In other words, $B \subset \mathcal{O}^m$.

Set $N = \bigcap_{m<\omega} \mathcal{O}^m$, to be interpreted in $H$. We have that $N$ is a $G_\delta$ null set inside $H$ with a code in $L$, and $B \subset N$. □ (Lemma 1.4)

**Theorem 1.5** *(D. Pincus, K. Prikry) In $H$, there is a Luzin set.*

*Proof.* Let $\Lambda \in L$ be such that $L \models$ "$\Lambda$ is a Luzin set." We aim to verify that $\Lambda$ is Luzin in $H$. $\Lambda$ is uncountable in $L$, so that also $H$ can see a bijection of $\Lambda$ with its own $\omega_1$, as $\omega_1^H = \omega_1^L$. In particular, $\Lambda$ is uncountable in $H$.

By Lemma 1.3, it suffices to verify that inside $H$,

$$\Lambda \setminus \mathcal{O} \text{ is at most countable,} \tag{5}$$

whenever $\mathcal{O}$ is a dense union of countably many open intervals with rational end–points.

Let $((p_n, q_n) \colon n < \omega)$ be an enumeration of all open intervals with rational end–points, and let $X \subset \omega$, $X \in H$, be such that

$$H \models \text{ "}\mathcal{O} = \bigcup\{(p_n, q_n) \colon n \in X\} \text{ is dense."}$$

Let us suppose that (5) were not true in $H$ for this fixed $\mathcal{O}$. As $\Lambda \in L$, inside $H$ there must then be a bijection from $\omega_1$ onto $\Lambda \setminus \mathcal{O}$, so that by $\omega_1^{L[g]} = \omega_1^H$ also

$$\Lambda \setminus \mathcal{O} \text{ is uncountable in } L[g]. \tag{6}$$

Let $\tau \in L^{\mathbb{C}(\omega)}$ be a name for $X$, and let $p \in g$ be such that

$$p \Vdash^{\mathbb{C}(\omega)}_L \text{ "}\Lambda \setminus \bigcup\{(p_n, q_n) \colon n \in \tau\} \text{ is uncountable."}$$

As $\mathbb{C}(\omega)$ is countable, we may work in $L[g]$ and find some $q \in g$, $q \leq p$, such that for uncountably many $x \in \mathbb{R} \cap L$,

$$q \Vdash^{\mathbb{C}(\omega)}_L \text{ "}\check{x} \in \Lambda \setminus \bigcup\{(p_n, q_n) \colon n \in \tau\}.\text{"} \tag{7}$$

Let us write $U$ for the set of all $x \in \mathbb{R} \cap L$ with (7), so that $U$ is an uncountable set of reals in $L$, and let

$$\mathcal{O}^* = \bigcup\{(p_n, q_n) \colon \exists r \leq q\, r \Vdash^{\mathbb{C}(\omega)}_L n \in \tau\},$$

as being defined in $L$.

Of course, $\mathcal{O}^* \supseteq \mathcal{O} \cap L$, so that $\mathcal{O}^*$ is open and dense in $L$. As $\Lambda$ is a Luzin set in $L$, $\Lambda \setminus \mathcal{O}^*$ must be countable in $L$.

We have a contradiction with (6). □ (Theorem 1.5)

**Lemma 1.6** *In $H$, there is no Sierpiński set.*

*Proof.* We shall prove that there is no set $S \in H$ of reals such that $S$ is not at most countable in $H$ and for each null set $N$ of $H$, $S \cap N$ is at most countable.

Let us suppose that $S \in H$ is such a set. By Lemma 1.4, we cannot have that $S \subseteq \mathbb{R}_a$ for some $a \in [A]^{<\omega}$, because if this were true, then $S \cap \mathbb{R}_a = S$ and $S$ itself would have to be at most countable.

Therefore, the set

$$F = \{a \in [A]^{<\omega} \colon S \cap \mathbb{R}_a^+ \neq \emptyset\}$$

is not finite. We may then inside $H$ define the function $f \colon F \to \mathbb{R} \cap H$ by setting $f(a)$ to be the $<_a$–least element of $S \cap \mathbb{R}_a^+$.

Write $B = \mathrm{ran}(f)$. Then $B \in H$, and $B$ is countable inside $L[g]$. By Lemma 1.4 (2), $B$ is then a null set in $H$. Therefore, $B = S \cap B$ must be countable in $H$, i.e., there is some bijective $g \in H$, $g \colon \omega \to B$.

However, $((a, \mathbb{R}_a^+) \colon a \in [A]^\omega) \in H$, so that $x \mapsto a(x)$ is in $H$, and hence $a \circ g \in H$, where $(a \circ g)(n) = a(g(n))$, $n < \omega$. Then $a \circ g \colon \omega \to [A]^{<\omega}$ is injective, which contradicts Lemma 1.2 (4). □ (Lemma 1.6)

**Theorem 1.7** *In $H$, there is a Bernstein set.*

*Proof.* In this proof, let us think of reals as elements of the Cantor space ${}^\omega 2$. Let us work in $H$.

We let

$$B = \{x \in \mathbb{R} \colon \exists \text{ even } n\, (2^n < \#(x) \leq 2^{n+1})\} \quad \text{and}$$
$$B' = \{x \in \mathbb{R} \colon \exists \text{ odd } n\, (2^n < \#(x) \leq 2^{n+1})\}.$$

Obviously, $B \cap B' = \emptyset$.

Let $P \subset \mathbb{R}$ be perfect. We aim to see that $P \cap B \neq \emptyset \neq P \cap B'$.

Say $P = [T] = \{x \in {}^\omega 2 \colon \forall n\, x \restriction n \in T\}$, where $T \subseteq {}^{<\omega}2$ is a perfect tree. Modulo some fixed natural bijection ${}^{<\omega}2 \leftrightarrow \omega$, we may identify $T$ with a real. By (3), we may pick some $a \in [A]^{<\omega}$ such that $T \in L[a]$. Say $\mathrm{Card}(a) < 2^n$, where $n$ is even.

Let $b \in [A]^{2^{n+1}}$, $b \supset a$, and let $x \in \mathbb{R}_b^+$. In particular, $\#(x) = 2^{n+1}$. It is easy to work in $L[b]$ and construct some $z \in [T]$ such that $x \leq_T z \oplus T$,[9] e.g., arrange that if $z \restriction m$ is the $k^{\text{th}}$ splitting node of $T$ along $z$, where $k \leq m < \omega$, then $z(m) = 0$ if $x(k) = 0$ and $z(m) = 1$ if $x(k) = 1$.

If we had $\#(z) \leq 2^n$, then $\#(z \oplus T) \leq \#(z) + \#(T) < 2^n + 2^n = 2^{n+1}$, so that $\#(x) < 2^{n+1}$ by $x \leq_T z \oplus T$. Contradiction! Hence $\#(z) > 2^n$. By $z \in L[b]$, $\#(z) \leq 2^{n+1}$. Therefore, $z \in P \cap B$.

The same argument shows that $P \cap B' \neq \emptyset$. $B$ (and also $B'$) is thus a Bernstein set. □ (Theorem 1.7)

# 2 A Hamel basis.

The following is the main theorem of the current paper. Recall that for any $a \in [A]^{<\omega}$, we write $\mathbb{R}_a = \mathbb{R} \cap L[a]$. Let us now also write $\mathbb{R}_{<a} = \mathrm{span}(\bigcup\{\mathbb{R}_b \colon b \subsetneq a\})$,

[9] Here, $(x \oplus y)(2n) = x(n)$ and $(x \oplus y)(2n + 1) = y(n)$, $n < \omega$.

and $\mathbb{R}_a^* = \mathbb{R}_a \setminus \mathbb{R}_{<a}$. In particular, $\mathbb{R}_{<\emptyset} = \{0\}$ by our above convention that $\mathrm{span}(\emptyset) = \{0\}$, and $\mathbb{R}_\emptyset^* = (\mathbb{R} \cap L) \setminus \{0\}$.

The proof of Claim 2.2 below will show that

$$\mathbb{R} \cap H = \mathrm{span}(\bigcup\{\mathbb{R}_a^* \colon a \in [A]^{<\omega}\}. \tag{8}$$

Also, we have that $\mathbb{R}_a^* \subset \mathbb{R}_a^+$, so that by (3),

$$\mathbb{R}_a^* \cap \mathbb{R}_b^* = \emptyset \text{ for } a, b \in [A]^{<\omega}, a \neq b. \tag{9}$$

**Theorem 2.1** *In $H$, there is a Hamel basis.*

*Proof.* We call $X \subset \mathbb{R}_a^*$ *linearly independent over* $\mathbb{R}_{<a}$ iff whenever

$$\sum_{n=1}^{m} q_n \cdot x_n \in \mathbb{R}_{<a},$$

where $m \in \mathbb{N}$, $m \geq 1$, and $q_n \in \mathbb{Q}$ and $x_n \in X$ for all $n$, $1 \leq n \leq m$, then $q_1 = \ldots = q_m = 0$. In other words, $X \subset \mathbb{R}_a^*$ is linearly independent over $\mathbb{R}_{<a}$ iff

$$\mathrm{span}(X) \cap \mathbb{R}_{<a} = \{0\}.$$

We call $X \subset \mathbb{R}_a^*$ *maximal linearly independent over* $\mathbb{R}_{<a}$ iff $X$ is linearly independent over $\mathbb{R}_{<a}$ and no $Y \supsetneq X$, $Y \subset \mathbb{R}_a$ is still linearly independent over $\mathbb{R}_{<a}$. In particular, $X \subset \mathbb{R}_\emptyset^* = (\mathbb{R} \cap L) \setminus \{0\}$ is linearly independet over $\mathbb{R}_{<\emptyset} = \{0\}$ iff $X$ is a Hamel basis for $\mathbb{R} \cap L$.

For any $a \in [A]^{<\omega}$, we let $b_a$ denote the $<_a$–least set $X \subset \mathbb{R}_a^*$, $X \in L[a]$, which is maximal linearly independent over $\mathbb{R}_{<a}$. By the above crucial fact, the function $a \mapsto b_a$ is well–defined and *exists inside* $H$. In particular,

$$B = \bigcup\{b_a \colon a \in [A]^{<\omega}\}$$

is an element of $H$.

We claim that $B$ is a Hamel basis for the reals of $H$, which will be established by Claims 2.2 and 2.3.

**Claim 2.2** $\mathbb{R} \cap H \subset \mathrm{span}(B)$.

*Proof* of Claim 2.2. Assume not, and let $n < \omega$ be the least size of some $a \in [A]^{<\omega}$ such that $\mathbb{R}_a^* \setminus \mathrm{span}(B) \neq \emptyset$. Pick $x \in \mathbb{R}_a^* \setminus \mathrm{span}(B) \neq \emptyset$, where $\mathrm{Card}(a) = n$.

We must have $n > 0$, as $b_\emptyset$ is a Hamel basis for the reals of $L$. Then, by the maximality of $b_a$, while $b_a$ is linearly independet over $\mathbb{R}_{<a}$, $b_a \cup \{x\}$ cannot be linearly independent over $\mathbb{R}_{<a}$. This means that there are $q \in \mathbb{Q}$, $q \neq 0$, $m \in \mathbb{N}$, $m \geq 1$, and $q_n \in \mathbb{Q} \setminus \{0\}$ and $x_n \in b_a$ for all $n$, $1 \leq n \leq m$, such that

$$z = q \cdot x + \sum_{n=1}^{m} q_n \cdot x_n \in \mathbb{R}_{<a}.$$

By the definition of $\mathbb{R}_{<a}$ and the minimality of $n$, $z \in \mathrm{span}(\bigcup\{b_c \colon c \subsetneq a\})$, which then clearly implies that $x \in \mathrm{span}(\bigcup\{b_c \colon c \subseteq a\}) \subset \mathrm{span}(B)$.

This is a contradiction! $\square$ (Claim 2.2)

**Claim 2.3** *B is linearly independent.*

*Proof* of Claim 2.3. Assume not. This means that there are $1 \leq k < \omega$, $a_i \in [A]^{<\omega}$ pairwise different, $m_i \in \mathbb{N}$, $m_i \geq 1$ for $1 \leq i \leq k$, and $q^i_n \in \mathbb{Q} \setminus \{0\}$ and $x^i_n \in b_{a_i}$ for all $n$, $1 \leq n \leq m$, and $i$, $1 \leq i \leq k$, such that

$$\sum_{n=1}^{m_1} q^1_n \cdot x^1_n + \ldots + \sum_{n=1}^{m_k} q^k_n \cdot x^k_n = 0. \tag{10}$$

By the properties of $b_{a_i}$, $\sum_{n=1}^{m_i} q^i_n \cdot x^i_n \in \mathbb{R}^*_{a_i}$, so that (10) buys us that there are $z_i \in \mathbb{R}^*_{a_i}$, $z_i \neq 0$, $1 \leq i \leq k$, such that

$$z_1 + \ldots + z_k = 0. \tag{11}$$

There must be some $i$ such that there is no $j$ with $a_j \supsetneq a_i$, which implies that $a_j \cap a_i \subsetneq a_i$ for all $j \neq i$. Let us assume without loss of generality that $a_j \cap a_1 \subsetneq a_1$ for all $j$, $1 < j \leq k$.

Let $a_1 = \{c_\ell \colon \ell \in I\}$, where $I \in [\omega]^{<\omega}$, and let $a_j \cap a_1 = \{c_\ell \colon \ell \in I_j\}$, where $I_j \subsetneq I$, for $1 < j \leq l$.

In what follows, a *nice name* $\tau$ for a real is a name of the form

$$\tau = \bigcup_{n,m<\omega} \{(n,m)^\vee\} \times A_{n,m}, \tag{12}$$

where each $A_{n,m}$ is a maximal antichain of conditions of the forcing in question deciding that $\tau(\check{n}) = \check{m}$.

We have that $z_1$ is $\mathbb{C}(I)$–generic over $L$, so that we may pick a nice name $\tau_1 \in L^{\mathbb{C}(I)}$ for $z_1$ with $(\tau_1)^{g\restriction I} = z_1$. Similarly, for $1 < j \leq k$, $z_k$ is $\mathbb{C}(I_j)$–generic over $L[g \restriction (\omega \setminus I)]$, so that we may pick a nice name $\tau_j \in L[g \restriction (\omega \setminus I)]^{\mathbb{C}(I_j)}$ for $z_j$ with $(\tau_j)^{g\restriction I_j} = z_j$. We may construe each $\tau_j$, $1 < j \leq k$, as a name in $L[g \restriction (\omega \setminus I)]^{\mathbb{C}(I)}$ by replacing each $p \colon I_j \to \mathbb{C}$ in an antichain as in (12) by $p' \colon I \to \mathbb{C}$, where $p'(\ell) = p(\ell)$ for $\ell \in I_j$ and $p'(\ell) = \emptyset$ otherwise. Let $p \in g \restriction I$ be such that

$$p \Vdash^{\mathbb{C}(I)}_{L[g\restriction(\omega\setminus I)]} \tau_1 + \tau_2 + \ldots + \tau_k = 0.$$

We now have that inside $L[g \restriction (\omega \setminus I)]$, there are nice $\mathbb{C}(I)$–names $\tau'_j$, $1 < j \leq k$ (namey, $\tau_j$, $1 < j \leq k$), such that still inside $L[g \restriction (\omega \setminus I)]$

1. $p \Vdash^{\mathbb{C}(I)} \tau_1 + \tau'_2 + \ldots + \tau'_k = 0$, and
2. for all $j$, $1 < j \leq k$ and for all $p$ in one of the antichains of the nice name $\tau'_j$, $\mathrm{supp}(p) \subseteq I_j$.

Both (1) and (2) are arithmetic in real codes for $\tau_1, \tau'_2, \ldots, \tau'_k$, so that by $\tau_1 \in L^{\mathbb{C}(I)}$ and $\Sigma^1_1$–absoluteness between $L$ and $L[g \restriction (\omega \setminus I)]$ there are inside $L$ nice $\mathbb{C}(I)$–names $\tau'_j$, $1 < j \leq k$, such that in $L$, (1) and (2) hold true. But then, writing $z'_j = (\tau'_j)^{g\restriction I}$, we have by (2) that $z'_j \in \mathbb{R}_{I_j}$ for $1 < j \leq k$, and $z_1 + z'_2 + \ldots + z'_k = 0$ by (1). But then $z_1 \in \mathbb{R}^*_I \cap \mathbb{R}_{<I}$, which is absurd. □ (Claim 2.3)

This finishes the proof of Theorem 2.1. □ (Theorem 2.1)

In the light of Lemma 1.1, Theorem 2.1 reproves Feferman's result.

**Corollary 2.4** *(S. Feferman) In $H$, there is a Vitali set.*

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