Sharps and the $\Sigma^1_3$ correctness of $K$

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The purpose of the present paper is to present a new, simple, and purely combinatorial proof of the following result.

**Theorem 0.1 (Steel-Welch 1993, [6, Theorem 4.1])** Let $A \subset \mathbb{R}$ be $\Pi^1_2$. Suppose that there is some sequence $(x_n : n < \omega)$ such that $x_0 \in A$ and for all $n < \omega$, $x_{n+1} = x_n^\#$. Suppose also that there is some $N < \omega$ such that there is no inner model with $N$ strong cardinals. Then $A \cap K \neq \emptyset$.

Here, $K$ denotes the core model; cf. the remark right after Definition 0.2. It is open whether Theorem 0.1 still holds if we replace the second sentence in its statement by “Suppose that there is some $x \in A$ such that $x^\#$ exists.” It is also open whether Theorem 0.1 still holds if we replace the third sentence in its statement by “Suppose also that there is no inner model with a Woodin cardinal, but $K$ exists” (cf. Definition 0.2).

We refer the reader to [4]. The current argument will exploit, among other things, the argument of [4].

**Definition 0.2** Let $\mathcal{A}$ be a transitive model of ZFC. Then by $K^\mathcal{A}$ we denote the model which is recursively constructed inside $\mathcal{A}$ in the manner of [5, §6], if it exists (otherwise we let $K^\mathcal{A}$ undefined). If $K^\mathcal{A} \downarrow$ then we say that $K^\mathcal{A}$ exists. If $K^V \downarrow$ then we write $K = K^V$ and say that $K$ exists.

It is shown in [3] that $K$ exists if $0^\dagger$ doesn’t exist. It is also shown in [5] that $K^M$ exists if $M = V^\mathcal{H}_\Omega$, where $\mathcal{H}$ is a transitive model of “ZFC$^- + \Omega$ is measurable + there is no inner model with a Woodin cardinal” (in this case we’ll sometimes also write $K^\mathcal{H}$ for $K^M$).

We shall prove Theorem 0.1 with the third sentence in its statement being replaced by “Suppose also that $0^\dagger$ doesn’t exist.” We’ll leave a proof of Theorem 0.1 as stated as an exercise to the reader.
Definition 0.3 Let (♣) denote the following assertion. Let \( x \in \mathbb{R} \) be such that \( x^\# \) exists. If \( K^{L[x]} \) and \( K^{L[x^\#]} \) both exist and are coiterable then there is some \( \alpha \in \text{OR} \) such that \( K^{L[x^\#]||\alpha} \) iterates past \( K^{L[x]} \).

Lemma 0.4 Suppose that \( 0^\# \) doesn’t exist. Then (♣) holds.

Proof. Suppose that \( x \in \mathbb{R} \) witnesses the failure of (♣). It is fairly easy too see that then \( K^{L[x^\#]} \models \text{“there is a strong cardinal.”} \) (Cf. the proof of [6, Lemma 3.3].) Let us now assume that \( 0^\# \) doesn’t exist. We aim to derive a contradiction. Let us work in \( L[x^\#] \).

Claim 1. \( \text{cf}(\kappa^+K^{L[x]}) = \omega \) for all \( \kappa \).

Proof. It is true that \( \text{cf}(\kappa^+L[x]) = \omega \) for all \( \kappa \). Let us thus fix some \( K^{L[x]}\)-cardinal \( \kappa \) such that \( \kappa^+K^{L[x]} < \kappa^+L[x] \). Let \( \lambda = \text{Card}^{L[x]}(\kappa) \leq \kappa \). Then \( \lambda \) is neither an \( x \)-indiscernible nor singular in \( L[x] \). Let \( \eta < \lambda \) be the largest \( x \)-indiscernible which is smaller than \( \lambda \).

Let \( \tau_n \) enumerate the Skolem terms of \( L[x] \). The sequence \( (\lambda_n; n < \omega) \), where

\[
\lambda_n = \sup(\{\tau_n(\bar{\xi}); \bar{\xi} < \eta \cap \lambda \} < \lambda)
\]

witnesses that \( \text{cf}(\lambda) = \omega \). But we’ll have that \( \text{cf}^{L[x]}(\kappa^+K^{L[x]}) = \lambda \) by weak covering applied inside \( L[x] \) (cf. [2]).

\( \square \) (Claim 1)

Claim 2. \( K^{L[x]} \) doesn’t move in the comparison with \( K^{L[x^\#]} \).

Proof sketch. The point is that by our assumption \( K^{L[x]} \) absorbs all coiterable set-sized premice which exist in \( L[x^\#] \). Jensen’s argument yielding that below \( 0^\# \) any universal weasel is an iterate of \( K \) then gives this Claim.

\( \square \) (Claim 2)

It is now easy to see that Claims 1 and 2, combined with an application of weak covering applied inside \( L[x^\#] \) (cf. [2]) yields the Lemma.

\( \square \) (Lemma 0.4)

The proof of Lemma 0.4 is certainly more interesting than its result. If we had assumed the existence of \( x^{##} \) then we could have just cited [6, Lemma
We conjecture that (♣) still holds under much weaker assumptions than the non-existence of \(0^\sharp\) (cf. [6, p. 188, Question3]).

We are now going to prove the following result, which will immediately give Theorem 0.1 (the third sentence in its statement being replaced by “Suppose also that \(0^\sharp\) doesn’t exist”) via Lemma 0.4.

We emphasize that Theorem 0.5 is not given by the results of [6]; the proof of [6, Theorem 4.1] which is given in [6] heavily uses universal iterations which are not known to exist significantly above \(0^\sharp\).

**Theorem 0.5** Let \(A \subset \mathbb{R}\) be \(\Pi^1_2\). Suppose that there is some sequence \(\vec{x} = (x_n: n < \omega)\) such that \(x_0 \in A\) and for all \(n < \omega\), \(x_{n+1} = x_n^\#\). Suppose also that there is no inner model with a Woodin cardinal, that \(K^L[\vec{x}]\) exists, and that (♣) holds. Then \(A \cap K^L[\vec{x}] \neq \emptyset\).

**Proof** of Theorem 0.5. Let \(A = \{z \in \mathbb{R}: \Phi(z)\}\) where \(\Phi(-)\) is \(\Pi^1_2\). There is a tree \(T \in K^L[\vec{x}]\) searching for a quadruple \((\vec{y}, \vec{M}, \vec{T}, \vec{\sigma})\) such that the following hold true.

- \(\vec{y} = (y_n: n < \omega) \in \omega^\omega\),
- \(\vec{M} = (M_n: n < \omega)\) such that for all \(n < \omega\) do we have the following:
  - (a) \(M_n = (J_{\alpha_n}[y_n]; \in, y_n, U_n)\) for some \(\alpha_n, U_n\),
  - (b) \(M_{n+1} \models "M_n = y_n^\#"\) (in particular, \(M_{n+1}\) thinks that \(M_n\) is iterable),
  - (c) \(y_{n+1}\) is the master code of \(M_n\),
  - (d) \(M_0 \models \Phi(y_0)\), and
  - (e) setting \(\kappa = \text{crit}(U_{n+1})\), there is an initial segment of \(K^{M_{n+1}}\) which iterates past \(K^{L[\vec{y}]}\), and

- \((\vec{T}, \vec{\sigma})\) witnesses that each individual \(K^{M_n}(n < \omega)\) is iterable (cf. [4]), i.e., \(\vec{T} = (T_n: n < \omega)\), \(\vec{\sigma} = (\sigma_n: n < \omega)\), and for all \(n < \omega\) do we have the following:
  - (a) \(T_n\) is a countable tree of successor length on \(K^{L[\vec{x}]}\), and
  - (b) \(\sigma_n: K^{M_n} \to M^T_\infty||\beta_n\), some \(\beta_n \leq M^T_\infty \cap \text{OR}\), is elementary.

We are now going to prove that

\(\emptyset \neq p[T] = \{y_0: \exists(y_1, y_2, ...) \exists \vec{M} \exists \vec{T} \exists \vec{\sigma}((y_0, y_1, ...), \vec{M}, \vec{T}, \vec{\sigma}) \in [T]\} \subset A\).

We may well leave the verification of \(p[T] \neq \emptyset\) as an exercise to the reader.
Now fix \((\vec{y}, \vec{M}, \vec{T}, \vec{\sigma}) \in T\). Let \(\vec{y} = (y_n : n < \omega)\) and \(\vec{M} = (M_n : n < \omega)\).
Let us prove that \(y_0 \in A\). Let \((^0_n)\) denote the assertion that the \(\alpha^{th}\) iterate of \(M_n\) is well-founded. It clearly suffices to prove the following.

**Main Claim.** For all \(\alpha\), for all \(n\), \((^n_\alpha)\) holds.

**Definition 0.6** Let \(n < \omega\). We write \((M_n^i, \pi_n^{ij} : i \leq j \leq \alpha)\) for the putative iteration of \(M_n\) of length \(\alpha + 1\), if it exists; and if so then for \(i < \alpha\) we write \(\kappa_n^i\) for the critical point of \(\pi_n^0(U_n)\), i.e., of the top extender of \(M_n^i\). We call \(\alpha\) a uniform indiscernible provided that for all \(n < \omega\), the putative iteration \((M_n^i, \pi_n^{ij} : i \leq j \leq \alpha)\) of \(M_n\) of length \(\alpha + 1\) exists and \(\{\kappa_n^i : i < \alpha\}\) is (closed and) unbounded in \(\alpha\).

**Proof** of the Main Claim. We’ll prove the Main Claim by induction on \(\alpha\).

**Case 1.** \(\alpha\) is not a uniform indiscernible.

Let \(n < \omega\). By our case assumption, there are some \(m > n\) and \(\beta < \alpha\) such that \(\alpha \in M_\beta^m\). But \(M_\beta^m \models "M_n = y_n^\#$", so that we may argue inside \(M_\beta^m\) and deduce that the \(\alpha^{th}\) iterate of \(M_n\), viz. \(M_\alpha^n\), is well-founded.

**Case 2.** \(\alpha\) is a uniform indiscernible.

Let \(n < \omega\). Let \(\kappa = \text{crit}(U_{n+1}) = \kappa_0^{n+1}\), and let \(P\) be the proper initial segment of \(K^{M_{n+1}} = K^{L_n[y_{n+1}]}\) which iterates past \(K^{L_n[y_n]}\). Let \((T, \mathcal{U})\) be the coiteration of \(K^{L_n[y_n]}\) with \(K^{L_n[y_{n+1}]}\). \((M_n^\alpha, \pi_n^{ij} : i \leq j < \alpha)\) is the putative iteration of \(M_n\) of length \(\alpha + 1\). Let

\[\sigma : M_{n+1}^{n+1} \to (U_{n+1}, M')\]

i.e., \(\sigma = \pi_n^{01}\) and \(M' = M_{n+1}^1\). Let \(X \in P(\kappa) \cap M_{n+1}^{\mathcal{U}}\). Then \(X = \pi_{\kappa^{n+1}}^{\sigma(\mathcal{U})}(\bar{X})\), some \(i < \kappa\), \(\bar{X}\), and \(\sigma(X) = \pi_{\kappa^{n+1}}^{\sigma(\mathcal{U})}(\bar{X}) = \pi_{\kappa^{n+1}}^{\sigma(\mathcal{U})}(X)\). Therefore, \(\pi_{\kappa^{n+1}}^{\sigma(\mathcal{U})} \upharpoonright \kappa^{+M_{n+1}^{\mathcal{U}}} = \sigma \upharpoonright \kappa^{+M_{n+1}^{\mathcal{U}}}\). The same argument shows that \(\pi_n^{\kappa^{\sigma(\mathcal{U})}} \upharpoonright \kappa^{+M_n^{\mathcal{U}}} = \sigma \upharpoonright \kappa^{+M_n^{\mathcal{U}}}\); we construe \(M_n\) in such a way that \(\text{crit}(U_n)^{+M_n} = M_n \cap \text{OR}\), so that this latter equality means that \(\pi_n^{\kappa^{\sigma(\mathcal{U})}} \upharpoonright M_n^{\kappa \cap \text{OR}} = \sigma \upharpoonright M_n^{\kappa \cap \text{OR}}\). Let us write \(\lambda = \sigma(\kappa) = \kappa^{n+1}\).
Now \((\sigma(T), \sigma(U))\) is the coiteration of \(K_{\lambda}[y_n] \) with \(K_{\lambda}[y_{n+1}]\). We’ll have that \(\kappa^+M^{\sigma(U)}_\infty = \kappa^+M^{\sigma(T)}_\infty \geq \kappa^+K_{\lambda}[y_n] = \kappa^+M^n_\infty\), so that we get that
\[
\pi_{\kappa\lambda}^n \upharpoonright M^n \cap \OR = \pi_{\kappa\infty}^{\sigma(U)} \upharpoonright M^{\kappa}_n \cap \OR.
\]
This now buys us that if we let \((T^*, U^*)\) denote the coiteration of \(K_{\lambda}[y_n]\) with \(K_{\lambda}[y_{n+1}]\) then for typical \(i \leq j < \alpha\) (namely, for all \(i \leq j \in \{\kappa^\beta_{n+1}; \beta < \alpha\}\)) we’ll have that
\[
\pi_{ij}^n \upharpoonright M^n_i \cap \OR = \pi_{ij}^{U^*} \upharpoonright M^n_i \cap \OR.
\]
Moreover, \(U^*\) may be construed as an iteration of \(\mathcal{P}\). As \(\mathcal{P}\) is iterable, we may thus conclude that the \(\alpha\)th iterate of \(M_n\), viz. \(M^\alpha_n\), is well-founded (cf. the argument of [4]).

□ (Main Claim)
□ (Theorem 0.5)

Using [1] it can be verified that Theorem 0.5 still holds if the assumption that \(K_{\lambda}[\vec{x}]\) is being crossed out and the conclusion is being replaced by “Then there is some lightface iterable premouse \(\mathcal{M}\) with \(A \cap \mathcal{M} \neq \emptyset\).

References