Sharps and the Σ^1_3 correctness of K

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The purpose of the present paper is to present a new, simple, and purely combinatorial proof of the following result.

Theorem 0.1 (Steel-Welch 1993, [6, Theorem 4.1]) *Let $A \subset \mathbb{R}$ be Π^1_2. Suppose that there is some sequence $(x_n : n < \omega)$ such that $x_0 \in A$ and for all $n < \omega$, $x_{n+1} = x_n^\#$. Suppose also that there is some $N < \omega$ such that there is no inner model with N strong cardinals. Then $A \cap K \neq \emptyset$.*

Here, K denotes the core model; cf. the remark right after Definition 0.2. It is open whether Theorem 0.1 still holds if we replace the second sentence in its statement by "Suppose that there is some $x \in A$ such that $x^\#$ exists." It is also open whether Theorem 0.1 still holds if we replace the third sentence in its statement by "Suppose also that there is no inner model with a Woodin cardinal, but K exists" (cf. Definition 0.2).

We refer the reader to [4]. The current argument will exploit, among other things, the argument of [4].

Definition 0.2 *Let $\mathcal{A}$ be a transitive model of ZFC. Then by $K^{\mathcal{A}}$ we denote the model which is recursively constructed inside $\mathcal{A}$ in the manner of [5, §6], if it exists (otherwise we let $K^{\mathcal{A}}$ undefined). If $K^{\mathcal{A}} \downarrow$ then we say that $K^{\mathcal{A}}$ exists. If $K^V \downarrow$ then we write $K = K^V$ and say that K exists.*

It is shown in [3] that K exists if $0^¶$ doesn't exist. It is also shown in [5] that K^M exists if $M = V^{\mathcal{H}}_\Omega$, where $\mathcal{H}$ is a transitive model of "$\mathsf{ZFC}^- + \Omega$ is measurable + there is no inner model with a Woodin cardinal" (in this case we'll sometimes also write $K^{\mathcal{H}}$ for K^M).

We shall prove Theorem 0.1 with the third sentence in its statement being replaced by "Suppose also that $0^¶$ doesn't exist." We'll leave a proof of Theorem 0.1 as stated as an exercise to the reader.

Definition 0.3 *Let* (♣) *denote the following assertion. Let* $x \in \mathbb{R}$ *be such that* $x^\#$ *exists. If* $K^{L[x]}$ *and* $K^{L[x^\#]}$ *both exist and are coiterable then there is some* $\alpha \in \mathrm{OR}$ *such that* $K^{L[x^\#]}||\alpha$ *iterates past* $K^{L[x]}$*.*

Lemma 0.4 *Suppose that* $0^\P$ *doesn't exist. Then* (♣) *holds.*

PROOF. Suppose that $x \in \mathbb{R}$ witnesses the failure of (♣). It is fairly easy too see that then $K^{L[x^\#]} \models$ "there is a strong cardinal." (Cf. the proof of [6, Lemma 3.3].) Let us now assume that $0^\P$ doesn't exist. We aim to derive a contradiction. Let us work in $L[x^\#]$.

Claim 1. $\mathrm{cf}(\kappa^{+K^{L[x]}}) = \omega$ for all κ.

PROOF. It is true that $\mathrm{cf}(\kappa^{+L[x]}) = \omega$ for all κ. Let us thus fix some $K^{L[x]}$-cardinal κ such that $\kappa^{+K^{L[x]}} < \kappa^{+L[x]}$. Let $\lambda = \mathrm{Card}^{L[x]}(\kappa) \leq \kappa$. Then λ is neither an x-indiscernible nor singular in $L[x]$. Let $\eta < \lambda$ be the largest x-indiscernible which is smaller than λ.

Let τ_n enumerate the Skolem terms of $L[x]$. The sequence $(\lambda_n : n < \omega)$, where

$$\lambda_n = \sup(\{\tau_n(\vec{\xi}) : \vec{\xi} < \eta\} \cap \lambda) < \lambda$$

witnesses that $\mathrm{cf}(\lambda) = \omega$. But we'll have that $\mathrm{cf}^{L[x]}(\kappa^{+K^{L[x]}}) = \lambda$ by weak covering applied inside $L[x]$ (cf. [2]).

□ (Claim 1)

Claim 2. $K^{L[x]}$ doesn't move in the comparison with $K^{L[x^\#]}$.

PROOF SKETCH. The point is that by our assumption $K^{L[x]}$ absorbs all coiterable set-sized premice which exist in $L[x^\#]$. Jensen's argument yielding that below $0^\P$ any universal weasel is an iterate of K then gives this Claim.

□ (Claim 2)

It is now easy to see that Claims 1 and 2, combined with an application of weak covering applied inside $L[x^\#]$ (cf. [2]) yields the Lemma.

□ (Lemma 0.4)

The proof of Lemma 0.4 is certainly more interesting than its result. If we had assumed the existence of $x^{\#\#}$ then we could have just cited [6, Lemma

3.3]. We conjecture that (♣) still holds under much weaker assuptions than the non-existence of $0^{¶}$ (cf. [6, p. 188, Question3]).

We are now going to prove the following result, which will immediately give Theorem 0.1 (the third sentence in its statement being replaced by "Suppose also that $0^{¶}$ doesn't exist") via Lemma 0.4.

We emphasize that Theorem 0.5 is not given by the results of [6]; the proof of [6, Theorem 4.1] which is given in [6] heavily uses universal iterations which are not known to exist significantly above $0^{¶}$.

Theorem 0.5 *Let $A \subset \mathbb{R}$ be Π^1_2. Suppose that there is some sequence $\vec{x} = (x_n : n < \omega)$ such that $x_0 \in A$ and for all $n < \omega$, $x_{n+1} = x_n^{\#}$. Suppose also that there is no inner model with a Woodin cardinal, that $K^{L[\vec{x}]}$ exists, and that (♣) holds. Then $A \cap K^{L[\vec{x}]} \neq \emptyset$.*

Proof of Theorem 0.5. Let $A = \{z \in \mathbb{R} : \Phi(z)\}$ where $\Phi(-)$ is Π^1_2. There is a tree $T \in K^{L[\vec{x}]}$ searching for a quadruple $(\vec{y}, \vec{M}, \vec{\mathcal{T}}, \vec{\sigma})$ such that the following hold true.

- $\vec{y} = (y_n : n < \omega) \in {}^{\omega}\mathbb{R}$,
- $\vec{M} = (M_n : n < \omega)$ such that for all $n < \omega$ do we have the following:
  (a) $M_n = (J_{\alpha_n}[y_n]; \in, y_n, U_n)$ for some α_n, U_n,
  (b) $M_{n+1} \models$ "$M_n = y_n^{\#}$" (in particular, M_{n+1} thinks that M_n is iterable),
  (c) y_{n+1} is the master code of M_n,
  (d) $M_0 \models \Phi(y_0)$, and
  (e) setting $\kappa = \text{crit}(U_{n+1})$, there is an initial segment of $K^{M_{n+1}}$ which iterates past $K^{L_\kappa[y_n]}$, and
- $(\vec{\mathcal{T}}, \vec{\sigma})$ witnesses that each individual K^{M_n} ($n < \omega$) is iterable (cf. [4]), i.e., $\vec{\mathcal{T}} = (\mathcal{T}_n : n < \omega)$, $\vec{\sigma} = (\sigma_n : n < \omega)$, and for all $n < \omega$ do we have the following:
  (a) $\mathcal{T}_n$ is a countable tree of successor length on $K^{L[\vec{x}]}$, and
  (b) $\sigma_n : K^{M_n} \to \mathcal{M}^{\mathcal{T}_n}_\infty || \beta_n$, some $\beta_n \leq \mathcal{M}^{\mathcal{T}_n}_\infty \cap \text{OR}$, is elementary.

We are now going to prove that

$$\emptyset \neq p[T] = \{y_0 : \exists (y_1, y_2, ...) \exists \vec{M} \exists \vec{\mathcal{T}} \exists \vec{\sigma} ((y_0, y_1, ...), \vec{M}, \vec{\mathcal{T}}, \vec{\sigma}) \in [T]\} \subset A.$$

We may well leave the verification of $p[T] \neq \emptyset$ as an exercise to the reader.

Now fix $(\vec{y}, \vec{M}, \vec{\mathcal{T}}, \vec{\sigma}) \in T$. Let $\vec{y} = (y_n : n < \omega)$ and $\vec{M} = (M_n : n < \omega)$. Let us prove that $y_0 \in A$. Let $\binom{\alpha}{n}$ denote the assertion that the α^{th} iterate of M_n is well-founded. It clearly suffices to prove the following.

Main Claim. For all α, for all n, $\binom{\alpha}{n}$ holds.

Definition 0.6 *Let $n < \omega$. We write $(M_n^i, \pi_n^{ij} : i \leq j \leq \alpha)$ for the putative iteration of M_n of length $\alpha + 1$, if it exists; and if so then for $i < \alpha$ we write κ_n^i for the critical point of $\pi_n^{0i}(U_n)$, i.e., of the top extender of M_n^i. We call α a* uniform indiscernible *provided that for all $n < \omega$, the putative iteration $(M_n^i, \pi_n^{ij} : i \leq j \leq \alpha)$ of M_n of length $\alpha + 1$ exists and $\{\kappa_i : i < \alpha\}$ is (closed and) unbounded in α.*

Proof of the Main Claim. We'll prove the Main Claim by induction on α.

Case 1. α is not a uniform indiscernible.

Let $n < \omega$. By our case assumption, there are some $m > n$ and $\beta < \alpha$ such that $\alpha \in M_m^\beta$. But $M_m^\beta \models$ "$M_n = y_n^\#$," so that we may argue inside M_m^β and deduce that the α^{th} iterate of M_n, viz. M_n^α, is well-founded.

Case 2. α is a uniform indiscernible.

Let $n < \omega$. Let $\kappa = \text{crit}(U_{n+1}) = \kappa_{n+1}^0$, and let $\mathcal{P}$ be the proper initial segment of $K^{M_{n+1}} = K^{L_\kappa[y_{n+1}]}$ which iterates past $K^{L_\kappa[y_n]}$. Let $(\mathcal{T}, \mathcal{U})$ be the coiteration of $K^{L_\kappa[y_n]}$ with $K^{L_\kappa[y_{n+1}]}$. $(M_n^i, \pi_n^{ij} : i \leq j < \alpha)$ is the putative iteration of M_n of length $\alpha + 1$. Let

$$\sigma : M_{n+1} \to_{U_{n+1}} M',$$

i.e., $\sigma = \pi_{n+1}^{01}$ and $M' = M_{n+1}^1$.

Let $X \in \mathcal{P}(\kappa) \cap \mathcal{M}_\infty^\mathcal{U}$. Then $X = \pi_{i\infty}^\mathcal{U}(\bar{X})$, some $i < \kappa$, $\bar{X}$, and $\sigma(X) = \pi_{i\infty}^{\sigma(\mathcal{U})}(\bar{X}) = \pi_{\kappa\infty}^{\sigma(\mathcal{U})}(X)$. Therefore, $\pi_{\kappa\infty}^{\sigma(\mathcal{U})} \restriction \kappa^{+\mathcal{M}_\infty^\mathcal{U}} = \sigma \restriction \kappa^{+\mathcal{M}_\infty^\mathcal{U}}$. The same argument shows that $\pi_n^{\kappa\sigma(\kappa)} \restriction \kappa^{+M_n^\kappa} = \sigma \restriction \kappa^{+M_n^\kappa}$; we construe M_n in such a way that $\text{crit}(U_n)^{+M_n} = M_n \cap \text{OR}$, so that this latter equality means that $\pi_n^{\kappa\sigma(\kappa)} \restriction M_n^\kappa \cap \text{OR} = \sigma \restriction M_n^\kappa \cap \text{OR}$. Let us write $\lambda = \sigma(\kappa) = \kappa_{n+1}^1$.

Now $(\sigma(\mathcal{T}), \sigma(\mathcal{U}))$ is the coiteration of $K^{L_\lambda[y_n]}$ with $K^{L_\lambda[y_{n+1}]}$. We'll have that $\kappa^{+\mathcal{M}_\infty^{\mathcal{U}}} = \kappa^{+\mathcal{M}_\kappa^{\sigma(\mathcal{U})}} = \kappa^{+\mathcal{M}_\kappa^{\sigma(\mathcal{T})}} \geq \kappa^{+K^{L_\lambda[y_n]}} = \kappa^{+L_\lambda[y_n]} = \kappa^{+M_n^\kappa}$, so that we get that

$$\pi_n^{\kappa\lambda} \restriction M_n^\kappa \cap \mathrm{OR} = \pi_{\kappa\infty}^{\sigma(\mathcal{U})} \restriction M_n^\kappa \cap \mathrm{OR}.$$

This now buys us that if we let $(\mathcal{T}^*, \mathcal{U}^*)$ denote the coiteration of $K^{L_\alpha[y_n]}$ with $K^{L_\alpha[y_{n+1}]}$ then for typical $i \leq j < \alpha$ (namely, for all $i \leq j \in \{\kappa_{n+1}^\beta : \beta < \alpha\}$) we'll have that

$$\pi_n^{ij} \restriction M_n^i \cap \mathrm{OR} = \pi_{ij}^{\mathcal{U}^*} \restriction M_n^i \cap \mathrm{OR}.$$

Moreover, $\mathcal{U}^*$ may be construed as an iteration of $\mathcal{P}$. As $\mathcal{P}$ is iterable, we may thus conclude that the α^{th} iterate of M_n, viz. M_n^α, is well-founded (cf. the argument of [4]).

□ (Main Claim)
□ (Theorem 0.5)

Using [1] it can be verified that Theorem 0.5 still holds if the assumption that $K^{L[\vec{x}]}$ is being crossed out and the conclusion is being replaced by "Then there is some lightface iterable premouse $\mathcal{M}$ with $A \cap \mathcal{M} \neq \emptyset$.

References

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