Remarkable cardinals

Ralf Schindler*

Institut für Mathematische Logik und Grundlagenforschung, Westfälische Wilhelms-Universität Münster, Münster, Germany

E-mail: `rds@wwu.de`

1 Introduction

To a large extent, the scientific work of both Peter Koepke and Philip Welch has always been inspired by Ronald Jensen's fine structure theory and his Covering Lemma for $\mathbf{L}$, the constructible universe. Cf., e.g., their joint papers [5] and [6].

In this paper we aim to play with the theme that large cardinals compatible with "$\mathbf{V}$=$\mathbf{L}$," specifically: remarkable cardinals, allow us to create situations below $\aleph_2$ which above $\aleph_2$ can only occur if Jensen's Covering fails.

Large cardinals are a central tool in set theory. A cardinal κ is called supercompact iff for every λ there is an elementary embedding $j\colon V \to M$ such that M is transitive, κ is the critical point of j, $j(\kappa) > \lambda$, and ${}^{\lambda}M \subset M$. The following elegant characterization of supercompact cardinals is due to Magidor, cf. [7].

Lemma 1.1. Let κ be a cardinal. Then κ is supercompact iff for every cardinal $\lambda > \kappa$ there is some $X \prec \mathbf{H}_\lambda$ such that $\overline{\overline{X}} < \kappa$, $X \cap \kappa \in \kappa$, and there is some cardinal $\bar{\lambda}$ such that X condenses to $\mathbf{H}_{\bar{\lambda}}$, i.e., $X \cong \mathbf{H}_{\bar{\lambda}}$.

It is well-known that supercompact cardinals cannot exist in $\mathbf{L}$, the smallest inner model of set theory. The situation changes if we don't require X as in Lemma 1.1 to exist in $\mathbf{V}$, but rather in the extension of $\mathbf{V}$ obtained by Lévy collapsing κ to become $\aleph_1$:

Definition 1.2. Let κ be a cardinal. Then κ is called *remarkable* iff in $\mathbf{V}^{\mathrm{Col}(\omega,<\kappa)}$, for every cardinal $\lambda > \kappa$ there is some $X \prec (\mathbf{H}_\lambda)^{\mathbf{V}}$ such that $\overline{\overline{X}} = \aleph_0$, $X \cap \kappa \in \kappa$, and there is some $\mathbf{V}$-cardinal $\bar{\lambda}$ such that X condenses to $(\mathbf{H}_{\bar{\lambda}})^{\mathbf{V}}$, i.e., $X \cong (\mathbf{H}_{\bar{\lambda}})^{\mathbf{V}}$.

Notice that if κ is remarkable, then in $\mathbf{V}^{\mathrm{Col}(\omega,<\kappa)}$, for every cardinal $\lambda > \kappa$ the set

$$S_\lambda = \{X \prec (\mathbf{H}_\lambda)^{\mathbf{V}} \colon \overline{\overline{X}} = \aleph_0 \wedge X \cap \kappa \in \kappa \wedge \exists \bar{\lambda} \in \mathrm{Card}^{\mathbf{V}}\, X \cong (\mathbf{H}_{\bar{\lambda}})^{\mathbf{V}}\} \quad (1)$$

is in fact *stationary* in $[(\mathbf{H}_\lambda)^{\mathbf{V}}]^\omega$. To see this, let us fix a cardinal $\lambda > \kappa$, and let us work in $V[G]$, where G is $\mathrm{Col}(\omega, <\kappa)$-generic over $\mathbf{V}$. Set

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$\lambda^* = (2^{<\lambda})^+$, and pick $Y \in S_{\lambda^*}$. Notice that $Y[G] \prec (\mathbf{H}_{\lambda^*})^{\mathbf{V}}[G]$; moreover, $Y[G] \cap (\mathbf{H}_{\lambda^*})^{\mathbf{V}} = Y$, as $\mathrm{Col}(\omega, <\kappa)$ has the κ-c.c. If S_λ is not stationary, then there is some $\mathbf{V}$-cardinal $\lambda' \in Y$ with $2^{<\lambda'} < \lambda^*$ and there is also some club $C \in \wp([(\mathbf{H}_{\lambda'})^{\mathbf{V}}]^\omega) \cap Y[G]$ such that $S_{\lambda'} \cap C = \varnothing$. But then $Y \cap (\mathbf{H}_{\lambda'})^{\mathbf{V}} \in S_{\lambda'} \cap C$. Contradiction!

Remarkable cardinals were introduced by the author in [12], cf. also [11] and [9]. There is a characterization of remarkable cardinals which does not mention forcing and which is in fact taken in [12, Definition 1.1] to be the official definition of remarkability; [12, Lemma 1.6] shows the equivalence of Definition 1.2 and [12, Definition 1.1]. It is shown in [12] that remarkable cardinals relativize down to $\mathbf{L}$, cf. [12, Lemma 1.7]; let us sketch the argument, for the reader's convenience.

Let κ be remarkable, and let $\lambda > \kappa$ be a cardinal. In $\mathbf{V}^{\mathrm{Col}(\omega,<\kappa)}$, there is then some $\mathbf{V}$-cardinal $\bar{\lambda} < \kappa$ and some elementary embedding $\pi\colon (\mathbf{H}_{\bar{\lambda}})^{\mathbf{V}} \to (\mathbf{H}_\lambda)^{\mathbf{V}}$ with critical point $\pi^{-1}(\kappa)$. We have that $\pi{\restriction}\mathbf{L}_{\bar{\lambda}}\colon \mathbf{L}_{\bar{\lambda}} \to \mathbf{L}_\lambda$ is elementary. But $\{\mathbf{L}_{\bar{\lambda}}, \mathbf{L}_\lambda\} \subset L$, and $\mathbf{L}_{\bar{\lambda}}$ is countable in $\mathbf{L}^{\mathrm{Col}(\omega,<\kappa)}$, so that by absoluteness[1] between $\mathbf{L}^{\mathrm{Col}(\omega,<\kappa)}$ and $\mathbf{V}^{\mathrm{Col}(\omega,<\kappa)}$ there is some elementary embedding $\sigma\colon \mathbf{L}_{\bar{\lambda}} \to \mathbf{L}_\lambda$ with $\sigma \in \mathbf{L}^{\mathrm{Col}(\omega,<\kappa)}$. We have verified that κ is remarkable in $\mathbf{L}$.

It is shown in [12, Lemmas 1.2 and 1.4] that consistency-wise remarkable cardinals lie strictly between ineffable and ω-Erdős cardinals. Gitman and Welch, cf. [3, Theorems 4.8 and 4.11], produce better bounds for their strength by showing that they lie strictly between "1-iterable" and "2-iterable" cardinals; one should think about this result in terms of inner model theory, as follows.

Because remarkable cardinals don't yield $0^\#$ (as they relativize down to $\mathbf{L}$), they cannot be used to prove the existence of an ω_1-iterable premouse. (Cf., e.g., [10, Chapter 10] on the relevant notion of a premouse.) However, they yield the existence of premice which are 1-iterable, i.e., whose ultrapower is well-founded, but they don't yield the existence of a premouse which is 2-iterable, i.e., whose second ultrapower is also well-founded.

Remarkable cardinals were not introduced because the author of [12] was full of mischief. Rather, they appear quite naturally; let us state a theorem.

Theorem 1.3. The following theories are equiconsistent:

(i) ZFC + proper forcing cannot change the theory of $\mathbf{L}(\mathbb{R})$.
(ii) ZFC + semi-proper forcing cannot change the theory of $\mathbf{L}(\mathbb{R})$.
(iii) Third order number theory + Harrington's principle ("there is a real x such that every x-admissible ordinal is an $\mathbf{L}$-cardinal").
(iv) ZFC + there is a remarkable cardinal.

[1] There is a tree T in $\mathbf{L}^{\mathrm{Col}(\omega,<\kappa)}$ of height ω searching for some such σ; T is ill-founded in $\mathbf{V}^{\mathrm{Col}(\omega,<\kappa)}$, hence in $\mathbf{L}^{\mathrm{Col}(\omega,<\kappa)}$. Cf., e.g., [12, Lemma 0.1].

We refer the reader to [12, Theorem 2.4 and Lemma 3.5], [9, Theorem 1.1], and [2, Theorem 3.2] for a proof of Theorem 1.3. One may also formulate an anti-coding theorem, cf. [12, Definition 2.5 and Corollary 3.6], which may be shown to be equi-consistent with a remarkable cardinal.

In the next section of this paper, we shall consider the combinatorial heart of the situation represented by (i),(ii), and (iii) of Theorem 1.3; $(\aleph_1)^{\mathbf{V}}$ will have to be a remarkable cardinal in $\mathbf{L}$.

Another use of remarkable cardinals is:

Theorem 1.4. The following theories are equiconsistent:

(i) ZFC + for every cardinal $\lambda > \aleph_2$ the set

$$\{X \prec \mathbf{H}_\lambda \colon \overline{\overline{X}} = \aleph_1 \wedge X \cap \omega_2 \in \omega_2 \wedge \mathrm{otp}(X \cap \lambda) \in \mathrm{Card}^{\mathbf{L}}\}$$

is stationary.

(ii) ZFC + there is a remarkable cardinal.

This theorem is produced as [8, Theorem 4]. We shall discuss the situation of this theorem in the second next section; (i) of Theorem 1.4 is realized by having $(\aleph_2)^{\mathbf{V}}$ be a remarkable cardinal in $\mathbf{L}$.

In the last section of this paper we shall produce a new result, Theorem 4.1, making use of *two* remarkable cardinals. This will involve taking another look at the notion of *subcomplete forcing*, cf. [4].

2 Why $\aleph_2$ is a threshold—turning a remarkable cardinal into $\aleph_1$

Let us write for an $\mathbf{L}$-cardinal $\lambda > (\aleph_1)^{\mathbf{V}}$,[2]

$$S_\lambda(\mathbf{L}) = \{X \prec \mathbf{L}_\lambda \colon \overline{\overline{X}} = \aleph_0 \wedge X \cap \omega_1 \in \omega_1 \wedge \exists \bar{\lambda} \in \mathrm{Card}^{\mathbf{L}}\, X \cong \mathbf{L}_{\bar{\lambda}}\}. \qquad (2)$$

Hence $(\aleph_1)^{\mathbf{V}}$ is remarkable in $\mathbf{L}$ iff $S_\lambda(\mathbf{L})$ is stationary in $\mathbf{L}^{\mathrm{Col}(\omega,<(\aleph_1)^{\mathbf{V}})}$ for every $\mathbf{L}$-cardinal $\lambda > (\aleph_1)^{\mathbf{V}}$. An absoluteness argument as above immediately gives the following sufficient criterion.

Lemma 2.1. Suppose that in $\mathbf{V}$, $S_\lambda(\mathbf{L})$ is stationary for every $\mathbf{L}$-cardinal $\lambda > (\aleph_1)^{\mathbf{V}}$. Then $(\aleph_1)^{\mathbf{V}}$ is remarkable in $\mathbf{L}$.

Let us address the question if $S_\lambda(\mathbf{L})$ can contain a club. The situation for $\lambda \leq (\aleph_2)^{\mathbf{V}}$ differs significantly from the situation for $\lambda > (\aleph_2)^{\mathbf{V}}$.

[2] We here understand that in a model of set theory, $S_\lambda(\mathbf{L})$ is to denote the set given by the right hand side of (2) as computed in that model. The same remark applies to the principles $S^\kappa_\lambda(\mathbf{L})$ and $\tilde{S}_\lambda(\mathbf{L})$ which will be defined later.

Lemma 2.2. Let $\lambda > (\aleph_2)^{\mathbf{V}}$ be an $\mathbf{L}$-cardinal. Then $S_\lambda(\mathbf{L})$ contains a club iff $0^\#$ exists.

Proof. If $0^\#$ exists, we get that if $\pi\colon \mathbf{L}_{\bar\lambda}[0^\#] \to \mathbf{L}_\lambda[0^\#]$ is elementary with $\lambda > \aleph_0$ being an $\mathbf{L}$-cardinal, then $\bar\lambda$ must be an $\mathbf{L}$-cardinal also.

Let us now assume that $\lambda > (\aleph_2)^{\mathbf{V}}$ is an $\mathbf{L}$-cardinal and $0^\#$ does not exist. We want to show that $S_\lambda(\mathbf{L})$ does not contain a club. Let us write $\tau = ((\aleph_2)^{\mathbf{V}})^{+\mathbf{L}} \leq \lambda$. We in fact show that $S_\tau(\mathbf{L})$ does not contain a club.[3] We are going to use Kueker's lemma.

Let $(\mathbf{L}_\tau; \in, \ldots) \in V$ be any model in a countable language with universe $\mathbf{L}_\tau$. By the proof of the Jensen Covering Lemma, cf., e.g., [10, Section 11.2], we may pick some elementary embedding

$$\pi\colon \bar H \to (\mathbf{H}_{\omega_4})^{\mathbf{V}},$$

where $\bar H$ is transitive and of size $\aleph_1$ with $(\mathbf{L}_\tau; \in, \ldots) \in \mathrm{ran}(\pi)$, such that if $\delta = \pi^{-1}((\aleph_2)^{\mathbf{V}})$ is the critical point of π, $\eta = \bar H \cap \mathrm{OR}$, and $\bar\tau = \pi^{-1}(\tau)$, then for every $\mu > \eta$ with $\wp(\delta) \cap \mathbf{L}_\mu \subset \mathbf{L}_\eta$,

$$\mathrm{ult}(\mathbf{L}_\mu; E_{\pi\restriction \mathbf{L}_{\bar\tau}}) \text{ is well–founded.} \tag{3}$$

(Cf. [10, Claim 11.58].) If $\bar\tau$ is an $\mathbf{L}$-cardinal, then $\wp(\delta) \cap L \subset \mathbf{L}_\eta$, so that (3), applied with $\mu = \infty$, yields a non–trivial elementary embedding from $\mathbf{L}$ to $\mathbf{L}$, i.e., the existence of $0^\#$.

Therefore, $\bar\tau$ is not an $\mathbf{L}$-cardinal. Let

$$\sigma\colon H \to (\mathbf{H}_{\omega_2})^{\mathbf{V}}$$

be an elementary embedding such that H is countable and transitive and $\{\bar\tau, \pi^{-1}((\mathbf{L}_\tau; \in, \ldots))\} \subset \mathrm{ran}(\sigma)$. Then $\sigma^{-1}(\bar\tau) = (\pi \circ \sigma)^{-1}(\tau) = \mathrm{otp}(\mathrm{ran}(\pi \circ \sigma) \cap \tau)$ is not an $\mathbf{L}$-cardinal, but[4]

$$\mathrm{ran}(\pi \circ \sigma) \cap \mathbf{L}_\tau \prec (\mathbf{L}_\tau; \in, \ldots).$$

As $(\mathbf{L}_\tau; \in, \ldots)$ was arbitrary, $S_\tau(\mathbf{L})$ does not contain a club. Q.E.D.

If $S_{(\aleph_2)^{\mathbf{V}}}(\mathbf{L})$ contains a club, then $(\aleph_2)^{\mathbf{V}}$ must be inaccessible in $\mathbf{L}$. This is because if $(\mathbf{L}_{(\aleph_2)^{\mathbf{V}}}; \in, \ldots) \in V$ is a model in a countable language with universe $\mathbf{L}_{(\aleph_2)^{\mathbf{V}}}$ and if $(\aleph_2)^{\mathbf{V}} = \varrho^{+\mathbf{L}}$, then there is some η, $\varrho < \eta < (\aleph_2)^{\mathbf{V}}$, such that $\mathbf{L}_\eta \prec (\mathbf{L}_{(\aleph_2)^{\mathbf{V}}}; \in, \ldots)$. But then if $X \prec (\mathbf{H}_{\omega_3})^{\mathbf{V}}$ is such that $\overline{\overline{X}} = \aleph_0$ and $\{\eta, (\mathbf{L}_{(\aleph_2)^{\mathbf{V}}}; \in, \ldots)\} \subset X$, then $X \cap \mathbf{L}_\eta \prec (\mathbf{L}_{(\aleph_2)^{\mathbf{V}}}; \in, \ldots)$, but $\mathrm{otp}(X \cap \eta)$ is not an $\mathbf{L}$-cardinal. The following is now easy to verify.

[3] If $\bar\lambda \leq \lambda$ and $S_{\bar\lambda}(\mathbf{L})$ does not contain a club, then $S_\lambda(\mathbf{L})$ does not contain a club either.

[4] Notice that $(\mathrm{ran}(\sigma) \cap \mathbf{L}_{\bar\tau}) \cup \{\mathbf{L}_{\bar\tau}\} \subset \mathrm{dom}(\pi)$, so that $\pi \circ \sigma \restriction (\mathbf{L}_{\sigma^{-1}(\bar\tau)} \cup \{\mathbf{L}_{\sigma^{-1}(\bar\tau)}\})$ makes sense.

Lemma 2.3. The set $S_{(\aleph_2)^{\mathbf{V}}}(\mathbf{L})$ contains a club iff [$(\aleph_2)^{\mathbf{V}}$ is inaccessible in $\mathbf{L}$ and $S_\lambda(\mathbf{L})$ contains a club for every $\mathbf{L}$-cardinal λ such that $(\aleph_1)^{\mathbf{V}} < \lambda < (\aleph_2)^{\mathbf{V}}$].

In the light of Lemmas 2.1 and 2.3, the large cardinal hypothesis of the next lemma is optimal.

Lemma 2.4. It is consistent, relative to the existence of an inaccessible cardinal κ such that $\mathbf{V}_\kappa \models$ "there is a remarkable cardinal," that $S_{(\aleph_2)^{\mathbf{V}}}(\mathbf{L})$ contains a club. In particular, the fact that $S_{(\aleph_2)^{\mathbf{V}}}(\mathbf{L})$ contains a club does not yield the existence of $0^\#$.

Proof. Let κ be inaccessible in $\mathbf{L}$, and let $\mathbf{L}_\kappa \models$ "μ is remarkable." We aim to produce a generic extension of $\mathbf{L}$ in which $\kappa = \aleph_2$ and $S_\kappa(\mathbf{L})$ contains a club.

Let G be $\mathrm{Col}(\omega, < \mu)$-generic over $\mathbf{L}$, and let H be $\mathrm{Col}(\mu, < \kappa)$-generic over $\mathbf{L}[G]$. In $\mathbf{L}[G]$, $S_\lambda(\mathbf{L})$ is stationary for every cardinal $\lambda > \mu$, $\lambda < \kappa$. As $\mathrm{Col}(\mu, < \kappa)$ is ω-closed, in $V[G, H]$, $S_\lambda(\mathbf{L})$ is still stationary for every $\mathbf{L}$-cardinal $\lambda > \mu$, $\lambda < \kappa$. For the record, $\aleph_1 = \mu$ and $\aleph_2 = \kappa$ in $\mathbf{L}[G, H]$.

Now let, for an $\mathbf{L}$-cardinal λ with $\mu < \lambda < \kappa$, $\mathbb{Q}_\lambda$ be defined in $\mathbf{L}[G, H]$ to be the set of all strictly increasing and continuous sequences $(X_i : i \le \alpha)$ of length some $\alpha < \mu$ consisting of elements of $S_\lambda(\mathbf{L})$. We order $\mathbb{Q}_\lambda$ by end-extension. Hence $\mathbb{Q}_\lambda$ shoots a club through $S_\lambda(\mathbf{L})$.

Let us work in $\mathbf{L}[G, H]$ until further notice. We let $\mathbb{Q}$ be the countable support product of all $\mathbb{Q}_\lambda$ for $\mathbf{L}$-cardinals λ strictly between μ and κ. If $p \in \mathbb{Q}$, then we may write

$$p = \{(X_i^\lambda(p) : i \le \alpha_\lambda(p)) : \lambda \in \mathrm{supp}(p)\}.$$

We claim that $\mathbb{Q}$ is ω-distributive. To this end, let $\vec{D} = (D_n : n < \omega)$ be a sequence of open dense sets, and let $p \in \mathbb{Q}$. Let us pick some $Y \prec \mathbf{H}_{\omega_3}$ such that $\mu \cup \{p, \vec{D}, \mathbb{Q}\} \subset Y$, $Y \cap \kappa \in \kappa$, and Y has size $\aleph_1$. Writing $\gamma = Y \cap \kappa$, γ must then be an $\mathbf{L}$-cardinal. Exploiting the fact that $S_\gamma(\mathbf{L})$ is stationary, we may pick some countable $X \prec \mathbf{H}_{\omega_3}$ such that $\{p, \vec{D}, \mathbb{Q}, Y, \gamma\} \subset X$ and $X \cap \mathbf{L}_\gamma \in S_\gamma(\mathbf{L})$. We have that

$$\{p, \vec{D}, \mathbb{Q}\} \subset X \cap Y \prec Y \prec \mathbf{H}_{\omega_3}.$$

We may therefore build a descending sequence $(p_n : n < \omega)$ of conditions $p_n \in \mathbb{Q}$ such that $p_0 = p$, $\{p_n : n < \omega\} \subset X \cap Y$, $p_{n+1} \in D_n$, and for every $\mathbf{L}$-cardinal $\lambda \in X \cap \gamma$ and every $x \in X \cap \mathbf{L}_\gamma$ there is some $n < \omega$ such that $\lambda \in \mathrm{supp}(p_n)$ and $x \in X_i^\lambda(p_n)$ for some $i \le \alpha_\lambda(p_n)$. Let us write $\alpha = X \cap \omega_1$ and

$$q = \{(X_i^\lambda : i \le \alpha) : \lambda \in X \cap \gamma \text{ is an } \mathbf{L}\text{-cardinal}\},$$

where for every **L**-cardinal $\lambda \in X \cap \gamma$, if $i < \alpha$, then $X_i^\lambda = X_i^\lambda(p_n)$ for some (all) sufficiently large $n < \omega$, and $X_\alpha^\lambda = X \cap \mathbf{L}_\lambda$. It is then straightforward to verify that $q \in \mathbb{Q}$, $q \leq_{\mathbb{Q}} p$, and $q \in D_n$ for all $n < \omega$. We have seen that $\mathbb{Q}$ is ω-distributive.

Using CH, $\overline{\overline{\mathbb{Q}_\lambda}} = \aleph_1$ for every $\lambda < \kappa$. An easy application of the Δ-system Lemma then gives that $\mathbb{Q}$ has the $\aleph_2$-chain condition.

Let us now step outside of $\mathbf{L}[G, H]$, and let K be $\mathbb{Q}$-generic over $\mathbf{L}[G, H]$. In $\mathbf{L}[G, H, K]$, $\mu = \aleph_1$, $\kappa = \aleph_2$, and if λ is an **L**-cardinal strictly between $\aleph_1$ and $\aleph_2$, then $S_\lambda(\mathbf{L})$ contains a club. Q.E.D.

A weaker version of Lemma 2.4 is presented in [1], and the full version of Lemma 2.4 is implicit in [2]. Theorem 1.4 is shown in [11, 9, 2] by further exploiting the arguments of this section.

3 Uncountable substructures—turning a remarkable cardinal into $\aleph_2$

The set $S_\lambda(\mathbf{L})$, as defined in (2), consists of countable substructures of $\mathbf{L}_\lambda$. As in [8], we now aim to talk about uncountable such substructures.

Let us write, for an uncountable regular cardinal κ and an **L**-cardinal $\lambda > \kappa$,

$$S_\lambda^\kappa(\mathbf{L}) = \{X \prec \mathbf{L}_\lambda \colon \overline{\overline{X}} < \kappa \wedge X \cap \kappa \in \kappa \wedge \exists \bar{\lambda} \in \mathrm{Card}^{\mathbf{L}}\ X \cong \mathbf{L}_{\bar{\lambda}}\}. \tag{4}$$

Obviously, $S_\lambda^{(\aleph_1)^{\mathbf{V}}}(\mathbf{L}) = S_\lambda(\mathbf{L})$. The following is easy to verify, cf. [8] and also the proof of Lemma 3.3 below.

Lemma 3.1. Let κ be regular, $\kappa \geq (\aleph_3)^{\mathbf{V}}$, and let $\lambda > \kappa$ be an **L**-cardinal. The following are equivalent:

1. $S_\lambda^\kappa(\mathbf{L})$ is stationary.
2. $S_\lambda^\kappa(\mathbf{L})$ contains a club.
3. $0^\#$ exists.

We shall thus now focus on $S_\lambda^{(\aleph_2)^{\mathbf{V}}}(\mathbf{L})$, which we shall now denote by $\tilde{S}_\lambda(\mathbf{L})$. Compare this with the set from (i) of Theorem 1.4.

Lemma 3.2. Suppose that for every **L**-cardinal $\lambda > (\aleph_2)^{\mathbf{V}}$, $\tilde{S}_\lambda(\mathbf{L}) \neq \varnothing$. Then $(\aleph_2)^{\mathbf{V}}$ is remarkable in **L**.

Proof. Write $\kappa = (\aleph_2)^{\mathbf{V}}$. Let $\lambda > \kappa$ be an **L**-cardinal, and let $X \in \tilde{S}_\lambda(\mathbf{L})$. In particular, $X \cong \mathbf{L}_{\bar{\lambda}}$ for some **L**-cardinal $\bar{\lambda} < \kappa$. We have that $\mathbf{L}_{\bar{\lambda}}$ is countable in $\mathbf{L}^{\mathrm{Col}(\omega,<\kappa)}$, so that by absoluteness between $\mathbf{L}^{\mathrm{Col}(\omega,<\kappa)}$ and $\mathbf{V}^{\mathrm{Col}(\omega,<\kappa)}$ there is some $\sigma \colon \mathbf{L}_{\bar{\lambda}} \to \mathbf{L}_\lambda$ in $\mathbf{L}^{\mathrm{Col}(\omega,<\kappa)}$. The set $\mathrm{ran}(\sigma)$ is then in S_λ as defined in $\mathbf{L}^{\mathrm{Col}(\omega,<\kappa)}$, cf. (1) on p. 299 and Definition 1.2. We have verified that κ is remarkable in **L**. Q.E.D.

Lemma 3.3. Set $\lambda = ((\aleph_2)^{\mathbf{V}})^{+\mathbf{L}}$. If $\tilde{S}_\lambda(\mathbf{L})$ contains a club, then $0^\#$ exists.

Proof. This immediately follows from the first part of the proof of "$\Longrightarrow$" of Lemma 2.2. Q.E.D.

Lemma 3.4. It is consistent, relative to the existence of a remarkable cardinal, that for every $\mathbf{L}$-cardinal $\lambda > (\aleph_2)^{\mathbf{V}}$, $\tilde{S}_\lambda(\mathbf{L})$ be stationary. In particular, the fact that for every $\mathbf{L}$-cardinal $\lambda > (\aleph_2)^{\mathbf{V}}$ the set $\tilde{S}_\lambda(\mathbf{L})$ is stationary does not yield the existence of $0^\#$.

Proof. We give a streamlined version of the proof of [8, Theorem 4]. We shall make use of Jensen's theory of subcomplete forcings, cf. [4] where also an iteration theorem is shown for RCS (revised countable support) iterations of subcomplete forcings.

Let us assume that $\mathbf{L} \models$ "κ is a remarkable cardinal," but there is no $\mathbf{L}$-inaccessible cardinal μ such that $\mathbf{L}_\mu \models$ "there is a remarkable cardinal." Let us perform an RCS iteration of legth κ over $\mathbf{L}$, as follows.

Let $\mathbb{N}$ denote Namba forcing. By [4], $\mathbb{N}$ is subcomplete. At stage $i < \kappa$ of the iteration, we let $\mu(i)$ denote the least $\mathbf{L}$-inaccessible above the current $\aleph_2$, and we force with

$$\mathrm{Col}(\omega_2, \mu(i)) * \mathbb{N} * \mathrm{Col}(\omega_1, (\mu(i))^{+\mathbf{L}}),$$

as defined in the current model. At limit stages $i \leq \kappa$, we take the revised limit. Let us denote by $\mathbb{P}$ the resulting RCS iteration of length κ.

By [4], $\mathbb{P}$ is subcomplete, so that in particular forcing with $\mathbb{P}$ preserves ω_1. Also, $\mathbb{P}$ has the κ-c.c., so that κ will be turned into $\aleph_2$.

Let G be $\mathbb{P}$-generic over $\mathbf{L}$. We claim that in $\mathbf{L}[G]$, $\tilde{S}_\lambda(\mathbf{L})$ is stationary for every ($\mathbf{L}$-)cardinal $\lambda > \kappa$.

Let us fix a regular cardinal $\lambda > \kappa$. As κ is remarkable in $\mathbf{L}$, inside $\mathbf{L}^{\mathrm{Col}(\omega,<\kappa)}$ we may pick some

$$\pi \colon \mathbf{L}_{\bar{\lambda}} \to \mathbf{L}_\lambda \tag{5}$$

with critical point $\pi^{-1}(\kappa)$ such that $\bar{\lambda} < \kappa$ is a cardinal in $\mathbf{L}$; as λ was chosen to be regular, $\bar{\lambda}$ will also be regular in $\mathbf{L}$. Let us write $\bar{\kappa} = \pi^{-1}(\kappa)$. We must have that $\bar{\kappa}$ is inaccessible in $\mathbf{L}$, so that the initial segment of $\mathbb{P}$ given by the first $\bar{\kappa}$ steps of the iteration has the $\bar{\kappa}$-c.c. Hence $\bar{\kappa} = (\aleph_2)^{V[G \restriction \bar{\kappa}]}$. Moreover, as $\mathbb{P}$ has the κ-c.c., $\mathrm{ran}(\pi)[G] \cap \mathbf{L}_\lambda = \mathbf{L}_\lambda$, so that inside $\mathbf{L}[G]^{\mathrm{Col}(\omega,<\kappa)}$, we may actually extend π to an embedding

$$\tilde{\pi} \colon \mathbf{L}_{\bar{\lambda}}[G \restriction \bar{\kappa}] \to \mathbf{L}_\lambda[G].$$

By our smallness hypothesis on $\mathbf{L}$, $\mu(\bar{\kappa}) > \bar{\lambda}$. The first two components of the $\bar{\kappa}$th stage of the iteration $\mathbb{P}$ will therefore add a surjection $f \colon \bar{\kappa} \to \mathbf{L}_{\bar{\lambda}}$

such that $f\restriction\xi \in \mathbf{L}[G\restriction\bar\kappa]$ for all $\xi < \bar\kappa$ and some (Namba) sequence $(\kappa_n : n < \omega)$ which is cofinal in $\bar\kappa$. Writing $X_n = f``\kappa_n$, we therefore have $X_n \in \mathbf{L}[G\restriction\bar\kappa]$ for every $n < \omega$ and $\mathbf{L}_{\bar\lambda} = \bigcup_{n<\omega} X_n$. By replacing X_n with the Skolem hull of X_n inside $\mathbf{L}_{\bar\lambda}$, we may in addition assume that $X_n \prec \mathbf{L}_{\bar\lambda}$ for every $n < \omega$. Of course $(X_n : n < \omega) \in \mathbf{L}[G]$.

Let $n < \omega$. As $\bar\lambda$ is regular in $\mathbf{L}[G\restriction\bar\kappa]$, $f\restriction\kappa_n \in \mathbf{L}_{\bar\lambda}[G\restriction\bar\kappa]$. Then $\pi\restriction X_n \in \mathbf{L}[G]$, as it may be computed in $\mathbf{L}[G]$ via $(\pi\restriction X_n)((f\restriction\kappa_n)(\xi)) = \tilde\pi(f\restriction\kappa_n)(\xi)$ for every $\xi < \kappa_n$. Also, $(\pi\restriction n : n < \omega) \in \mathbf{L}[G]^{\mathrm{Col}(\omega,<\kappa)}$.

Now let $T \in \mathbf{L}[G]$ be the tree of attempts to find a map as in (5), more precisely, let T be the set of all $\sigma \in \mathbf{L}[G]$ for which there is some $n < \omega$ such that $\sigma : X_n \to \mathbf{L}_\lambda$ is elementary, ordered by end-extension. As $(\pi\restriction n : n < \omega) \in \mathbf{L}[G]^{\mathrm{Col}(\omega,<\kappa)}$ and $\pi\restriction X_n \in \mathbf{L}[G]$ for every $n < \omega$, T is ill-founded in $\mathbf{L}[G]^{\mathrm{Col}(\omega,<\kappa)}$ and hence in $\mathbf{L}[G]$. There is therefore in $\mathbf{L}[G]$ a system $(\sigma_n : n < \omega)$ such that for $n < \omega$, $\sigma_n : X_n \to \mathbf{L}_\lambda$ is elementary, and if $n \le m$, then $\sigma_m \supset \sigma_n$. Therefore, in $\mathbf{L}[G]$ we get an elementary embedding

$$\sigma : \mathbf{L}_{\bar\lambda} \to \mathbf{L}_\lambda,$$

where for each $x \in \mathbf{L}_{\bar\lambda}$,

$$\sigma(x) = \sigma_n(x)$$

for some (all) sufficently large $n < \omega$. But then $\mathrm{ran}(\sigma) \in \tilde S_\lambda(\mathbf{L}) \cap \mathbf{L}[G]$.

We have verified that $\tilde S_\lambda(\mathbf{L})$ is stationary in $\mathbf{L}[G]$. Q.E.D.

4 Two remarkable cardinals

We now aim to explore the argument for Lemma 3.4 further by trying to arrange that for each $\mathbf{L}$-cardinal $\lambda > (\aleph_2)^{\mathbf{V}}$, $S_\lambda(\mathbf{L})$ and $\tilde S_\lambda(\mathbf{L})$ are simultaneouly stationary, without $0^\#$. It is worth mentioning that if $\tilde S_\lambda(\mathbf{L})$ is true for every $\mathbf{L}$-cardinal $\lambda > \aleph_2$ and $S_{\bar\lambda}(\mathbf{L})$ is true for every $\mathbf{L}$-cardinal $\bar\lambda$ between $\aleph_1$ and $\aleph_2$, then in fact $S_\lambda(\mathbf{L})$ is true for every $\mathbf{L}$-cardinal λ. This corresponds to the fact that if $\mathbf{V}_\kappa \models$ "μ is remarkable," and κ is remarkable, then μ is remarkable.

By Lemmas 2.1 and 3.2, if both $S_\lambda(\mathbf{L})$ and $\tilde S_\lambda(\mathbf{L})$ are stationary for every $\mathbf{L}$-cardinal λ, then $(\omega_1)^{\mathbf{V}}$ and $(\omega_2)^{\mathbf{V}}$ are both remarkable in $\mathbf{L}$. We shall therefore now work with *two* remarkable cardinals.

Theorem 4.1. It is consistent, relative to the existence of two remarkable cardinals, that for every $\mathbf{L}$-cardinal $\lambda > (\aleph_2)^{\mathbf{V}}$, both $S_\lambda(\mathbf{L})$ and $\tilde S_\lambda(\mathbf{L})$ are stationary, and that $S_{(\aleph_2)^{\mathbf{V}}}(\mathbf{L})$ contains a club.

Proof. Let us assume that in $\mathbf{L}$ there are two remarkable cardinals, $\mu < \kappa$. Let G be $\mathrm{Col}(\omega, < \mu)$-generic over $\mathbf{L}$, so that in $\mathbf{L}[G]$, $S_\lambda(\mathbf{L})$ is stationary for every $\mathbf{L}$-cardinal $\lambda > \mu$. As the remarkability of κ is indestructible by small forcing, κ is still remarkable in $\mathbf{L}[G]$.

Now let $\mathbb{P}$ be the forcing defined in the proof of Lemma 3.4, but defined over $\mathbf{L}[G]$ instead of over $\mathbf{L}$. We may and shall assume that there is no $\mathbf{L}$-inaccessible cardinal $\varrho > \kappa$. Let H be $\mathbb{P}$-generic over $\mathbf{L}[G]$. We shall then have that in $\mathbf{L}[G,H]$, $\aleph_1 = \mu$, $\aleph_2 = \kappa$, and $\tilde{S}_\lambda(\mathbf{L})$ is stationary for every $\mathbf{L}$-cardinal $\lambda > \aleph_2$.

We claim that in $\mathbf{L}[G,H]$, $S_\lambda(\mathbf{L})$ is stationary for every $\mathbf{L}$-cardinal $\lambda > \aleph_1$. We shall make use of the fact that $\mathbb{P}$ is subcomplete. Suppose that there is $p \in \mathbb{P}$ and $\tau \in \mathbf{L}[G]^{\mathbb{P}}$ such that

$$p \Vdash \tau \text{ is a model with universe } \mathbf{L}_\lambda,$$

and also

$$p \Vdash \text{ if } X \prec \tau, \overline{\overline{X}} = \aleph_0, \text{ and } X \cap \omega_1 \in \omega_1, \text{ then } \mathrm{otp}(X \cap \check{\lambda}) \notin \mathrm{Card}^{\mathbf{L}}. \quad (6)$$

Let $\vartheta \gg \max(\lambda, \kappa)$ be a cardinal. Working in $\mathbf{L}[G]$, where $S_{(2^{<\vartheta})^+}(\mathbf{L})$ is stationary, we may pick some

$$Y \prec (\mathbf{H}_{(2^{<\vartheta})^+})^{\mathbf{L}[G]}$$

such that Y is countable, $Y \cap \mu \in \mu$, $\{\mathbb{P}, p, \tau, \lambda\} \subset Y$, and $\mathrm{otp}(Y \cap (2^{<\vartheta})^+) \in \mathrm{Card}^{\mathbf{L}}$. Let

$$\sigma \colon H' \cong Y,$$

where H' is transitive, and write $H = \sigma^{-1}(\mathbf{H}_\vartheta)$, $\bar{\mathbb{P}}, \bar{p}, \bar{\tau}, \bar{\lambda} = \sigma^{-1}(\mathbb{P}, p, \tau, \lambda)$. Because $\mathbb{P}$ is subcomplete (we in fact only need that $\mathbb{P}$ is subproper, cf. [4]), there are k, K, and σ' such that $\bar{p} \in k$, k is $\bar{\mathbb{P}}$-generic over H, K is $\mathbb{P}$-generic over $\mathbf{V}$, $\sigma' \in \mathbf{L}[G,K]$,

$$\sigma' \colon H[k] \to \mathbf{H}_\vartheta[K]$$

is an elementary embedding, and $\sigma'(\bar{\mathbb{P}}, \bar{p}, \bar{\tau}, \bar{\lambda}) = \mathbb{P}, p, \tau, \lambda$. Setting $X = \mathbf{L}_\lambda \cap \mathrm{ran}(\sigma')$, $X \prec \tau^H$ and $\bar{\lambda} = \mathrm{otp}(X \cap \lambda)$ is an $\mathbf{L}$-cardinal. As $p \in H$, this contradicts (6).

Now let $\mathbb{Q}$ be the forcing from the proof of Lemma 2.4 for simultaneously shooting clubs through all $S_\lambda(\mathbf{L})$, where λ is an $\mathbf{L}$-cardinal between μ and κ, and let I be $\mathbb{Q}$-generic over $\mathbf{L}[G,H]$. In $\mathbf{L}[G,H,I]$, $S_\kappa(\mathbf{L})$ will contain a club. It is easy to verify that the proof that $\mathbb{Q}$ is ω-distributive also gives that $\mathbb{Q}$ preserves the stationarity of every $S_\vartheta(\mathbf{L})$, ϑ any $\mathbf{L}$-cardinal above κ. (In the proof of Lemma 2.4, pick $X \subset \mathbf{H}_\vartheta$ with $X \cap \mathbf{L}_\vartheta \in S_\vartheta(\mathbf{L})$, rather than $X \prec \mathbf{H}_{\omega_3}$ and $X \cap \mathbf{L}_\gamma \in S_\gamma(\mathbf{L})$.) Moreover, $\mathbb{Q}$ has the κ-c.c., which implies that every $\tilde{S}_\lambda$, λ any $\mathbf{L}$-cardinal above κ, remains stationary in $\mathbf{L}[G,H,I]$. Therefore, $\mathbf{L}[G,H,I]$ is a model as desired. Q.E.D.

References

[1] Y. Cheng. *Analysis of Martin–Harrington Theorem in Higher Order Arithmetic*. PhD thesis, National University of Singapore, 2012.

[2] Y. Cheng and R. Schindler. Harrington's principle in higher order arithmetic, 2014. Submitted.

[3] V. Gitman and P. D. Welch. Ramsey-like cardinals II. *Journal of Symbolic Logic*, 76(2):541–560, 2011.

[4] R. B. Jensen. Iteration theorems for subcomplete and related forcings. Handwritten notes.

[5] P. Koepke and P. Welch. On the strength of mutual stationarity. In J. Bagaria and S. Todorcevic, editors, *Set theory, Centre de Recerca Matemàtica, Barcelona, 2003–2004*, Trends in Mathematics, pages 309–320. Birkhäuser Verlag, 2006.

[6] P. Koepke and P. D. Welch. Global square and mutual stationarity at the $\aleph_n$. *Annals of Pure and Applied Logic*, 162(10):787–806, 2011.

[7] M. Magidor. On the role of supercompact and extendible cardinals in logic. *Israel Journal of Mathematics*, 10:147–157, 1971.

[8] T. Räsch and R. Schindler. A new condensation principle. *Archive for Mathematical Logic*, 44(2):159–166, 2005.

[9] R. Schindler. Semi-proper forcing, remarkable cardinals, and bounded Martin's maximum. *Mathematical Logic Quarterly*, 50(6):527–532, 2004.

[10] R. Schindler. *Set theory, Exploring independence and truth.* Springer-Verlag, 2014. To appear.

[11] R.-D. Schindler. Proper forcing and remarkable cardinals. *Bulletin of Symbolic Logic*, 6(2):176–184, 2000.

[12] R.-D. Schindler. Proper forcing and remarkable cardinals. II. *Journal of Symbolic Logic*, 66(3):1481–1492, 2001.