Coding over Core Models

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Early in their careers, both Peter Koepke and Philip Welch made major contributions to two important areas of set theory, core model theory (cf. [10]) and coding (cf. [1]), respectively. In this article we aim to survey some of the work that has been done which combines these two themes, extending Jensen's original Coding Theorem from $\mathbf{L}$ to core models witnessing large cardinal properties.

The original result of Jensen can be stated as follows.

Theorem 1. (Jensen, cf. [1]) Suppose that (V, A) is a transitive model of $\mathsf{ZFC} + \mathsf{GCH}$ (i.e., V is a transitive model of $\mathsf{ZFC} + \mathsf{GCH}$ and replacement holds in V for formulas mentioning A as an additional unary predicate). Then there is a (V, A)-definable, cofinality-preserving class forcing P such that if G is P-generic over (V, A) we have:

(a) For some real R, $(V[G], A) \models \mathsf{ZFC}$ + the universe is $\mathbf{L}[R]$ and A is definable with parameter R.

(b) The typical large cardinals properties consistent with $\mathbf{V}{=}\mathbf{L}$ are preserved from V to $V[R]$: inaccessible, Mahlo, weak compact, Π^1_n indescribable, subtle, ineffable, α-Erdős for countable α.

Corollary 2. It is consistent to have a real R such that $\mathbf{L}$ and $\mathbf{L}[R]$ have the same cofinalities but R belongs to no set-generic extension of $\mathbf{L}$.

The theme of this article is to consider the following question: To what extent is it possible to establish an analagous result when $\mathbf{L}$ is replaced by a core model $\mathbf{K}$ and the large cardinal properties in (b) are strengthened to those consistent with $\mathbf{V}{=}\mathbf{K}$ (measurable, hypermeasurable, strong, Woodin)?

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A brief summary of the situation is as follows. Coding up to one measurable cardinal is unproblematic (cf. [4]), although already in this case there are some issues with condensation and the interesting new phenomenon of "ultrapower codings" arises. At the level of hypermeasurable cardinals there are serious condensation issues which obstruct a fully general result; nevertheless variants of Corollary 2 can be established and very special predicates A as in Theorem 1 can be coded (such as a generic for a Příkrý product, cf. [7]). In addition, although one is able to lift enough of the total extenders on the hierarchy of a core model witnessing hypermeasurability, it requires extra effort to lift more than one total extender for the same critical point (and it is not in general possible to lift all of the extenders (partial and total) on a fixed critical point κ satisfying $o(\kappa) = \kappa^{+++}$; we conjecture that this can be improved to $o(\kappa) = \kappa^{++}$). At the level of Woodin cardinals, even Corollary 2 is not possible if the aim is to lift all total extenders in a witness to Woodinness via the "A-strong" definition of this notion; however this obstacle is removed by instead considering witnesses to the definition of Woodinness in terms of "$j(f)(\kappa)$ strength" (cf. [6]).

There are a number of applications of coding over core models. In addition to those found in [5] based on Jensen's original method, we mention two other examples.

Theorem 3. (Friedman-Schrittesser, [9]) Relative to a Mahlo cardinal it is consistent that every set of reals in $L(\mathbb{R})$ is Lebesgue measurable but some projective (indeed lightface Δ^1_3) set of reals does not have the Baire property.

Theorem 4. (Friedman-Golshani, [7]) Relative to a strong cardinal (indeed relative to a cardinal κ that is $\mathbf{H}(\kappa^{+++})$-strong) it is consistent to have transitive models $V \subseteq V[R]$ of ZFC where R is a real, GCH holds in V and GCH fails at every infinite cardinal in $V[R]$. One can further require that V, $V[R]$ have the same cardinals.

1 About Jensen coding

To make what follows more intelligible it is worthwhile to first review the case of Jensen coding. No matter how you look at it, even this argument is complicated, although major simplifications can be made if one assumes the nonexistence of $0^\#$ in the ground model V. Our aim here however is not to delve into the fine points of the proof (and in particular we shall not reveal how the nonexistence of $0^\#$ can be exploited), but rather to give the architecture of the argument in order to facilitate a later discussion of generalisations.

For simplicity consider the special case where the cardinals of the ground model V are the same as those in $\mathbf{L}$ and the ground model is $(\mathbf{L}[A], A)$ where

A is a class of ordinals such that $\mathbf{H}(\alpha) = \mathbf{L}_\alpha[A]$ for each infinite cardinal α (the latter can be arranged using the fact that the GCH holds in V).

Coding is based on the method of almost disjoint forcing. Suppose that A is a subset of ω_1. Then we can code A into a real as follows: For each countable ordinal ξ attach a subset b_ξ of ω (so that the b_ξ's are almost disjoint) and force a real R such that R is almost disjoint from b_ξ iff ξ belongs to A. Actually it is convenient to modify this to: R almost contains b_ξ iff ξ belongs to A (where *almost contains* means contains with only finitely many exceptions). The conditions to achieve this are pairs (s, s^*) where s is an ω-Cohen condition (i.e., element of ${}^{<\omega}2$) and s^* is a finite subset of A; when extending to (t, t^*) we extend s to t, enlarge s^* to t^* and insist that if $s(n)$ is undefined but $t(n)$ equals 0 then n does not belong to b_ξ for any ξ in s^*. Then the generic G is determined by the union G_0 of the s for (s, s^*) in G and we can take R to be the set of n such that $G_0(n)$ equals 1. The forcing has the c.c.c. and ensures that A belongs to $\mathbf{L}[R]$ using the hypothesis $\omega_1 = \omega_1^{\mathbf{L}}$ to produce the b_ξ's in $\mathbf{L}$ (and therefore also in $\mathbf{L}[R]$).[1]

There is nothing to stop us from coding a subset A of ω_2 into a real in a similar fashion: First we use the hypothesis $\omega_2 = \omega_2^{\mathbf{L}}$ to choose subsets b_ξ of ω_1 to set up a forcing to code A into a subset B of ω_1 via the equivalence $\xi \in A$ iff B almost contains b_ξ, and then we code B into a real as in the previous paragraph. It is pretty clear how to do this for a subset of any ω_n, n finite.

If we have a subset A of $\aleph_\omega$ then we have to force subsets A_n of ω_n for each n so that A_n codes both A_{n+1} and $A \cap \omega_n$. At first this is confusing because there is no "top", i.e., no largest n to begin with, but further reflection reveals that there is no problem at all, as we don't need to know all of A_{n+1} to talk about conditions to add A_n. More precisely, a condition p will assign to each n a pair (s_n, s_n^*) so that s_n is an ω_n-Cohen condition and s_n^* is a size less than ω_n subset of the set of ξ such that $s_{n+1}(\xi)$ is defined with value 1. This makes sense even though s_{n+1} is not defined on all of ω_{n+1}. We also insist that all of the b_ξ's consist of even ordinals and that each $A \cap \omega_n$ is coded into the union of the s_n's using its values at odd ordinals. In the end A gets coded into a real and cofinalities are preserved since for any n the forcing factors into an ω_n-closed forcing (the nth upper part) followed by an ω_n-c.c. forcing (the nth lower part).

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[1] In a more general setting we have to worry about how to find the b_ξ's in $\mathbf{L}[R]$. Jensen's trick to achieve this is to "reshape" A into a stronger predicate A' with the property that any countable ordinal ξ is in fact countable in $\mathbf{L}[A' \cap \xi]$; then after R decodes $A' \cap \xi$ it can find b_ξ and continue the decoding. A clever argument shows that such an A' can be added over $\mathbf{L}[A]$ by an ω-distributive forcing; when A is not just a subset of ω_1 but a subset of some larger cardinal or even a proper class of ordinals, then the "reshaping" forcing must be woven into the coding forcing itself.

Coding a subset A of $\aleph_{\omega+1}$ into a real requires a new idea. Actually by the previous paragraph it's enough to see how to code A into a subset of $\aleph_\omega$. Again we would like to assign a subset b_ξ of $\aleph_\omega$ to each $\xi < \aleph_{\omega+1}$ and then hope to force a subset B of $\aleph_\omega$ which almost contains b_ξ iff ξ belongs to A; how are we going to do that? The conditions to add B cannot be built from "$\aleph_\omega$-Cohen conditions" as this makes no sense for the singular cardinal $\aleph_\omega$. Instead they should look like conditions in the product of the ω_n-Cohen forcings, i.e., of the form $(s_n \mid n \in \omega)$ where each s_n is an ω_n-Cohen condition (as in the previous paragraph but without the "restraints" s^*_n). Actually it is very convenient to instead write $(s_{\omega_n} \mid n \in \{-1\} \cup \omega)$ where $\omega_{-1} = 0$ and s_{ω_n} is an ω_{n+1}-Cohen condition for each $n \geq -1$, and to think of s_{ω_n} as an ω_{n+1}-Cohen condition on the interval $[\omega_n, \omega_{n+1})$ rather than on ω_{n+1}, to separate the domains of the different s_{ω_n}'s. Thus the characteristic function of the generic subset B of $\aleph_\omega$ is the union of all of the s_{ω_n}'s which appear in the generic.

As said above we'd like to choose the b_ξ's so that ξ belongs to A iff the generic subset B of $\aleph_\omega$ almost contains b_ξ (i.e., contains b_ξ with a set of exceptions which is bounded in $\aleph_\omega$). This is done using a *scale*, i.e., a sequence $(f_\xi \mid \xi < \aleph_{\omega+1})$ of functions in $\prod_{n\geq -1}[\omega_n, \omega_{n+1})$ which is cofinal mod finite. Then we take b_ξ to be the range of f_ξ. Again it is convenient to change notation: instead of writing $f_\xi(n)$ we write $f_\xi(\omega_n)$. So the coding is: ξ belongs to A iff $G_{\omega_n}(f_\xi(\omega_n)) = 1$ for sufficiently large n, where G_{ω_n} denotes the union of the s_n's which appear in the generic.

As we are using a scale we can arrange the following: if $p = (s_{\omega_n} \mid -1 \leq n < \omega)$ is a condition then for some ordinal $|p| < \aleph_{\omega+1}$ called the *height* of p, if ξ is less than $|p|$ then $\xi \in A$ iff $s_{\omega_n}(f_\xi(\omega_n)) = 1$ for sufficiently large n and if ξ is at least $|p|$ then $f_\xi(\omega_n)$ is not in the domain of s_{ω_n} for sufficiently large n. In other words, p already codes A below $|p|$ but provides no information about future coding on the interval $[|p|, \aleph_{\omega+1})$. Notice the difference from the successor coding case: a single condition will definitively code an initial segment of A, in the sense that its values on a final segment of b_ξ for ξ in an initial segment of $\aleph_{\omega+1}$ have already been fixed (restraints are not needed). Of course no condition will code all of A, so this initial segment of A is proper.

But how do we know that this coding of $A \subseteq \aleph_{\omega+1}$ into a subset of $\aleph_\omega$ preserves the cardinal $\aleph_{\omega+1}$? For each n we can factor the forcing as the part $\geq \omega_n$ followed by the part below ω_n, and as the latter is a small forcing it causes no problems with cardinal-preservation; so we want to show that the forcing $\geq \omega_n$ (using conditions $p = (s_{\omega_k} \mid k \geq n)$) is ω_{n+1}-distributive, i.e., does not add new ω_n-sequences. For simplicity suppose that n is 0, so we want to hit ω-many open dense sets below any condition $p = (s_{\omega_k} \mid k \geq 0)$. Here is the worry: maybe things are going fine with the sequence $p = p_0 \geq p_1 \geq \cdots$ with corresponding heights $|p_0| \leq |p_1| \leq \cdots$ so we can conclude that the limit p_ω of the p_n's will code A up to the limit $|p_\omega|$ of the $|p_n|$'s. But there is the danger that p_ω "overspills" in the sense that it already has assigned cofinally many values on b_ξ for some $\xi \geq |p_\omega|$. This unintended assignment may conflict with the desired coding of A at the ordinal ξ.

The solution is to guide the construction using sufficiently elementary submodels and to refine our concept of scale. Namely, when we build the p_n's we also build a definable ω-chain of size $\aleph_\omega$ sufficiently elementary submodels $M_0 \prec M_1 \prec \cdots$ of the universe which are transitive below $\aleph_{\omega+1}$; we ensure that the p_n's are chosen from the M_n's and have heights $|p_n|$ which interleave with the ordinals $\gamma_n = M_n \cap \aleph_{\omega+1}$. The result is that the supremum of the $|p_n|$'s is exactly $\gamma_\omega = M_\omega \cap \aleph_{\omega+1}$, where M_ω is the union of the M_n's. Now how does this help? The point is that we can arrange for p_ω, the limit of the p_n's, to be definable over M_ω and therefore also over its transitive collapse $\overline{M}_\omega$; if we can also arrange our scale so that f_{γ_ω} eventually dominates any function in $\prod_n [\omega_n, \omega_{n+1})$ which is definable over $\overline{M}_\omega$, then p_ω will leave a final segment of the range of f_{γ_ω} untouched, as the sequence $(|p_\omega(\omega_n)| \mid n \in \omega)$ is indeed definable over $\overline{M}_\omega$. Finally, arranging our scale in this way is not a problem, as $\overline{M}_\omega$ is an initial segment of $\mathbf{L}$ which is so short that it still thinks that γ_ω is a cardinal (it is the image of $\aleph_{\omega+1}$ under the transitive collapse of M_ω) and we can define f_ξ to eventually dominate any function in $\prod_n [\omega_n, \omega_{n+1})$ which belongs to a model which still thinks that ξ is a cardinal (f_ξ is defined using Skolem hulls inside some big initial segment which sees that ξ is not a cardinal).

The reason we discussed the fine point above about the coding of a subset of $\aleph_{\omega+1}$ into $\aleph_\omega$ is to note that there is some condensation involved (we needed that $\overline{M}_\omega$ is an initial segment of our hierarchy). This is unproblematic for $\mathbf{L}$ (and even for $\mathbf{L}[U]$ where U is a single normal measure) but is a serious problem for large core models. The use of condensation is even more substantial when looking at $\aleph_{\omega^2}$, where one needs to simultaneously consider transitive collapses of unions of chains of sufficiently elementary submodels of any fixed size $\aleph_{\omega \cdot n}$ and worry about their transitive collapses being initial segments of the hierarchy. Indeed it is this issue with condensation which obstructs a fully general coding result over core models as in Theorem 1. Nearly all of the successes with coding over core models are variants of the weaker Corollary 2.

Now the fact that the strategy to code a subset of $\aleph_{\omega+1}$ into $\aleph_\omega$ fits so nicely with the strategy to code a subset of $\aleph_\omega$ into a real means that we can combine the two codings into a single coding of a subset of $\aleph_{\omega+1}$ into a real. Thus a condition is a function p that for each finite n assigns a pair (s_n, s_n^*) as in the latter coding so that in addition the sequence of s_n's is a condition in the former coding. For later use we change notation slightly: the domain of p consists of 0 together with the ω_n's and for each α in the domain of p, $p(\alpha) = (p_\alpha, p^*_\alpha)$ where p_α is an α^+-Cohen condition on the interval $[\alpha, \alpha^+)$ (0^+ is taken to be ω). And of course the restraint p^*_α is a size at less than α^+ subset of the set of ξ such that $p_{\alpha^+}(\xi)$ is defined with value 1. We also require that p_α codes $A \cap |p_\alpha|$ where the domain of p_α is $[\alpha, |p_\alpha|)$, using its values at odd ordinals. Finally, for some $|p| < \aleph_{\omega+1}$, if ξ is less than $|p|$ then ξ belongs to A iff $p_\alpha(\eta) = 1$ for sufficiently large η in $b_\xi = \mathrm{ran}(f_\xi)$ and when ξ is at least $|p|$, sufficiently large η in b_ξ lie outside the domain of the p_α's.

This ends our introduction to Jensen coding. For arbitrary infinite cardinals α, the coding from a subset of α^{++} into a subset of α^+ is similar to the coding of a subset of ω_1 into a real and for arbitrary singular cardinals α, the coding of a subset of α^+ into a subset of α is similar to the above coding of a subset of $\aleph_{\omega+1}$ into a subset of $\aleph_\omega$. The final case of the coding of a subset of α^+ into a subset of α for inaccessible α uses either *full support* and thereby resembles the singular coding, or uses *Easton support* and thereby resembles the successor coding. In nearly all cases (including [1]) full support is used (it faciliates the preservation of large cardinals); Easton support coding is however needed in [9]. The reason is that in [9], we iterate Jensen coding to a length of κ, at the same time collapsing everything below κ; but we want to preserve κ itself. The usual strategy of "reducing to the lower part" fails below κ, since as we keep coding into ω, there are κ-many lower parts. Instead, a much more complex argument is needed, in which there is no fixed height where we cut into "upper" and "lower part": intuitively, we capture a given name by deciding it in different ways using larger and larger lower parts, catching our tail at an inaccessible below κ, where we shall have looked at all the relevant lower parts. For this to work, supports must be bounded below inaccessibles (we also have to assume κ is Mahlo).

2 One measurable cardinal

Suppose that there is a measurable cardinal κ in V. Can we code V into a real R preserving the measurability of κ?

Of course the model that results after coding into R cannot be $\mathbf{L}[R]$, but it could be $\mathbf{L}[U^R, R]$ where U^R is a normal measure on κ extending a given normal measure U on κ in V. As alluded to above there are serious issues with condensation when coding over core models and for this reason we shall only discuss here how to establish a version of Corollary 2: It is possible to force a real R over $\mathbf{L}[U]$ which preserves cofinalities, is not set-generic over $\mathbf{L}[U]$ and preserves the measurability of κ. Even in this special situation it is very helpful (and essential for further generalisations) to use a hierarchy for $\mathbf{L}[U]$ with good condensation properties, which we write as $\mathbf{L}[E]$. Note that the $\mathbf{L}[U]$-hierarchy does not obey even the weakest of consequences of condensation, the property that subsets of an infinite cardinal α appear in the hierarchy at a stage before α^+. The $\mathbf{L}[E]$ hierarchy inserts "partial measures" which ensure this property and more without altering the model: $\mathbf{L}[E] = \mathbf{L}[U]$. The measure U (or something very close to it) is placed on the $\mathbf{L}[E]$ hierarchy at an appropriate stage between κ^+ and κ^{++}, its *index* on the $\mathbf{L}[E]$-hierarchy, and there will be many approximations to it placed on the hierarchy at indices cofinal in any uncountable cardinal up to and including κ^+.

So proceed now to form conditions p in $\mathbf{L}[E]$ which resemble the coding conditions from Jensen coding: For α either 0 or an infinite cardinal, $p(\alpha)$ is a pair (p_α, p^*_α) where p_α is an α^+-Cohen condition on $[\alpha, \alpha^+)$ and p^*_α is a size at most α set of ξ such that $p_{\alpha^+}(\xi) = 1$. Also for limit cardinals λ we have a scale $(f_\xi \mid \xi \in [\lambda, \lambda^+))$ of functions in $\prod_{\alpha^+<\lambda}[\alpha^+, \alpha^{++})$ and for $\xi < |p_\lambda|$, $p_\lambda(\xi) = 1$ iff $p_{\alpha^+}(f_\xi(\alpha^+)) = 1$ for sufficiently large $\alpha^+ < \lambda$. And $p \restriction \lambda$ does not interfere with future coding on $[|p_\lambda|, \lambda^+)$ in the sense that for $\xi \geq |p_\lambda|$, $p_{\alpha^+}(f_\xi(\alpha^+))$ is not defined for sufficiently large $\alpha^+ < \lambda$. The previous applies both to inaccessible and singular limit cardinals λ.

Now we need a strategy for showing that this forcing preserves the measurability of κ. It is best to think of measurability in terms of embeddings: In the ground model $V = \mathbf{L}[U] = \mathbf{L}[E]$ there is an elementary embedding $j : V \to M = \mathrm{Ult}_U$ with critical point κ, derived from the ultrapower given by U. The hierarchy provided by E is defined so that we have $j : \mathbf{L}[E] \to \mathbf{L}[E^*]$ where E, E^* agree up to the index of U (an ordinal between κ^+ and κ^{++}); for the present discussion we only need to know that this agreement persists at least up to the κ^{++} of M, the ultrapower of V by U. This has the important consequence that our coding forcing P agrees with $P^* = j(P)$, the coding forcing of the ultrapower M, up to the κ^{++} of M. More precisely, a function p defined at 0 together with the infinite cardinals $\leq \kappa^+$ such that $p(\alpha) = (p_\alpha, p^*_\alpha)$ for each α and $p^*_{\kappa^+} = \varnothing$ belongs to P^* iff it belongs to P, $|p_{\kappa^+}|$ is less $(\kappa^{++})^M$ and p^*_κ is a subset of $(\kappa^{++})^M$.[2]

Now Silver taught us that if we want to preserve the measurability of κ we should lift the embedding $j : V \to M$ to an embedding $j^* : V[G] \to M[G^*]$ where G^* is generic over M for $P^* = j(P)$. The key is to choose G^* to contain the pointwise image $j[G]$ of G as a subclass. There are many examples of such liftings in the context of reverse Easton forcing, where there are typically many choices for G^*. But notice that with coding there is only one candidate for G^*, the P^*-generic coded into the same real R

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[2] This may not be entirely clear, as V has more subsets of κ^+ than M. However the coding is defined so that p_{κ^+} will belong to M provided its length is less than the κ^{++} of M.

that codes G. This is because $j^*(R)$ will equal R for any possible lifting j^* of j to $V[G]$.

Of course our desired generic G^* must include the image $j(p)$ of any condition p in G; it would be ideal if G^* were simply generated by these conditions in the sense that G^* is obtained as the class of all conditions extended by a condition in $j[G]$. This will however not be the case and it is instructive to see why not.

For G^* to be generic it must intersect all $\mathbf{L}[E^*]$-definable dense classes D on the forcing P^*. As $\mathbf{L}[E^*]$ is the ultrapower of $\mathbf{L}[E]$ by the measure U we can write D as $j(f)(\kappa)$ for some definable function f with domain κ in $\mathbf{L}[E]$ so that $f(\alpha)$ is dense on P for each α. Now our coding forcing P satisfies the following useful form of "diagonal distributivity": We say that a subclass D of P is γ-*dense* for a cardinal γ if any condition in P can be extended into D without changing its values below γ. Now suppose that $f(\alpha)$ is α^+-dense for each cardinal $\alpha < \kappa$ and p is a condition. Then p has an extension q which meets (i.e., extends an element of) each $f(\alpha)$. It follows that some condition p in G meets each $f(\alpha)$ and therefore on the ultrapower side, $j(p)$ will meet $j(f)(\kappa) = D$ provided D is κ^+-dense on P^*. In particular this means that the $j(p)$ for p in G will indeed provide us with a generic for the forcing P^* above κ^+, i.e., a generic subset of the κ^{++} of $\mathbf{L}[E]$ that in turn codes an entire generic class for the forcing P^* above κ^+. As the embedding j is the identity below κ, $j[G]$ also provides us with a generic below κ and indeed a generic subset G_κ of κ^+, as this is coded in both $\mathbf{L}[E]$ and $\mathbf{L}[E^*]$ into the generic below κ in the same way.

So $j[G]$ in fact gives us a subset $G^*_{\kappa^+}$ of $(\kappa^{++})^M$ which codes an entire P^*-generic above $(\kappa^{++})^M$, as well as a subset $G_\kappa = G^*_\kappa$ of κ^+ which is generically coded (in both the P and P^* forcings) into a real; what is missing is to ensure that G_κ, which generically codes G_{κ^+} over $V[G_{\kappa^+}]$, also generically codes $G^*_{\kappa^+}$ over $M[G^*_{\kappa^+}]$. We have to fit the "ultrapower coding" of $G^*_{\kappa^+}$ into G_κ together with the "V-coding" of G_{κ^+} into G_κ, in order to produce the desired P^*-generic G^*.

It is tempting now to make use of the fact that $V = \mathbf{L}[E]$ and $M = \mathbf{L}[E^*]$ actually agree up to $(\kappa^{++})^M$ in the sense that the hierarchies given by E and E^* are the same up to that point. Indeed it is natural to expect that G_κ will generically code $G^*_{\kappa^+}$ using $E \restriction (\kappa^{++})^M$, since it generically codes G_{κ^+} using E and $E^* \restriction (\kappa^{++})^M$ is an initial segment of E. This is encouraging, however it leads to a contradiction, as what G_κ codes below the ordinal $(\kappa^{++})^M$ using E is G_{κ^+} restricted to this ordinal, an element of V, whereas what we want G_κ to code over M, namely $G^*_{\kappa^+}$, cannot be an element of V (else both $G^*_{j(\kappa)}$ and its preimage G_κ would belong to V, reducing our class-forcing to a set-forcing).

Thus we need a different approach, in which the codings over $\mathbf{L}[E]$ and $\mathbf{L}[E^*]$ do not agree at κ^+, in the sense that the generic subset G_κ of κ^+ codes the generic subset G_{κ^+} of κ^{++} using E in a way which accomodates, but differs from, the way it codes the generic subset $G^*_{\kappa^+}$ using E^*. The solution is this: When defining conditions $p(\kappa) = (p_\kappa, p^*_\kappa)$ to almost disjoint code $p_{\kappa^+} : [\kappa^+, |p_{\kappa^+}|) \to 2$ we use sets b_ξ for $\xi < \kappa^{++}$ as before to ensure that $p_{\kappa^+}(\xi) = 1$ iff $p_\kappa(\delta) = 1$ for sufficiently large $\delta \in b_\xi$; however we additionally have sets b^*_ξ for $\xi < (\kappa^{++})^M$ to ensure that for $\xi < (\kappa^{++})^M$, $j(p)_{\kappa^+}(\xi) = 1$ iff $p_\kappa(\delta) = 1$ for sufficiently large $\delta \in b^*_\xi$. Thus there are two codings taking place simultaneously, one of p_{κ^+} and the other of $j(p)_{\kappa^+}$, with two different forms of restraint. To avoid conflicts between these codings we choose the b_ξ's to be very "thin" making use of the measure U. We choose a scale $(f_\xi \mid \xi \in [\kappa^+, \kappa^{++}))$ of functions from κ^+ to κ^+ so that the least function f_{κ^+} of this scale eventually dominates all functions from κ^+ to κ^+ in $M = \mathbf{L}[E^*]$; this is possible as there are only κ^+-many such functions in M. The net effect is that the resulting subset G_κ of κ^+ which is generic over $\mathbf{L}[E]$ will also be generic over $\mathbf{L}[E^*]$, as the thinness of the sets b_ξ allows us to show that conditions can be extended to meet the necessary dense sets from the $\mathbf{L}[E^*]$ coding without conflicting with the restraint imposed by the b_ξ for ξ in $p^*_{\kappa^+}$.

3 Measures of higher order

Suppose now that we are in a *Mitchell model* $\mathbf{L}[E]$ where we now have two normal measures U_0, U_1 on κ with U_0 below U_1 in the Mitchell order. Thus U_0 belongs to the ultrapower of V by the measure U_1. Can we create a real which is class-generic but not set-generic lifting both of the measures U_0 and U_1?

It is convenient to reformulate the situation of the last section (with a single measure U) as follows. Recall that at κ^+ we have two codings, that of $j(p)_{\kappa^+}$ into $G_\kappa \subseteq \kappa^+$ over the ultrapower Ult_U of V by U, and the other of p_{κ^+} into G_κ over V. As the latter coding takes place "above" the former (ultrapower) coding, it is natural to think of the κ^{++}-Cohen condition $p_{\kappa^+} : [\kappa^+, |p_{\kappa^+}|) \to 2$ in two parts: there is p_{κ^+} on $[\kappa^+, |j(p)_{\kappa^+}|)$ coinciding with $j(p)_{\kappa^+}$ and then p_{κ^+} on $[(\kappa^{++})^{\mathrm{Ult}_U}, |p_{\kappa^+}|)$, which is coded using the b_ξ's which "lie above" the ultrapower Ult_U. In this way there is in a sense just one coding, which uses restraints from Ult_U below the κ^{++} of Ult_U and restraints from V between the κ^{++} of Ult_U and the real κ^{++}. In fact $p \restriction \kappa$ is responsible for the coding below κ^{++} of Ult_U (via the embedding j) and p_{κ^+} is responsible for the coding above. But notice that viewed this way, the domain of p_{κ^+} is no longer an interval, but the union of two intervals, namely $[\kappa^+, |j(p)_{\kappa^+}|)$ and $[(\kappa^{++})^{\mathrm{Ult}_U}, |p_{\kappa^+}|)$. So p_{κ^+} is what one might call a "perforated string".

Now let us return to the more complex case of two measures U_0, U_1. At κ^+ the strings are doubly-perforated, since their domains consist of the union of three intervals: $[\kappa^+, |j_{U_0}(p)_{\kappa^+}|)$, $[(\kappa^{++})^{\mathrm{Ult}_{U_0}}, |j_{U_1}(p)_{\kappa^+}|)$ and $[(\kappa^{++})^{\mathrm{Ult}_{U_1}}, |p_{\kappa^+}|)$. For cardinals $\overline{\kappa}$ of Mitchell order 0 (i.e., carrying only normal measures concentrating on non-measurables), strings at $\overline{\kappa}^+$ will only be singly-perforated and at non-measurables we return to non-perforated strings. The situation is similar, but more complicated, when dealing with measurable κ of Mitchell order less than κ^{++} (the "real" coding takes place above the supremum of the $(\kappa^{++})^{\mathrm{Ult}_U}$ for U a normal measure on κ on the $\mathbf{L}[E]$ hierarchy).

But if we go as far as $\mathrm{o}(\kappa) = \kappa^{++}$, where $(\kappa^{++})^U$ can be arbitrarily large in κ^{++} for measures U on κ, and wish to lift all of these measures, then we have a problem, as it seems that there is no longer room to code, as the entire interval $[\kappa^+, \kappa^{++})$ has been covered with ultrapower codings which must be respected. In fact, we cannot expect to add a class-generic real which is not set-generic but lifts all extenders, partial and total, when $\mathrm{o}(\kappa) = \kappa^{+++}$:

To see this, work with $K = \mathbf{L}[E]$ and assume that $\mathrm{o}(\kappa) = \kappa^{+++}$, κ is the largest measurable cardinal, but $\mathbf{L}[E]$ is also closed under sharps. Let $\Gamma(\lambda)$ denote the theory "ZF^-, λ is the largest measurable cardinal, λ^{+++} exists, and $\mathrm{o}(\lambda) = \lambda^{+++}$," so that in particular $K \models \Gamma(\kappa)$. If M is a premouse, $M \models \Gamma(\lambda)$, and $M \models S = \{\xi < \lambda^{+++} \mid \mathrm{cf}(\xi) = \lambda^{++}\}$, then we may canonically and uniformly split S into two sets which are both stationary from the point of view of M; say $S = S^0_M \cup S^1_M$, where $M \models$ "S^0_M, S^1_M are both stationary," and S^0_M, S^1_M are both Δ_1-definable over $M|\lambda^{+++M}$ via defining formulas which are independent of M. For $i \in \{0,1\}$ we shall also write $T^i_M = S^i_M \cup \{\xi < \lambda^{+++M} \mid \mathrm{cf}^M(\xi) < \kappa^{++M}\}$ and $T^i = T^i_K$.

Assume now that for $i \in \{0,1\}$ there is a forcing P^i in K such that in any P^i-generic extension there exists a real r^i with the following property:

> For all ξ such that $K|\xi \models \Gamma(\lambda)$, there is a club through $T^i_{K|\xi}$ in $K|\xi[r^i]$. Moreover every (partial or total) extender on E lifts to $K[r^i]$. (∗)

By the last sentence we mean that every iteration tree on K can be lifted to one on $K[r]$. Now force with $P = P^0 \times P^1$, obtaining reals r^0, r^1 as above (note that P collapses κ^{+++}, but this is irrelevant), and let g be generic for the collapse of κ^{++K} to ω over $K[r^0, r^1]$. By an unpublished result of Woodin (cf. [2]), $K[g]$ is Σ^1_4-correct in $K[g, r^i]$; this makes vital use of the fact that enough extenders from K lift to $K[r^i]$ (cofinally many total extenders suffice here). The following Π^1_3 statement $\Psi^i(r^i)$ holds in $K[g, r^i]$:

> Every countable mouse M_0 such that $M_0 \models \Gamma(\lambda)$ has a simple countable iterate M_1 such that r^i lifts all extenders on the M_1-sequence, and if M_0 is a countable mouse such that $M_0 \models \Gamma(\lambda)$ and r^i lifts all extenders on the M_0-sequence, then there is a club through $T^i_{M_0}$ in $M_0[r^i]$. $(**)$

That this statement is Π^1_3 boils down to the fact that being a mouse, in our setting, is Π^1_2. By a simple iterate we mean one without any drops. To see that the first part of $(**)$ holds of r^i, let M_0 be a countable mouse. Co-iterate M_0 with K until you reach $M_1 \triangleleft K'$, where K' is an iterate of K. Since r^i lifts all extenders on E, we can push forward $(*)$ to K' in the sense that $(*)$ holds with initial segments of K replaced by those of K'. If M_1 is not countable, then by taking a countable hull we shall obtain a *countable* iterate of M_0 which is as desired. To see the second part of $(**)$, we may argue in a similar fashion, this time co-iterating $M_0[r^i]$ with $K[r^i]$.

By Σ^1_4-correctness, we find s^0, s^1 in $K[g]$ such that $K[g] \models \Psi^0(s^0) \wedge \Psi^1(s^1)$. We claim the following:[3]

> For $i \in \{0,1\}$, K has a simple iterate W such that s^i lifts all extenders on the W-sequence and there is a club through T^i_W in $W[s^i]$. $(* * *)$

If $(***)$ is false for some $K|\vartheta \models \Gamma(\lambda)$, then we may pick some cardinal $\Omega > \vartheta$ and some $\sigma \colon \overline{K}[s^i] \to K|\Omega[s^i]$ with $\vartheta \in \operatorname{ran}(\sigma)$ such that $\overline{K}[s^i] \in K[g]$ is transitive and countable in $K[g]$. Write $\overline{\vartheta} = \sigma^{-1}(\vartheta)$, and let $h \in K[g]$ be $\operatorname{Col}(\omega, \overline{\vartheta})$-generic over $\overline{K}[s^i]$. By our hypothesis that $\mathbf{L}[E]$ be closed under sharps, $\overline{K}[s^i][h]$ will be Σ^1_2-correct in $K[g]$. This means that if we look at the family $\mathcal{F}$ of all $M \in \overline{K}[s^i][h]$ which in $\overline{K}[s^i][h]$ are countable simple iterates of $\overline{K}|\overline{\vartheta}$, then by $(**)$ densely many $M \in \mathcal{F}$ will be such that r^i lifts all extenders on the M-sequence and there is a club through T^i_M in $M[r^i]$. But then if $\tilde{M}$ is the direct limit of all mice in $\mathcal{F}$, then $\tilde{M} \in \overline{K}[s^i]$ by the homogeneity of $\operatorname{Col}(\omega, \overline{\vartheta})$, r^i lifts all extenders on the $\tilde{M}$-sequence, and there is a club through $T^i_{\tilde{M}}$ in $\tilde{M}[r^i]$. Moreover, in $\overline{K}[s^i]$, $\tilde{M}$ can be absorbed by a simple iterate of $\overline{K}|\overline{\vartheta}$ which has the same properties. But the elementarity of σ then yields a contradiction. We have verified $(* * *)$.

We may now use $(* * *)$ to find a simple iterate W of K such that for both $i = 0$ and $i = 1$, s^i lifts all extenders on the W-sequence and there is a club through T^i_W. Let $D^i \in W[s^i]$ be a club through T^i_W, and let $j \colon K \to W$ be the iteration map. Let D denote the limit points of

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[3] We wish to thank the referee for pointing out a gap in our earlier presentation of this argument.

$j^{-1}[D^0 \cap D^1]$. Obviously, D is club in κ^{+++K} and $D \in K[g]$. Let $S \in K$ be such that $K \models S = \{\xi < \kappa^{+++} \mid \mathrm{cf}(\xi) = \kappa^{++}\}$. As S remains stationary in $K[g]$, we can find $\eta \in S \cap D$. Since j is continuous at points whose K-cofinality is bigger than κ, we have $j(\eta) \in D^0 \cap D^1$, and by elementarity $j(\eta) \in j(S)$. This is a contradiction, since $j(S) \cap D^0 \cap D^1 = \varnothing$. This finishes off the argument that no real r^i as in $(*)$ can exist.

We now argue that the problem with $(*)$ lies not in its first sentence, the coding part, but in its second sentence, the lifting of extenders. Consider the following weakening of $(*)$:

> For all ξ, *if* $K|\xi \models$ "λ is the greatest measurable and $\mathrm{o}(\lambda) = \lambda^{+++}$" and $K|\xi[r^i] \models$ "ZF^- and $\lambda^{+++} = (\lambda^{+++})^{K|\xi}$", then there is a club through $(T^i_\lambda)^{K|\xi}$ in $K|\xi[r^i]$.

We can produce a real r^i satisfying this by first shooting a club through T^i and then forcing to code it with localization (using a core model analogue of David's trick; cf. [5, Theorem 6.18]). The club is added by a κ^{+++} distributive forcing (a forcing adding no new κ^{++}-sequences) of K; then, the condensation provided by K suffices for the distributivity of the second forcing, as we can take Skolem hulls in K (as in [7]). In fact, if we weaken $(*)$ by just dropping the last requirement (i.e., that all extenders lift), we obtain a statement which should be forceable using a core model analogue of *strong coding* (cf. [3]). For these reasons we believe that the problem with $(*)$ lies with its second assertion, that all extenders lift.

In fact we conjecture that it is not possible to add a class-generic real which is not set-generic while lifting all normal measures on a measurable κ of order κ^{++}; in fact we conjecture that it is not even possible to do this while lifting all normal measures in a "cofinal" collection $\mathcal{S} \in \mathbf{L}[E]$ of such measures. (By "cofinal" we mean that the ordinals $(\kappa^{++})^{\mathrm{Ult}_U}$ for U in $\mathcal{S}$ are cofinal in κ^{++}.) But this does *not* mean that we cannot preserve the property $\mathrm{o}(\kappa) = \kappa^{++}$! As we shall see in the next section, it is possible to preserve cases of hypermeasurability, which in turn implies that the set of normal measures that are lifted is cofinal; it is not however clear that this set can contain a cofinal subset in the ground model.

4 Hypermeasurables

Can we preserve stronger forms of measurability? Suppose that κ is hypermeasurable in $V = \mathbf{L}[E]$ in the sense that some total extender F on the E-sequence with critical point κ witnesses that κ is $\mathbf{H}(\kappa^{++})$-strong, i.e., the ultrapower $j_F : V \to M$ has the property that $\mathbf{H}(\kappa^{++})$ is contained in M. (This is the same as saying that F is indexed past κ^{++} in the E-hierarchy.) Can we add a real which is class-generic but not set-generic and lifts F?

Again we want to set up our conditions so that the embedding $j = j_F$ can be lifted to $V[G]$. This time we have that the union of the $j(p)_{\kappa^{++}}$ is not in V yet like $p_{\kappa^{++}}$ must be coded into the same subset G_{κ^+} of κ^{++}. As in the one measure case this can be resolved by starting the coding of $p_{\kappa^{++}}$ above $(\kappa^{+++})^{\mathrm{Ult}_F}$, below which the former coding takes place. But we have a new problem: the union of the $j(p)_{\kappa^+}$ would appear to not belong to V and as V and Ult_F completely agree below κ^{++}, the set G_{κ^+} will code it in exactly the same way as it codes $G_{\kappa^{++}}$. This is a serious obstacle and the only way around it is to thin out the coding conditions to ensure that in fact the $j(p)_{\kappa^+}$ will be empty for each condition p.

To ensure the latter we require that for any condition p there is a closed unbounded subset C of κ such that for inaccessible α in C, p_{α^+} is the empty string. This ensures that $j(p)_{\kappa^+}$ will also be the empty string. The price one pays for this is that we only have a weaker form of diagonal distributivity: If $f(\alpha)$ is α^{++}-dense for each cardinal $\alpha < \kappa$ then any condition can be extended to meet each $f(\alpha)$. This only ensures that the pointwise image $j[G]$ will generate a generic over the ultrapower Ult_F above κ^{++}, coded into the subset $G^*_{\kappa^{++}}$ of $(\kappa^{+++})^M$ consisting of the union of the $j(p)_{\kappa^{++}}$ for p in G. G provides a generic below κ^{++} and now the task is to ensure that G_{κ^+} will code not only $G_{\kappa^{++}}$ but also $G^*_{\kappa^{++}}$. This is dealt with as in the one measure case, by starting the former coding "above" the latter, making use of an appropriate scale.

For a stronger total extender (of successor cardinal strength) the pattern is similar: Thin out the conditions to guarantee that $j(p)_\alpha$ is empty for cardinals α strictly between κ and the strength of the total extender. At the strength there are two codings which must be performed simultaneously, one over V and the other over the ultrapower. Conflicts between these codings are avoided by allowing the V-coding to make use of the total extender F when defining the coding sets b_ξ via an appropriate scale.

The above ideas are sufficient to lift a class $\mathcal{S}$ of total extenders (each of successor cardinal strength) which is *bounded* (the set of $(\alpha^+)^{\mathrm{Ult}_F}$ for total extenders F in $\mathcal{S}$ of strength exactly α is bounded in α^+ for each cardinal α) and *uniform (or coherent)* (if F belongs to $\mathcal{S}$ then $j_F(\mathcal{S})$ agrees with $\mathcal{S}$ below the index of F in the $\mathbf{L}[E]$-hierarchy), provided that in $\mathbf{L}[E]$ no inaccessible α is the stationary limit of cardinals which are strong up to α. This yields a version of Corollary 2 up to the level of a proper class of strong cardinals, but handling a stationary-limit of strong cardinals requires new ideas.

5 Woodin cardinals

As coding makes heavy use of condensation it is only reasonable to consider ground models for which a suitable core model theory is available, currently up to the level of Woodin cardinals. Recall that δ is *Woodin* if for each $A \subseteq \delta$ there is a $\kappa < \delta$ which is *A-strong in δ*, i.e., the critical point of embeddings $j : V \to M$ such that $j(A)$ agrees with A up to γ, for each $\gamma < \delta$. At first it appears that this indicates the end of the coding method, as Woodin proved the following (cf. [6] and [11, Theorem 7.14]): If $\mathcal{S}$ is a set of total extenders in V sufficient to witness Woodinness in this sense and R is a real such that each total extender in $\mathcal{S}$ lifts to $V[R]$, then in fact R is generic over V for a (δ-c.c.) forcing of size δ. So there appears to be no version of Corollary 2 in the context of a Woodin cardinal.

But actually there is another definition of Woodin cardinal with a different notion of witness: δ is Woodin if for each $f : \delta \to \delta$ there is a $\kappa < \delta$ closed under f which is *f-strong*, meaning that some embedding $j : V \to M$ with critical point κ is $j(f)(\kappa)$-strong (i.e., $\mathbf{H}(j(f)(\kappa))$ is contained in M). It is shown in [6] that if δ is Woodin in $V = \mathbf{L}[E]$ then in $\mathbf{L}[E]$ there is a witness T to Woodinness in this latter sense which can be lifted by a non-set-generic real R, thereby preserving the Woodinness of δ. And indeed this can be done simultaneously for all Woodin cardinals in $\mathbf{L}[E]$.

The proof of the latter result is much more involved than in the case of nonstationary limits of strongs. In that simpler setting, one can use the strength function $\alpha \mapsto$ (sup of the strengths of total extenders with critical point α) to thin out the codings uniformly below each inaccessible cardinal. In the Woodin cardinal setting one must instead use a uniform witness $\mathcal{T}$ to the Woodinness of each Woodin cardinal whose total extenders have non-Woodin critical point, and then thin out the codings using functions which witness the failure of these critical points to be Woodin. A major difference from the easier setting is that for total extenders F that are to be lifted and conditions p, it is no longer the case that $j_F(p)$ will be trivial between the critical point and strength of F; instead one must deal with this extra information at a cardinal α between the critical point and strength of F until reaching a condition which "recognises" that each of the finitely-many total extenders in $\mathcal{T}$ overlapping α has non-Woodin critical point; this is essential for showing that this extra information stabilises to a set in V.

6 Future work

The story is far from over regarding coding over core models. In terms of versions of Corollary 2, the current frontier is the preservation of measurable Woodinness, which will need a technique beyond what is sketched above for plain Woodinness. Going back all the way to hypermeasurable cardinals, there remains the difficult problem of condensation, which obstructs a satisfying version of Theorem 1. As mentioned, the special case of coding a generic for a Příkrý product is handled in [7], but this is an extremely special case and it is quite possible that there is a counterexample for the coding of more general predicates while preserving hypermeasurability. And of course it will be worthwhile to look at generalisations to the large cardinal setting of the many applications of Jensen coding (and its iterations), as found in [5, 9]. Finally, can one do something with coding at the level of supercompact cardinals? Of course the core model theory is not yet available there, but there has been considerable progress in showing that many of the nice features of $\mathbf{L}[E]$ models can be forced consistently with the strongest of large cardinal properties (cf., e.g., [8]). Are there coding theorems to be proved over such "pseudo" core models? A positive answer may have very interesting consequences.

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