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When is a given real generic over L?

Theorem. The following are equivalent for a given real $x \in V$.

(1) x is (set) generic over L.

(2) There is some $p \in L$ such that for all elementary $j: L_\alpha \longrightarrow L_\beta$ with a critical point, say κ, where $j \in L$, $L_\alpha \models ZFC^-$, and $p \in L_\kappa$ ($\subsetneq L_\alpha$), then is some $\hat{j}: L_\alpha[x] \longrightarrow L_\beta[x]$ with $\hat{j} \supset j$ and $L_\alpha[x] \models ZFC^-$.

In order to prove this theorem, we first present a version of Woodin's extender algebra for (partial) extenders which may exist in L.

Let $\delta \geq \aleph_1$ be regular, and let $\mathcal{L}$ be the inf. language with atomic fmlas "$\check{n} \in \dot{a}$" for all $n < \omega$ and closed under negation and disjunction of length $< \delta$. It will be convenient to assume $\delta^{<\delta} = \delta$. If $x \subset \omega$, x not nec. in V, and if φ is a fmla, then we define $x \models \varphi$ in the obvious rec. fashion. If $\Gamma \cup \{\varphi\}$ is a collection of fmlas, then $\Gamma \vdash \varphi$ iff inside $V^{\mathrm{Col}(\omega,\delta)}$, if $x \subset \omega$ and $x \models \psi$ for all $\psi \in \Gamma$, then $x \models \varphi$. Cf. "The long extender algebra."*)

Now let $\mathcal{E}$ be a collection of elementary embeddings $j : M \to N$ with a critical point, say $\kappa = \kappa(j)$ (depending on j), such that M, N are transitive, and $M \models ZFC^-, \bar{\bar{N}} < \delta$. We associate to $\mathcal{E}$ a set $A_{\mathcal{E}}$ of axioms as follows. Whenever $j \in \mathcal{E}$ and $\vec{\varphi} = (\varphi_i : i < \kappa(j)) \in M$ (where $j : M \to N$), then

$$\bigvee j(\vec{\varphi}) \longrightarrow \bigvee \vec{\varphi}$$

is in $A_{\mathcal{E}}$. We write $\mathbb{P}_{\mathcal{E}}$ for the set of all $[\varphi] = \{\psi : A_{\mathcal{E}} \vdash \psi \leftrightarrow \varphi\}$,**) and we construe $\mathbb{P}_{\mathcal{E}}$ as a poset by letting $[\varphi] \leq [\psi]$ iff $A_{\mathcal{E}} \vdash \varphi \to \psi$.

We say that δ is <u>weakly Woodin</u> ~~iff~~ ~~for all~~ as being witnessed by $\mathcal{E}$ iff

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*) wwwmath.uni-muenster.de/logic/Personen/rds/long_extender_algebra.pdf

**) where φ is consistent with $A_{\mathcal{E}}$

for all $A \subset \delta$ there is some $\kappa < \delta$ such that for all $\alpha < \delta$ there is some $j : M \longrightarrow N$ in $\mathcal{E}$ with $\kappa(j) = \kappa$, $A \cap \kappa \in M$, and $j(A \cap \kappa) \cap \alpha = A \cap \alpha$. By taking hulls, it is easy to see that every regular cardinal $\delta \geq \aleph_1$ is weakly Woodin, so "weak Woodinness" is not a large cardinal concept. Cf. below, Lemma 3.

Lemma 1. Let $\mathcal{E}$ witness that δ is weakly Woodin. Then $\mathbb{P}_{\mathcal{E}}$ has the δ-c.c. and is hence a cba.

Proof: Let $\{\varphi_i : i < \delta\}$ be an antichain. Write $A = (\varphi_i : i < \delta)$. We may pick $j : M \to N$ in $\mathcal{E}$ such that the κ^{th} element of the sequence $j(A)$ is equal to φ_κ, where $\kappa = \kappa(j)$. But then $A_{\mathcal{E}} \vdash \varphi_\kappa \longrightarrow \bigvee(\varphi_i : i < \kappa)$, so that $\{\varphi_i : i \leq \kappa\}$ is not an antichain. Contradiction! $\dashv$

Lemma 2. Let $\mathcal{E}$ witness that δ is weakly Woodin. Let $x \subset \omega$, x not nec. in V, and assume that $x \models A_{\mathcal{E}}$. Then x is

$\mathbb{P}_{\mathcal{E}}$ – generic over V.

Proof: Let $G_x = \{ [\varphi] : x \models \varphi \}$. Let $A \subset \mathbb{P}_{\mathcal{E}}$ be an antichain, and suppose that $G_x \cap A = \emptyset$. By Lemma 1, $\overline{\overline{A}} < \delta$, so that $\neg \bigvee A$ is in $\mathcal{L}$. By $G_x \cap A = \emptyset$, $x \models \neg \bigvee A$. By absoluteness then, $A \cup \{ \neg \bigvee A \} \supsetneq A$ is also an antichain. We proved that $G_x \cap A \neq \emptyset$ for all maximal antichains. $\dashv$

<u>Lemma</u> 3. Let $\delta \geq \aleph_1$ be regular. Let $p \in H_\delta$ and let $\mathcal{E}$ be the collection of all $j : M \longrightarrow N$ such that M, N are transitive, $M \models ZFC^-$, $\overline{\overline{N}} < \delta$, and $p \in H^M_{\kappa(j)}$. Then $\mathcal{E}$ witnesses that δ is weakly Woodin.

Proof: Let $A \subset \delta$, and let $X \prec H_{\delta^+}$ with $X \cap \delta \in \delta$, $p \in X$, $\overline{\overline{X}} < \delta$, and $A \in X$. Let $i : M \cong X$ where M is transitive. Write $\kappa = \text{crit}(i) = X \cap \delta < \delta$. Of course, $A \cap \kappa = i^{-1}(A)$.

Let $\alpha < \delta$. Let $Y = \text{Hull}^{H_{\delta^+}}(X \cup \{\kappa\} \cup (\alpha+1)) \prec H_{\delta^+}$, and let $k : N \cong Y$, where N is transitive. Setting $j = k^{-1} \circ i$, $j : M \longrightarrow N$ has critical point κ, and $j(A \cap \kappa) = k^{-1}(A)$ and $k^{-1}(A) \cap \alpha = A \cap \alpha$, as $k \restriction (\alpha+1) = \mathrm{id}$. So $j(A \cap \kappa) \cap \alpha = A \cap \alpha$. $\dashv$

<u>Lemma 4.</u> Let δ, p, $\mathcal{E}$ be as in the statement of Lemma 3. Let $x \subset \omega$, x not nec. in V, be such that for every $j : M \longrightarrow N$ in $\mathcal{E}$ there is some el. $\hat{j} : M[x] \longrightarrow N[x]$ with $\hat{j} \supset j$ and $M[x] \models ZFC^-$. Then x is $\mathbb{P}_{\mathcal{E}}$-generic over V.

Proof: By the previous lemmas, it suffices to show that $x \models A_{\mathcal{E}}$. So let $j : M \rightarrow N$ in $\mathcal{E}$, and let $\vec{\varphi} = (\varphi_i : i < \kappa(j)) \in M$. Let us assume that $x \models W_j(\vec{\varphi})$. Then $M[x] \models$ "$x \models W_j(\vec{\varphi})$", hence by the elementarity of $\hat{j}$, $M[x] \models$ "$x \models W\vec{\varphi}$", and so $x \models W\vec{\varphi}$. $\dashv$

Let us now prove the theorem. (2) $\Rightarrow$ (1) is now clear: Given p, let $\delta \geq \aleph_1$ be regular with $p \in L_\delta$. Let $\mathcal{E}$ be defined in L as in the statement of Lemma 3. Then if $\mathbb{P}_\Sigma$ is defined in L as above, x is $\mathbb{P}_\Sigma$-generic over L.

To show (1) $\Rightarrow$ (2), let $\mathbb{P} \in L$ be such that x is $\mathbb{P}$-generic over L. Write $\mu = \mathrm{Card}^L(\mathbb{P})$ and let $p = L_{\mu^+}$, where $\mu^+ = \mu^{+L}$. Let $j : L_\alpha \to L_\beta$ have critical point κ, $\mu^+ < \kappa$, $L_\alpha \models ZFC^-$. W.l.o.g., $\mathbb{P} \in p_{\not{z}} \in L_\kappa$, so that x is $\mathbb{P}$-generic over L_α. This gives $L_\alpha[x] \models ZFC^-$.

x is also $\mathbb{P}$-generic over L_β, and writing $X = \mathrm{ran}(j)$, $\mathbb{P} \in p \in X$. In order to see that there is $\hat{j} : L_\alpha[x] \to L_\beta[x]$ with $\hat{j} \supset j$ it suffices to verify that $X[x] \cap \beta = X \cap \beta$. *) Let $\tau \in X \cap L^{\mathbb{P}}$ be a name for an ordinal. $B = \{\gamma : \exists p \in \mathbb{P} \; p \Vdash^{\mathbb{P}}_{L_\beta} \tau = \check{\gamma}\} \in X$, and $\mathrm{otp}(B) < \mu^+$. So the order isomorphism $\pi : \mathrm{otp}(B) \cong B$ is in X, and as $\mu^+ < \kappa \subset X$, $\mathrm{otp}(B) \cup \{\mathrm{otp}(B)\} \subset X$, so that $B \subset X$. In particular $\tau^x \in X$.

*) We pretend x is the $\mathbb{P}$-generic filter.

Of course, the previous arguments don't have much to do with L, and after having written the first 6 pages of this note we realized that those arguments simply reprove Bukowský's result which we state as follows.

<u>Theorem</u> (Bukovský) Let $X \subset \mu$, X not necessarily in V, and let δ be a regular uncountable cardinal. The following are equivalent.

(1) There is some $\mathbb{P}$ such that $\mathbb{P}$ has the δ-c.c. and X is $\mathbb{P}$-generic over V.

(2) There is some $\theta > \mu$ and some club C of $Y \prec H_\theta$ with $Y \cap \delta \in \delta$, $\overline{\overline{Y}} < \delta$ such that if $j : M \cong Y$, where M is transitive, then there is some el. $\hat{j} : M[\bar{X}] \longrightarrow H_\theta[X]$ for some $\bar{X}$ and $H_\theta[X] \models ZFC^-$.

(3) If $f : \theta \longrightarrow OR$, some θ, $f \in V[X]$, then there is $g : \theta \longrightarrow \mathcal{P}(OR)$, $g \in V$ such that $\overline{\overline{g(\xi)}} < \delta$ in V and $f(\xi) \in g(\xi)$ for all $\xi < \theta$.

Proof of Bukowsky's Theorem:

(2) $\Rightarrow$ (1). Let $\mathcal{L}$ be the inf. language with atomic fmlae "$\check{\xi} \in \dot{a}$" for $\xi < \mu$ and closed under $\neg$ and $\bigvee$ of length $< \delta$. Let $\mathcal{E}$ be the collection of all $j : M \longrightarrow H_\theta$, when M is transitive and $\operatorname{ran}(j) \in C$ (C as in hypo. (2)). Whenever $j : M \to H_\theta$ is in $\mathcal{E}$ and Γ is a set of $\mathcal{L}$-fmlae such that $\Gamma \in \operatorname{ran}(j)$, then for all $\varphi \in \Gamma$

$$\varphi \longrightarrow \bigvee \Gamma \cap \operatorname{ran}(j)$$

is an axiom of $A_{\mathcal{E}}$. (Notice that $\Gamma \cap \operatorname{ran}(j)$ is a collection of fmlae of size $< \delta$, so this makes sense.) We let $\mathbb{P}_{\mathcal{E}}$ be basically as before.

$\mathbb{P}_{\mathcal{E}}$ has the δ-c.c. This is because if Γ is a set of $\mathcal{L}$-fmlae, $\overline{\overline{\Gamma}} \geq \delta$, then we may pick $j : M \to H_\theta$ from $\mathcal{E}$ such that $\Gamma \in \operatorname{ran}(j)$. But then if $\varphi \in \Gamma \setminus \operatorname{ran}(j)$, $\varphi \longrightarrow \bigvee \Gamma \cap \operatorname{ran}(j)$ and this shows that

Γ is not an antichain.

It can be shown exactly as before (cf. Lemma 2) that if $X \models A_\Sigma$, then X is $\mathbb{P}_\Sigma$-generic over V. However, the hypo. (2) buys us that $X \models A_\Sigma$: this is because if $j : M \longrightarrow H_\theta$ is in Σ and $\Gamma \in \operatorname{ran}(j)$ and $\varphi \in \Gamma$ is such that $X \models \varphi$, then

$H_\theta[X] \models$ "$\exists \varphi' \in \Gamma \; X \models \varphi'$," so

$M[\bar{X}] \models$ "$\exists \varphi' \in j^{-1}(\Gamma) \; \bar{X} \models \varphi'$," where $\bar{X}$ is as in (2), and if φ' is a witness, then

$M[\bar{X}] \models$ "$\varphi' \in j^{-1}(\Gamma) \wedge \bar{X} \models \varphi'$," so

$H_\theta[X] \models$ "$j(\varphi') \in \Gamma \wedge X \models j(\varphi')$", hence

$H_\theta[X] \models$ "$X \models \bigvee \Gamma \cap \operatorname{ran}(j)$," i.e.

$X \models \bigvee \Gamma \cap \operatorname{ran}(j)$.

We have verified (2) $\Rightarrow$ (1).

It is easy to see that (3) $\Rightarrow$ (2). Let $\theta > \mu$ be suff. big, and let $h \in V[X]$ be a function such that for all $A \subset H_\theta$ there is some $Y \prec H_\theta[X]$ with $h'' A = Y \cap H_\theta$. Using (3), we may then find some $h^* \in V$ such that for all $A \subset H_\theta$ of size $< \delta$ in V, $h'' A \subset h^{*}{}'' A$, so that if $Y \prec H_\theta$ is h^*-closed, $\overline{\overline{Y}} < \delta$, then there is some $Z \prec H_\theta[X]$ with $h^{*}{}'' A = Z \cap H_\theta$. We may then let C be the collection of all h^*-closed $Y \prec H_\theta$ with $Y \cap \delta \in \delta$ and $\overline{\overline{Y}} < \delta$.

(1) $\Rightarrow$ (3) is a standard argument. $\dashv$

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