The strength of choiceless patterns of singular and weakly compact cardinals

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ABSTRACT

We extend the core model induction technique to a choiceless context, and we exploit it to show that each one of the following two hypotheses individually implies that AD, the Axiom of Determinacy, holds in the $L(\mathbb{R})$ of a generic extension of HOD: (a) ZF + every uncountable cardinal is singular, and (b) ZF + every infinite successor cardinal is weakly compact and every uncountable limit cardinal is singular.

CONTENTS

1. INTRODUCTION

In this paper we suppose that the universe V is a model of ZF. We want to show that the axiom of determinacy is consistent relative to the hypotheses "each uncountable successor cardinal is weakly compact and each uncountable limit cardinal is singular" and "each uncountable cardinal is singular", respectively.

It is natural first to ask if these hypotheses themselves are consistent. Moti Gitik has shown that it is possible that all uncountable cardinals are singular. He proved this in [Git80]:

Theorem 1.1. *If "$\mathsf{ZFC} + \forall\alpha \in \mathsf{On}\ \exists\kappa > \alpha\ \kappa$ is a strongly compact cardinal" is consistent, then "$\mathsf{ZF} + \forall\alpha \in \mathsf{On}\ \operatorname{cof}(\aleph_\alpha) = \aleph_0$" is consistent, too.*

Note that if each uncountable cardinal is singular, then the axiom of choice is violated badly. In this case neither DC nor AC_ω hold, because these kinds of choice force ω_1 to be regular. This is why we may only demand that V be a model of ZF, rather than a model of ZFC.

In [Git85], Gitik has generalized his result by showing the following.

Theorem 1.2. *If "$\mathsf{ZFC}+\exists\kappa\ \kappa$ is an almost huge cardinal" is consistent, then there is a model M of ZFC such that for every class A of M consisting of nonlimit ordinals there exists a model N_A of ZFC such that its regular alephs are exactly $\{\aleph_\alpha : \alpha \in A \cup \{0\}\}$.*

Also the hypothesis "every uncountable successor cardinal is weakly compact and each uncountable limit cardinal is singular" is relatively consistent. Arthur W. Apter has generalized Gitik's method in [Apt85] to obtain the following result.

Theorem 1.3. *Suppose V is a model of ZFC such that*

1. $\mathsf{V} \models$ *"κ is a 3-huge cardinal" and*
2. $\mathsf{V} \models$ *"A and B are disjoint subsets of the successor ordinals $< \kappa$ with*

$$A \cup B = \{\alpha < \kappa : \alpha \text{ is a successor ordinal}\}."$$

Then there is a symmetric submodel N_A of a generic extension of V such that N_A is a model of $\mathsf{ZF} + \neg\mathsf{AC}$ (in fact of $\mathsf{ZF} + \neg\mathsf{AC}_\omega$), the ordinals of N_A have height κ, and

1. *for $\alpha \in A$, $N_A \models$ "$\aleph_\alpha$ is a Ramsey cardinal",*
2. *for $\beta \in B$, $N_A \models$ "$\aleph_\beta$ is a singular Rowbottom cardinal which carries a Rowbottom filter", and*
3. *for γ a limit ordinal, $N_A \models$ "$\aleph_\gamma$ is a Jónsson cardinal which carries a Jónsson filter".*

Moreover, it follows from the construction of N_A that all limit cardinals are singular, so all uncountable regular cardinals in N_A are Ramsey cardinals in N_A.

If we use this theorem with $A = \{\alpha < \kappa : \alpha \text{ is a successor ordinal}\}$ and $B = \emptyset$, then we get a model N_A of ZF which satisfies

1. every uncountable successor cardinal is a Ramsey cardinal and
2. every uncountable limit cardinal is singular.

So we even have that there is a model of "ZF + each uncountable successor cardinal is Ramsey" rather than a model of "ZF + each uncountable successor cardinal is weakly compact", but since we only need weak compactness we don't demand more.

The main results of our paper are the following.

Theorem 1.4. *Let V be a model of ZF. Suppose that each uncountable successor cardinal is weakly compact and each uncountable limit cardinal is singular.*

Then there is a cardinal μ and a set of ordinals X such that $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ holds in $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu^{+\mathsf{V}})}$.

Theorem 1.5. *Let* V *be a model of* ZF *in which each uncountable cardinal is singular.*

Then there is a cardinal μ *and a set of ordinals* X *such that* $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ *holds in* $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu^{+\mathsf{HOD}_X})}$.

Remark. Note that we use the Lévy collapse of $\mu^{+\mathsf{V}}$ in Theorem 1.4 whereas we use the Lévy collapse of $\mu^{+\mathsf{HOD}_X}$ in Theorem 1.5. In both cases the set X is just a technical proviso which ensures that certain structures do not compute the cardinal successor of a special cardinal κ correctly.

Theorem 1.5 answers a question which arises from a corollary of the main theorem Apter proved in [Apt96]. He derived the consistency of "ZF + each uncountable cardinal below θ is singular" from "$\mathsf{ZF} + \mathsf{AD}$," where θ is the least ordinal onto which the set of reals cannot be mapped. So it is natural to ask if it is possible to find a model of "ZF + each uncountable cardinal is singular" from "$\mathsf{ZF} + \mathsf{AD}$". This question has a negative answer, because if V were a model of "ZF + each uncountable cardinal is singular", then the proof of Theorem 1.5 would show that each set in HOD_X, and even in the generic extension $\mathsf{HOD}_X[g]$, where g is $\mathrm{Col}(\omega, <\mu^{+\mathsf{HOD}_X})$-generic, would have a sharp; in particular $\mathbf{R}^\sharp$ would exist for $\mathbf{R} := \mathbb{R}^{\mathsf{HOD}_X[g]}$. But then there would be an α such that $J_\alpha(\mathbf{R}) \prec \mathsf{L}(\mathbf{R})$. Since $\mathsf{L}(\mathbf{R}) \models \mathsf{AD}$ by Theorem 1.5, we would therefore have $J_\alpha(\mathbf{R}) \models \mathsf{AD}$. But $J_\alpha(\mathbf{R})$ has set size, so in the end, if the answer of the question were "yes", we would have that "$\mathsf{ZF}+\mathsf{AD}$" implies $\mathit{Con}(\mathsf{ZF} + \mathsf{AD})$, which contradicts Gödel's second incompleteness theorem.

We will use the so-called *core model induction* to prove our theorems. The core model induction, originally developed by W. Hugh Woodin and enhanced by John R. Steel, is an induction along the $J_\alpha(\mathbb{R})$-hierarchy of $\mathsf{L}(\mathbb{R})$. The goal is to show that at each stage α, the axiom of determinacy holds true in $J_\alpha(\mathbb{R})$, i. e. $J_\alpha(\mathbb{R}) \models \mathsf{AD}$, so that in the end one gets $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$.

In the induction we don't show explicitly that $J_\alpha(\mathbb{R})$ is a model of the axiom of determinacy, we rather show by induction that for all α, a condition denoted by $(W_\alpha^\star)$ holds true. This condition demands that if there is a set of reals U such that there are scales on U and $\mathbb{R} \setminus U$, whose associated sequences of prewellorderings are both in $J_\alpha(\mathbb{R})$, then there are structures, called Woodin mice, which are "correct" for that level of the $\mathsf{L}(\mathbb{R})$-hierarchy, i. e. the existence of these mice ensures that $J_\alpha(\mathbb{R})$ satisfies AD. We use

Steel's core model theory to build these mice.

Since $(W^\star_\alpha)$ only mentions sets of reals such that there are sequences of prewellorderings in $J_\alpha(\mathbb{R})$ coming from a scale, we only need to prove $(W^\star_{\alpha+1})$ for those α for which there are a set $U \subseteq \mathbb{R}$ and scales on U and $\mathbb{R} \setminus U$ whose associated sequences of prewellorderings are new in $J_{\alpha+1}(\mathbb{R})$, i. e. there is no scale (resp. no sequence of associated prewellorderings) on U in $J_\alpha(\mathbb{R})$. Such an ordinal is called *critical*. If α is not critical, then $(W^\star_{\alpha+1})$ follows trivially.

Descriptive set theory is used to handle these critical ordinals. For this we need the concept of Σ_1-gaps. First let $\theta^{\mathsf{L}(\mathbb{R})}$ be θ computed in $\mathsf{L}(\mathbb{R})$, i. e. the least ordinal γ such that there is no surjection $f\colon \mathbb{R} \to \gamma$ with $f \in \mathsf{L}(\mathbb{R})$. A Skolem hull argument then yields:

$$\theta^{\mathsf{L}(\mathbb{R})} \text{ is the least } \gamma \text{ such that } \mathrm{Pow}(\mathbb{R}) \cap \mathsf{L}(\mathbb{R}) \subseteq J_\gamma(\mathbb{R})$$

So we only need to prove $(W^\star_\alpha)$ for $\alpha < \theta^{\mathsf{L}(\mathbb{R})}$, since each subset of the reals in $\mathsf{L}(\mathbb{R})$ appears before $\theta^{\mathsf{L}(\mathbb{R})}$. Now a Σ_1-gap is a maximal interval $[\alpha, \beta]$ such that $J_\alpha(\mathbb{R})$ is a Σ_1-substructure of $J_\beta(\mathbb{R})$ for statements with parameters in $\mathbb{R}$. One can show that these gaps partition $\theta^{\mathsf{L}(\mathbb{R})}$. It follows that each scale (resp. the associated sequence of prewellorderings), shows up within a Σ_1-gap $[\alpha, \beta]$. In [Ste83] John R. Steel has analyzed precisely at which levels α there are new scales. For this he used the concept of Σ_1-gaps.

It turns out that the induction consists of various cases. The base for the induction is the case $(W^\star_0) \Rightarrow (W^\star_1)$.[1] Thus, in this case we show that projective determinacy holds. The specific method for proving $(W^\star_\alpha) \Rightarrow (W^\star_{\alpha+1})$ for $\alpha > 0$ depends on the properties of α:

1. α begins a Σ_1-gap, is $\mathbb{R}$-inadmissible[2] and successor of a critical ordinal.
2. α begins a Σ_1-gap, is $\mathbb{R}$-inadmissible and has uncountable cofinality.
3. α begins a Σ_1-gap, is $\mathbb{R}$-inadmissible and has countable cofinality.
4. α begins a Σ_1-gap, is $\mathbb{R}$-inadmissible and successor of a *non*-critical ordinal.

[1] $(W^\star_0)$ holds trivially.

[2] $\mathbb{R}$-admissibility is just the translation of the concept of admissibility from the L- to the $\mathsf{L}(\mathbb{R})$-context.

5. α ends a weak Σ_1-gap.

The difference in handling these cases is that in the first three cases we work with *ordinary* premice as for example introduced in [Steb]. The cases four and five are different, since we work with so-called *hybrid* premice, which are premice with an additional predicate for some iteration strategy.

Our paper builds upon [Ste05], in which John R. Steel uses the core model induction to show the following result [Ste05, Theorem 0.1]:

Theorem 1.6. *Suppose there is a singular strong limit cardinal κ such that $\square_\kappa$ fails; then AD holds in $\mathsf{L}(\mathbb{R})$.*

The main difference between Steel's [Ste05] and the current paper is that Steel can work in V and generic extensions of V. In our situation we unfortunately don't have the axiom of choice in V and therefore no choice in $\mathsf{V}^{\mathrm{Col}(\omega, <\mu^+)}$. Since we have no choice in $\mathsf{V}^{\mathrm{Col}(\omega, <\mu^+)}$, we see no a priori reason why there should be any kind of choice in $\mathsf{L}(\mathbb{R})$. However, the core model induction is apparently in need of DC to hold in $\mathsf{L}(\mathbb{R})$. So we decided to work with a canonical inner model of ZFC, HOD, and generic extensions thereof. As mentioned above, though, we don't literally use the pure HOD for technical reasons; we need to work with some relativized HOD_X.

The paper is organized as follows.

The first part of the second chapter introduces basic definitions and theorems such as Vopěnka's theorem, which ensures that each set of ordinals in V is generic over HOD. Furthermore we introduce Jensen's J-hierarchy, the concept of Σ_1-gaps, weak and strong, the concept of scales, and $\mathbb{R}$-admissibility. We introduce premice, iteration trees and iteration strategies. A paragraph about capturing terms follows, which we use to handle the "successor of a critical" and "countable cofinality" cases (see above), and a few notes about weakly compact cardinals.

In the second part we specify some important cardinals and sets of ordinals to define the model for which we prove $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$; for example the cardinal μ and the set of ordinals X of our main theorems.

The third part is devoted to the framework of the core model induction. We introduce two special kinds of premice, coarse Woodin mice and $\langle\varphi, z\rangle$-witnesses, where φ is a Σ_1-formula and $z \in \mathbb{R}$ a real. We further make precise the induction hypothesis $(W^\star_\alpha)$ and define a new fine structural hypothesis

(W_α). Moreover, we specify what we mean by "critical ordinal" and explain why it suffices to consider only these ordinals.

The third chapter contains the proof of Theorem 1.4.

The first part is the proof of the projective case. We show inductively that $M_n^\sharp(A)$ exists for each set of ordinals $A \in \mathsf{V}$. For this we prove first that under the assumption "$M_{n-1}^\sharp(A)$ exists for all sets of ordinals $A \in \mathsf{V}$" the existence of $M_n^\sharp(A)$ for any set of ordinals $A \in \mathsf{V}$ is equivalent to the fact $\mathsf{HOD}_B \models$ "$\forall A\ M_n^\sharp(A)$ exists" for each set of ordinals $B \in \mathsf{V}$. Then we use the proof of [Sch99, Theorem 2] to get the existence of $M_n^\sharp(A)$ for each set of ordinals $A \in \mathsf{V}$ under the hypothesis that each uncountable successor cardinal is weakly compact.

In the second part we prove the cases 1 – 3 from the above enumeration. For this we define a *mouse closure operator* $\mathcal{M}$ which serves as a basis for "projective like" induction. We show that $\mathcal{M}(A)$ exists for each bounded subset A of κ^+, where κ is a sufficiently closed cardinal as will be defined in Definition 2.49. To show this we use a different technique from the one used by John R. Steel in [Ste05]. He entirely works in V, so he "only" needs first to show that $\mathcal{M}(A)$ exists for each bounded subset of μ^+, and then to use a covering argument to show that $\mathcal{M}(A)$ exists for each bounded $A \subseteq \kappa^+$. In our case we first get that $\mathcal{M}(A)$ exists for sets A bounded in $\mu^{+\mathsf{V}}$ which are in HOD_X rather than in V. We then use a covering argument to show the existence for bounded $A \subseteq \kappa$ in HOD, then that $\mathcal{M}(A)$ exists for all bounded $A \subseteq \kappa$ in V, and finally we use another covering argument to prove this for bounded subsets $A \subseteq \kappa^+$ in V. Then we can build the least $\mathcal{M}$-closed model and use the covering argument to show that a sharp for it exists. Inductively we show that for each $n < \omega$ there is a least active $\mathcal{M}$-closed mouse having n Woodin cardinals. Then we can utilize these mice to go one step further in our efforts to show $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$.

In the last part of this chapter, we prove the fourth and fifth case of the above enumeration. In these cases we need hybrid premice. From a theorem of Woodin and the results in [Ste05] we have a so-called suitable premouse $\mathcal{N}$ together with an iteration strategy Σ which condenses well. This property enables us to build models which have fine structure, satisfy condensation, contain $\mathcal{N}$, and know how to iterate it. The difference to [Ste05] is this time to lift Σ to an iteration strategy for larger trees which also condenses well. Woodin's theorem just gives us a μ^+-iteration strategy on the suitable $\mathcal{N}$

for trees which live in HOD_X, rather than for arbitrary trees in V. So we cannot simply use Steel's proof [Ste05, Lemma 1.25] to lift this strategy to a κ^+-strategy which one needs, but we have to adapt it to this situation. In the end we get an iteration strategy which works for arbitrary trees in HOD_X, and this is sufficient. Then we can again build the least hybrid model which contains $\mathcal{N}$ and use covering arguments to show the existence of a sharp for it. We get inductively for each $n < \omega$ the least active hybrid mouse containing n Woodin cardinals, which we use to show AD at the next stage of the $J_\alpha(\mathbb{R})$-hierarchy.

The fourth chapter deals with the proof of $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ in a $\mathrm{Col}(\omega, < \mu^{+\mathsf{HOD}_X})$-generic extension $\mathsf{HOD}_X[g]$ if each uncountable cardinal is singular. The arguments are essentially the same as in the weakly compact case, with the difference that we can show that in the inadmissible case $\mathcal{M}(A)$ exists for each set of ordinals $A \in \mathsf{V}$, not only for sets which are bounded in κ^+.

The material of this paper is contained in the first author's dissertation, "The core model induction in a choiceless context." The authors would like to thank John R. Steel for his constant willingness to answer questions concerning details in his paper [Ste05].

2. FRAMEWORK

2.1 Some definitions and notations

Notations

As usual in set theory, we let $\mathbb{R} := {}^{\omega}\omega$ denote the Baire space, i. e. the set of functions from ω to ω.

We often need to control the cardinality of the power set of the power set of a given ordinal computed in some inner model W. To get an upper bound for all inner models we define a function $\Theta\colon \mathsf{On} \to \mathsf{On}$ by

$$\Theta(\alpha) := \sup\{\gamma : \exists f\colon \mathrm{Pow}(\mathrm{Pow}(\alpha)) \to \gamma \text{ surjective}\}$$

Of course $\Theta(\alpha)$ depends on the model in which it is computed; the larger the model the larger is $\Theta(\alpha)$, i. e. if W and W' are models of ZF with $W \subseteq W'$, then $\Theta(\alpha)^W \leq \Theta(\alpha)^{W'}$. In the presence of AC we have $\Theta(\alpha) = (2^{2^{\alpha}})^+$.

For any ordinal κ, let $\mathrm{Col}(\omega, <\kappa)$ be the *Lévy collapse*, i. e. $\mathrm{Col}(\omega, <\kappa)$ is the set of all finite partial functions $p\colon \omega \times \kappa \to \kappa$ such that $p(n, \alpha) < \alpha$ for all $(n, \alpha) \in \mathrm{dom}(p)$. $\mathrm{Col}(\omega, \kappa)$ in contrast denotes the simple *collapse of* κ, i. e. the forcing consisting of all finite partial functions $p\colon \omega \to \kappa$.

As usual one can see that

$$\mathrm{Col}(\omega, <\kappa) = \bigcup_{\alpha<\kappa} \mathrm{Col}(\omega, <\alpha) \quad \text{and} \quad \mathrm{Col}(\omega, \kappa) = \bigcup_{\alpha<\kappa} \mathrm{Col}(\omega, \alpha)$$

Moreover, there is the connection

$$\mathrm{Col}(\omega, <\kappa) \cong \prod_{\substack{\text{finite support}\\ \alpha<\kappa}} \mathrm{Col}(\omega, \alpha)$$

So if g is generic for $\mathrm{Col}(\omega, <\kappa)$, then g induces for each $\alpha < \kappa$ a $\mathrm{Col}(\omega, <\alpha)$-generic filter $g{\restriction}\alpha := g \cap \mathrm{Col}(\omega, <\alpha)$.

Now let q be a condition in $\mathrm{Col}(\omega, <\kappa)$. We associate to each condition $p \in \mathrm{Col}(\omega, <\kappa)$ an element $p_q \in \mathrm{Col}(\omega, <\kappa)$ with the same domain as p such that:

$$p_q(n, \alpha) = \begin{cases} q(n, \alpha) & \text{if } (n, \alpha) \in \mathrm{dom}(q) \\ p(n, \alpha) & \text{otherwise} \end{cases}$$

One can easily see that if g is $\mathrm{Col}(\omega, <\kappa)$-generic over V then also

$$g_q := \{p_q : p \in g\}$$

is $\mathrm{Col}(\omega, <\kappa)$-generic over V and the appropriate generic extensions are the same, $\mathsf{V}[g] = \mathsf{V}[g_q]$. We call g_q a *finite variant of* g. Also if $\dot{x}^g$ is an element in $\mathsf{V}[g]$, then it is easy to compute a term $\dot{x_q}$ such that $\dot{x_q}^g = \dot{x}^{g_q}$. This can be done in a uniform way.

As usual, we let HOD be the inner model which consists of all A such that any element of the transitive closure of $\{A\}$, $\mathrm{tc}(\{A\})$[1], is ordinal definable in V. For sets $X_0, \ldots, X_n$ let $\mathsf{HOD}_{X_0,\ldots,X_n}$ denote the class consisting of the sets A such that each element of $\mathrm{tc}(\{A\})$ is ordinal definable in V using $X_0, \ldots, X_n$ as additional parameters. Elements of any X_i are not allowed as parameters. Note that each $\mathsf{HOD}_{X_0,\ldots,X_n}$ is a model of ZFC.

In this paper we often use Vopěnka's theorem [Jec03, Theorem 15.46]:

Theorem 2.1 (Vopěnka). *Suppose* ZF *holds. Let* A *be a set of ordinals with* $\sup(A) = \mu$.

Then there is an ordinal $\alpha < \Theta(\mu)$ *and a partial ordering* $\preccurlyeq \in \mathsf{HOD}$ *with support* α *such that* A *is* $(\alpha, \preccurlyeq)$*-generic over* HOD. *We denote this forcing by* Vop_μ.

If $B \in \mathsf{V}$ *is a set of ordinals, then this theorem relativizes easily to* HOD_B.

Proof. Let $\mathbb{P} := \mathrm{Pow}(\mathrm{Pow}(\mu)) \cap \mathsf{OD} \setminus \{\emptyset\}$. If $f : \alpha \to \mathbb{P}$ is an enumeration of $\mathbb{P}$ in OD, then we define the Vopěnka forcing $\mathrm{Vop}_\mu := (\alpha, \preccurlyeq)$ by $\xi \preccurlyeq \xi'$ iff $f(\xi) \subseteq f(\xi')$. Now $G := \{\xi \in \alpha : A \in f(\xi)\}$ is Vop_μ-generic over HOD and

$$\xi \in A \Leftrightarrow f^{-1}(\{Y \subseteq \mu : \xi \in Y\}) \in G \qquad \square$$

The reason why we introduced the function Θ is that if λ is a Θ-closed cardinal, then we have that each bounded subset of λ is not only Vopěnka-generic over HOD, but already Vopěnka-generic over H_λ^{HOD}.

[1] $\mathrm{tc}(\{A\})$ is the smallest transitive set containing A as an element.

The structure of $\mathsf{L}(\mathbb{R})$

For fine structural reasons we use Jensen's J-hierarchy instead of Gödel's L-hierarchy to build models like L, $\mathsf{L}(\mathbb{R})$ or $\mathsf{L}[A]$. Ronald Jensen introduced this hierarchy in [Jen72].

In order to define the J-hierarchy we need the concept of rudimentary functions, cf. [Jen72, p. 233].

Definition 2.2. Let B be a set or proper class. A function $f\colon \mathsf{V}^k \to \mathsf{V}$ is called *rudimentary in* B or rud_B iff it is generated by the following schemata:

1. $f(x_1, \dots, x_k) = x_i$
2. $f(x_1, \dots, x_k) = x_i \setminus x_j$
3. $f(x_1, \dots, x_k) = \{x_i, x_j\}$
4. $f(x_1, \dots, x_k) = h(g_1(x_1, \dots, x_k), \dots, g_l(x_1, \dots, x_k))$ where $h, g_1, \dots, g_l$ are rudimentary.
5. $f(x_1, \dots, x_k) = \bigcup_{z \in x_1} g(z, x_2, \dots, x_k)$ where g is rudimentary.
6. $f(x) = x \cap B$

Lemma 2.3. *There is a finite list of* rud_B*-functions such that each function rudimentary in B is a combination of them. This list is called a* basis.

Jensen defined nine functions as a basis. In [SZ] this list is enlarged to 16 functions to ensure that the pointwise image of some transitive set under all these functions is also transitive.

Rudimentary functions are simple in the following sense

Definition 2.4. A function f is called *simple* iff whenever $R(z, \vec{y})$ is a Σ_0-relation the relation $R(f(\vec{x}), \vec{y})$ is also Σ_0.

Lemma 2.5. *A function f is simple iff*

1. *there is a Σ_0-formula ψ_f such that $z \in f(\vec{x}) \Leftrightarrow \psi_f(z, \vec{x})$ and*
2. *whenever $\psi(z, \vec{y})$ is Σ_0, then there is a Σ_0-formula $\chi_{f,\psi}$ such that*

$$\exists z \in f(\vec{x})\ \psi(z, \vec{y}) \quad \Leftrightarrow \quad \chi_{f,\psi}(\vec{x}, \vec{y}).$$

Lemma 2.6. *Every rudimentary function f is simple and therefore Σ_0.*

Both these lemmata are shown in [Dev84, pp. 230 – 232].

Now if f is rudimentary and $\varphi(x)$ is a Σ_0-formula, we get that also $\varphi(f(x))$ is a Σ_0-formula.

Lemma 2.7. *Let $f\colon \mathsf{V}^k \to \mathsf{V}$ be rudimentary and $\varphi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ be a Σ_0-formula.*

Then there is a Σ_0-formula φ_f such that for any set U and for any $\vec{y}_1, \ldots, \vec{y}_n \in U^k$, $\vec{x} \in U^m$ the following equivalence holds:

$$\varphi(f(\vec{y}_1), \ldots, f(\vec{y}_n), \vec{x}) \quad \Leftrightarrow \quad \varphi_f(\vec{y}_1, \ldots, \vec{y}_n, \vec{x})$$

Proof. We construct φ_f by induction on φ. If each parameter in φ is in U, i. e. $n = 0$, then let $\varphi_f := \varphi$.

If $\varphi \equiv x \in f(\vec{y})$ for $\vec{y} \in U^k$ and $x \in U$, then, since f is simple, let φ_f be the formula $\psi_f(x, \vec{y})$ from Lemma 2.5.

Suppose $\varphi \equiv x = f(\vec{y})$. Since f is rudimentary this is a Σ_0-formula.

For $\varphi \equiv f(\vec{y}) \in x$ for $\vec{y} \in U^k$ and $x \in U$, let φ_f be $\exists z \in x\ z = f(\vec{y})$. Since f is Σ_0 this is a Σ_0-formula.

For $\varphi \equiv \psi \wedge \chi$ and $\varphi \equiv \neg\psi$ let $\varphi_f := \psi_f \wedge \chi_f$ and $\varphi_f := \neg\psi_f$, respectively.

If $\varphi \equiv \exists z \in x\ \psi$ we let $\varphi_f := \exists z \in x\ \psi_f$.

Finally suppose $\varphi \equiv \exists z \in f(\vec{y})\ \psi$. Then let φ_f be the Σ_0-formula $\chi_{f,\psi}(\vec{y})$ from Lemma 2.5.

From Lemma 2.5 we get the desired property. $\square$

Now we can define the J-hierarchy.

Definition 2.8. Let B be a set or a proper class and A be a set.

$$\begin{aligned} J_0^B(A) &:= \mathrm{tc}(\{A\}), \\ J_{\alpha+1}^B(A) &:= \mathrm{rud}_B(J_\alpha^B(A) \cup \{J_\alpha^B(A)\}), \\ J_\lambda^B(A) &:= \bigcup_{\alpha<\lambda} J_\alpha^B(A) \text{ for } \lambda \text{ limit,} \end{aligned}$$

where $\mathrm{rud}_B(X)$ is the closure of X under rud_B functions.

We also define a subhierarchy of the J^B-hierarchy to have more control over the successor steps of the J^B-hierarchy.

$$\begin{aligned} S_0^B(A) &:= \mathrm{tc}(\{A\}), \\ S_{\alpha+1}^B(A) &:= \mathbf{S}_B(S_\alpha^B(A)), \\ S_\lambda^B(A) &:= \bigcup_{\alpha<\lambda} S_\alpha^B(A) \text{ for } \lambda \text{ limit.} \end{aligned}$$

Here $\mathbf{S}_B$ is an operator which, applied to a set U, adds images of members of $U \cup \{U\}$ under the basic functions rudimentary in B.[2]

In the special case $A = \mathbb{R}$ and $B = \emptyset$ we abuse our notation of $J_\alpha(\mathbb{R})$ by starting the hierarchy with $J_0(\mathbb{R}) := V_{\omega+1}$ rather than $J_0(\mathbb{R}) = \mathrm{tc}(\{\mathbb{R}\})$.

In particular, $J_1(\mathbb{R}) \cap \mathrm{Pow}(\mathbb{R})$ is the collection of all projective sets.

First observe that $J_\alpha^B(A) = S_{\omega\alpha}^B(A)$ and for $\gamma := \mathrm{tc}(\{A\}) \cap \mathsf{On}$ we have $J_\alpha^B(A) \cap \mathsf{On} = \gamma + \omega\alpha$.

One can easily see that $\bigcup_{\alpha\in\mathsf{On}} J_\alpha^B(A)$ is the minimal inner model of ZF which contains A as an element and which is closed under functions being rudimantary in B. If A is easily wellorderable, then $\bigcup_{\alpha\in\mathsf{On}} J_\alpha^B(A)$ is even a model of AC. Of course, we drop B and A whenever $B = \emptyset$ and $A = \emptyset$; for example $\mathsf{L}(\mathbb{R}) = \bigcup_{\alpha\in\mathsf{On}} J_\alpha(\mathbb{R})$ and $\mathsf{L}[B] = \bigcup_{\alpha\in\mathsf{On}} J_\alpha^B$.

We need definable subsets of stages of the $J_\alpha(\mathbb{R})$-hierarchy, but we don't want that quantification over the set of reals increases the complexity of the formulae. So we follow the notions from [Ste83] to define $\Sigma_n(J_\alpha(\mathbb{R}))$- and $\Pi_n(J_\alpha(\mathbb{R}))$-sets of reals.

Definition 2.9. Let α be an ordinal. Then we let $\Sigma_n(J_\alpha(\mathbb{R}), x_1, \ldots, x_k)$ be the set of subsets of the reals which are Σ_n-definable over $J_\alpha(\mathbb{R})$ from the parameters $x_1, \ldots, x_k$ and the parameter $V_{\omega+1}$[3]. I.e., the quantifiers $\exists x \in \mathbb{R}$, $\forall x \in \mathbb{R}$ are counted as bounded quantifiers. We use the notation $\underset{\sim}{\Sigma}_n(J_\alpha(\mathbb{R}))$ for the set of subsets of the reals which are Σ_n-definable over $J_\alpha(\mathbb{R})$ from arbitrary parameters of $J_\alpha(\mathbb{R})$. The terms $\Pi_n(J_\alpha(\mathbb{R}), x_1, \ldots, x_k)$ and $\underset{\sim}{\Pi}_n(J_\alpha(\mathbb{R}))$ are defined accordingly.

[2] Here we use the enlarged list from [SZ] to ensure that each stage of the S^B-hierarchy is transitive.

[3] Note that we do *not* allow elements of $V_{\omega+1}$ (other than $x_1, \ldots, x_k$) as parameters! For example, individual reals are forbidden!

Definition 2.10. Let M and N be such that $X \cup \{V_{\omega+1}\} \subseteq M \subseteq N$. We write $M \prec_n^X N$ if M and N satisfy the same Σ_n-formulae with parameters in $X \cup \{V_{\omega+1}\}$. We write $M \prec_n N$ for $M \prec_n^M N$. $M \prec^X N$ means that M and N satisfy the same formulae of arbitrary complexity with parameters in $X \cup \{V_{\omega+1}\}$. Finally $M \prec N$ means that M *is an elementary substructure of* N, i. e. $M \prec^M N$.

We now introduce Σ_1-gaps, which are needed to characterize critical ordinals.

Definition 2.11. Let $\alpha, \beta \in \mathsf{On}$ be such that $\alpha \leq \beta$. The interval $[\alpha, \beta]$ is called a Σ_1*-gap* iff it is a maximal interval in which no new Σ_1-facts about reals are verified, i. e.

1. $J_\alpha(\mathbb{R}) \prec_1^{\mathbb{R}} J_\beta(\mathbb{R})$,
2. $\forall \alpha' < \alpha \; J_{\alpha'}(\mathbb{R}) \not\prec_1^{\mathbb{R}} J_\alpha(\mathbb{R})$, and
3. $\forall \beta' > \beta \; J_\beta(\mathbb{R}) \not\prec_1^{\mathbb{R}} J_{\beta'}(\mathbb{R})$.

Remark. We have $J_\alpha(\mathbb{R}) \prec_1^{\mathbb{R}} J_\gamma(\mathbb{R}) \prec_1^{\mathbb{R}} J_\beta(\mathbb{R})$ for each $\gamma \in [\alpha, \beta]$, since Σ_1-formulae are upward absolute, $J_\alpha(\mathbb{R}) \prec_1^{\mathbb{R}} J_\beta(\mathbb{R})$, and since we only allow parameters from $\mathbb{R} \subseteq J_\alpha(\mathbb{R})$.

One divides the Σ_1-gaps in two parts: strong and weak.

Definition 2.12. Let $\Sigma_{a,\alpha}^n$ be the Σ_n*-type realized by* a *in* $J_\alpha(\mathbb{R})$, i. e.

$$\Sigma_{a,\alpha}^n := \{\varphi : \varphi \in \Sigma_n \cup \Pi_n \wedge J_\alpha(\mathbb{R}) \models \varphi(a)\}$$

An ordinal β is called *strongly* Π_n*-reflecting* iff each Σ_n-type which is realized in $J_\beta(\mathbb{R})$ is already realized in $J_\alpha(\mathbb{R})$ for some $\alpha < \beta$, i. e.

$$\forall b \in J_\beta(\mathbb{R}) \; \exists \alpha < \beta \; \exists a \in J_\alpha(\mathbb{R}) \; \Sigma_{b,\beta}^n = \Sigma_{a,\alpha}^n$$

A Σ_1-gap $[\alpha, \beta]$ is called *strong* iff β is strongly Π_n-reflecting, where n is least such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$.[4] Otherwise $[\alpha, \beta]$ is called *weak*.

For our proof of AD we heavily use the concept of scales in $\mathsf{L}(\mathbb{R})$.

[4] $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$ means that there is partial $\underset{\sim}{\Sigma}_n(J_\beta(\mathbb{R}))$-map $f\colon \mathbb{R} \to J_\beta(\mathbb{R})$ which is onto; cf. [Ste83, Definition 1.13].

Definition 2.13. Let $A \subseteq \mathbb{R}^k$ and $\lambda \in \mathsf{On}$. A function $\varphi\colon A \to \lambda$ is called a *norm.*

A *scale on* A is a sequence of norms $(\varphi_i : i < \omega)$, $\varphi_i\colon A \to \lambda$ with the following property: Suppose $\{x_i : i < \omega\} \subseteq A$ with $\lim_{i\to\omega} x_i = x$ and $g\colon \omega \to \lambda$ is a function such that for every n and for all but finitely many i the equality $\varphi_n(x_i) = g(n)$ holds. Then $x \in A$ and $\varphi_n(x) \leq g(n)$ for every n.

Let $\vec{\varphi}$ be a scale on $A \subseteq \mathbb{R}^k$. We can associate a sequence of prewellorderings $\leq_{\vec{\varphi}} := (\leq_{\varphi_i} : i < \omega)$ to $\vec{\varphi}$ such that for all $x, y \in A$:

$$x \leq_{\varphi_i} y \quad \Leftrightarrow \quad \varphi_i(x) \leq \varphi_i(y)$$

In $\mathsf{L}(\mathbb{R})$ scales are closely connected with Σ_1-gaps. In [Ste83] it is analyzed at which stages in the $J_\alpha(\mathbb{R})$ hierarchy new scales appear.

Definition 2.14. An ordinal α is called $\mathbb{R}$-*admissible* iff for no $D \in J_\alpha(\mathbb{R})$ there is a total cofinal map $f\colon D \to \omega\alpha$ which is $\underset{\sim}{\Sigma}_1$-definable over $J_\alpha(\mathbb{R})$.[5] Otherwise α is called $\mathbb{R}$-*inadmissible.*

Remark. Note that if α is an $\mathbb{R}$-admissible ordinal, then $J_\alpha(\mathbb{R})$ is a model of $\underset{\sim}{\Delta}_1$-separation. For this let $D, w, z \in J_\alpha(\mathbb{R})$, $\varphi \in \Sigma_1$, and $\psi \in \Pi_1$ be such that

$$A := \{x \in D : J_\alpha(\mathbb{R}) \models \varphi(x, w)\} = \{x \in D : J_\alpha(\mathbb{R}) \models \psi(x, z)\}$$

To show $A \in J_\alpha(\mathbb{R})$, we define $f\colon D \to \omega\alpha$ by

$$f(x) := \min\{\gamma : D \in S_\gamma(\mathbb{R}) \wedge S_\gamma(\mathbb{R}) \models \varphi(x, w) \vee \neg\psi(x, z)\}$$

Since φ and $\neg\psi$ are Σ_1-formulae and $J_\alpha(\mathbb{R}) = \bigcup_{\gamma<\omega\alpha} S_\gamma(\mathbb{R})$, and since each $x \in D$ is either in A or $D \setminus A$, we have that this function is well defined and total on D. Moreover, f is $\underset{\sim}{\Sigma}_1$-definable over $J_\alpha(\mathbb{R})$. As α is $\mathbb{R}$-admissible, f is not cofinal, so let $\gamma < \omega\alpha$ such that $f''D \subseteq \gamma$. But then

$$A = \{x \in D : S_\gamma(\mathbb{R}) \models \varphi(x, w)\},$$

so $A \in \mathrm{rud}(S_\gamma(\mathbb{R})) \subseteq J_\alpha(\mathbb{R})$.[6]

[5] This is equivalent to the statement $J_\alpha(\mathbb{R}) \models \mathsf{KP}$.

[6] Note that if U is transitive, then $\mathrm{rud}(U) \cap \mathrm{Pow}(U)$ are exactly the definable subsets of U.

Premice

For definitions and proofs which are not given see for example [Steb] (or in more detail [MS94b]) or [Zem02].

Definition 2.15. Let $\vec{E}$ be a fine extender sequence.[7] A *potential premouse* is a J-structure of the form $(J_\alpha^{\vec{E}}; \in, \vec{E}\restriction\alpha, E_\alpha)$. A *premouse* is a potential premouse $\mathcal{M}$ all of whose proper initial segments are ω-sound. We often identify $(J_\alpha^{\vec{E}}; \in, \vec{E}\restriction\alpha, E_\alpha)$ with its support $J_\alpha^{\vec{E}}$. We also have $\mathsf{L}[\vec{E}] = \bigcup_{\alpha \in \mathsf{On}} J_\alpha^{\vec{E}}$.

If $\mathcal{M}$ is a potential premouse we write $ht(\mathcal{M}) := \mathsf{On}^{\mathcal{M}}$, i. e. if $\mathcal{M} = J_\alpha^{\vec{E}}$ then $ht(\mathcal{M}) = \omega\alpha$.

Remark. If $\vec{E}$ is a sequence, then we abuse notation by writing $J_\alpha^{\vec{E}}$ for J_α^B, where $B = \{(\beta, z) : z \in E_\beta\}$.

We also use generalized premice, so-called *A-premice*, i. e. the construction of the premice begins with the transitive closure of $\{A\}$.

Definition 2.16. Let $\vec{E}$ be a fine extender sequence over some set A.[8] A *potential premouse over* A is a J-structure of the form $(J_\alpha^{\vec{E}}(A); \in, A, \vec{E}\restriction\alpha, E_\alpha)$. A *premouse over* A or *A-premouse* is a potential premouse over A all of whose proper initial segments are ω-sound.

See [Stec] for more details on relativized premice. Note that if $\mathcal{M}$ is an A-premouse we demand that all critical points of extenders on the $\mathcal{M}$-sequence are larger than $\sup(\mathrm{tc}(\{A\}) \cap \mathsf{On})$. Moreover, we have that each Skolem hull contains $\{A\} \cup A$. For example the first core of $\mathcal{M}$, $\mathfrak{C}_1(\mathcal{M})$, is the transitive collapse of the Σ_1-hull in $\mathcal{M}$ generated by $\rho_1(\mathcal{M}) \cup \{p_1(\mathcal{M})\} \cup \mathrm{tc}(\{A\})$.

Definition 2.17. Let $\mathcal{M} := (J_\alpha^{\vec{E}}(A); \in, A, \vec{E}\restriction\alpha, E_\alpha)$ be an A-premouse and $\beta \leq \alpha$. Then we call

$$\mathcal{M}\|\beta := (J_\beta^{\vec{E}}(A); \in, A, \vec{E}\restriction\beta, E_\beta)$$

and

$$\mathcal{M}|\beta := (J_\beta^{\vec{E}}(A); \in, A, \vec{E}\restriction\beta, \emptyset)$$

[7] See [Steb, Definition 2.4].
[8] See [Stec, Definition 2.6].

initial segments of $\mathcal{M}$.

Let $\mathcal{N}$ and $\mathcal{M}$ be A-premice. We write $\mathcal{N} \trianglelefteq \mathcal{M}$ iff $\mathcal{N} = \mathcal{M}\|\beta$ for some $\beta \leq \alpha$. Moreover, we say $\mathcal{N}$ and $\mathcal{M}$ are *compatible* or *lined up* iff $\mathcal{N} \trianglelefteq \mathcal{M}$ or $\mathcal{M} \trianglelefteq \mathcal{N}$.

So if $\mathcal{X}$ is a set of compatible premice we can build the "union" of them.

Definition 2.18. Let A be a set and $\mathcal{X}$ be a set of pairwise compatible A-premice, i. e. for each $\mathcal{M}, \mathcal{N} \in \mathcal{X}$ we have $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$. Then we write $\bigtriangledown \mathcal{X}$ for the premouse $\mathcal{N}$ of least height such that for all $\mathcal{M} \in \mathcal{X}$ there is an α with $\mathcal{M} = \mathcal{N}\|\alpha$.

If S is a set and $(\mathcal{M}_\alpha : \alpha \in S)$ is a sequence of pairwise compatible A-premice, we write $\bigtriangledown_{\alpha \in S} \mathcal{M}_\alpha$ for $\bigtriangledown\{\mathcal{M}_\alpha : \alpha \in S\}$.

If $\mathcal{X}$ is a set of A-premice such that for each $\mathcal{M}, \mathcal{N} \in \mathcal{X}$ there is an α with $\mathcal{M} = \mathcal{N}|\alpha$ or $\mathcal{N} = \mathcal{M}|\alpha$, then we use the notation $\bigtriangledown \mathcal{X}$ for the premouse $\mathcal{N}$ of least height such that for all $\mathcal{M} \in \mathcal{X}$ there is an α with $\mathcal{M} = \mathcal{N}|\alpha$.

Iteration trees

Definition 2.19. A *tree order* on an ordinal θ is a partial ordering $<_\mathcal{T} \subseteq <$ such that 0 is the least element of $<_\mathcal{T}$ and for any $\gamma < \theta$:

1. $\{\beta : \beta <_\mathcal{T} \gamma\}$ is wellordered by $<_\mathcal{T}$,
2. γ is a $<_\mathcal{T}$-successor iff γ is a successor ordinal, and
3. if γ is a limit ordinal, then the set $\{\beta : \beta <_\mathcal{T} \gamma\}$ is $<$-cofinal in γ.

A set $b \subseteq \theta$ which is downward closed under $<_\mathcal{T}$ and wellordered by $<_\mathcal{T}$ is called a *branch in* $\mathcal{T}$.

If $<_\mathcal{T}$ is a tree order, then we define

$$[\beta, \gamma]_\mathcal{T} := \{\eta : \beta \leq_\mathcal{T} \eta \leq_\mathcal{T} \gamma\}$$

as usual. $(\beta, \gamma]_\mathcal{T}$, $[\beta, \gamma)_\mathcal{T}$, and $(\beta, \gamma)_\mathcal{T}$ are defined similarly.

Now we briefly describe the *iteration game* $\mathcal{G}_k(\mathcal{M}, \theta)$ where $\mathcal{M}$ is a k-sound A-premouse, $k \leq \omega$, and θ is an ordinal. For a more detailed version see [Steb].

During the game the players produce

1. a tree order $<_{\mathcal{T}}$ on θ,
2. A-premice $\mathcal{M}^{\mathcal{T}}_\alpha$ for $\alpha < \theta$ with $\mathcal{M}^{\mathcal{T}}_0 = \mathcal{M}$,
3. an extender $E^{\mathcal{T}}_\alpha$ from the $\mathcal{M}^{\mathcal{T}}_\alpha$-sequence for $\alpha < \theta$, and
4. a set $D^{\mathcal{T}} \subseteq \theta$ and embeddings $e_{\alpha,\beta} \colon \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{T}}_\beta$ for each $\alpha <_{\mathcal{T}} \beta$ with $D^{\mathcal{T}} \cap (\alpha, \beta]_{\mathcal{T}} = \emptyset$.

The game is played as follows. Suppose we are at move $\alpha + 1$ and the players have produced $(E^{\mathcal{T}}_\xi : \xi < \alpha)$, $(\mathcal{M}^{\mathcal{T}}_\xi : \xi \leq \alpha)$, $<_{\mathcal{T}} \restriction \alpha{+}1$ and $D^{\mathcal{T}} \cap \alpha{+}1$. Then player I has to pick an extender $E^{\mathcal{T}}_\alpha$ from the $\mathcal{M}^{\mathcal{T}}_\alpha$-sequence such that $lh(E^{\mathcal{T}}_\xi) < lh(E^{\mathcal{T}}_\alpha)$ for all $\xi < \alpha$. If he does not, then the game is over and I has lost. Let $\beta \leq \alpha$ be least with $cr(E^{\mathcal{T}}_\alpha) < \nu(E^{\mathcal{T}}_\beta)$ and let $\eta \leq ht(\mathcal{M}^{\mathcal{T}}_\beta)$ be largest such that $E^{\mathcal{T}}_\alpha$ is an extender over $\mathcal{M}^{\mathcal{T}*}_{\alpha+1} := \mathcal{M}^{\mathcal{T}}_\beta \| \eta$. Then we define

$$\alpha + 1 \in D^{\mathcal{T}} \Leftrightarrow \eta < ht(\mathcal{M}^{\mathcal{T}}_\beta) \quad \text{and} \quad \mathcal{M}^{\mathcal{T}}_{\alpha+1} := \mathrm{ult}_n(\mathcal{M}^{\mathcal{T}*}_{\alpha+1}, E^{\mathcal{T}}_\alpha)$$

where $n \leq \omega$ is largest such that

1. $cr(E^{\mathcal{T}}_\alpha) < \rho_n(\mathcal{M}^{\mathcal{T}*}_{\alpha+1})$ and
2. if $D^{\mathcal{T}} \cap [0, \alpha + 1]_{\mathcal{T}} = \emptyset$ then $n \leq k$.

If this ultrapower is not wellfounded, then the game is over and II has lost. Finally we set $\beta <_{\mathcal{T}} \alpha + 1$. If $\alpha + 1 \notin D^{\mathcal{T}}$ then let $e_{\beta,\alpha+1} \colon \mathcal{M}^{\mathcal{T}}_\beta \to \mathcal{M}^{\mathcal{T}}_{\alpha+1}$ be the canonical ultrapower embedding and let $e_{\gamma,\alpha+1} := e_{\beta,\alpha+1} \circ e_{\gamma,\beta}$ for any $\gamma <_{\mathcal{T}} \beta$ such that $D^{\mathcal{T}} \cap (\gamma, \beta]_{\mathcal{T}} = \emptyset$.

Now suppose we are at move λ, where λ is a limit ordinal. Then II has to pick a branch b in $<_{\mathcal{T}}$. We demand that b is $\in$-cofinal in λ and that $D^{\mathcal{T}} \cap b$ is bounded in λ, so that we can build the direct limit of the ultrapower embeddings. Set

$$\mathcal{M}^{\mathcal{T}}_\lambda := \lim \mathrm{dir}_{\alpha \in b \setminus \sup(D^{\mathcal{T}} \cap b)} \mathcal{M}^{\mathcal{T}}_\alpha$$

If II does not pick a branch such that $D^{\mathcal{T}} \cap b$ is bounded in λ and $\mathcal{M}^{\mathcal{T}}_\lambda$ is wellfounded, then the game is over and II has lost. Finally we set $\alpha <_{\mathcal{T}} \lambda$ iff

$\alpha \in b$, and we let $e_{\alpha,\lambda}$ be the direct limit of the embeddings $e_{\alpha,\beta}$ for $\beta \in b \setminus \alpha$.[9] We often write e_b or e_λ for $e_{\alpha,\lambda}$ where α is least such that $e_{\alpha,\lambda}$ is defined.

If no one has lost after θ many moves, then II wins.

Definition 2.20. A *k-maximal iteration tree $\mathcal{T}$ on $\mathcal{M}$ of length θ* is a partial run of the game $\mathcal{G}_k(\mathcal{M}, \theta)$ in which no one has lost. We often identify $\mathcal{T}$ with the tree order $<_\mathcal{T}$ and write shortly *iteration tree* for *k-maximal iteration tree.*

An iteration tree $\mathcal{T}$ is determined by the tree order $<_\mathcal{T}$, the mouse $\mathcal{M}$, and the sequence $(E_\alpha^\mathcal{T} : \alpha < \theta)$ of the extenders played by player I. We say $lh(\mathcal{T}) := \theta$ is the *length of $\mathcal{T}$*. $\mathcal{M}_\alpha^\mathcal{T}$ is a structure which is built from the fine extender sequence $\vec{E}^{\mathcal{M}_\alpha^\mathcal{T}}$. Note that the rules of the iteration game ensure that $\mathcal{M}_\alpha^\mathcal{T}$ is a premouse. Moreover, for each $\beta \geq \alpha$ the equation

$$\vec{E}^{\mathcal{M}_\beta^\mathcal{T}} \upharpoonright lh(E_\alpha^\mathcal{T}) = \vec{E}^{\mathcal{M}_\alpha^\mathcal{T}} \upharpoonright lh(E_\alpha^\mathcal{T})$$

holds true. So we have

$$\mathcal{M}_\beta^\mathcal{T} | lh(E_\alpha^\mathcal{T}) = \mathcal{M}_\alpha^\mathcal{T} | lh(E_\alpha^\mathcal{T}).$$

Definition 2.21. If $\mathcal{T}$ is an iteration tree on $\mathcal{M}$ of limit length then we define

$$\delta(\mathcal{T}) := \sup_{\alpha < lh(\mathcal{T})} lh(E_\alpha^\mathcal{T}) \quad \text{and} \quad \mathcal{M}(\mathcal{T}) := \bigtriangledown_{\alpha < lh(\mathcal{T})} \mathcal{M}_\alpha^\mathcal{T} | lh(E_\alpha^\mathcal{T})$$

$\mathcal{M}(\mathcal{T})$ is called the *common part model* of the tree $\mathcal{T}$.

So $\mathcal{M}(\mathcal{T})$ is the premouse built from the fine extender sequence

$$\vec{E} := \bigcup_{\alpha < lh(\mathcal{T})} \vec{E}^{\mathcal{M}_\alpha^\mathcal{T}} \upharpoonright lh(E_\alpha^\mathcal{T})$$

Definition 2.22. Let $\mathcal{T}$ be a k-maximal iteration tree on $\mathcal{M}$ and $\alpha < lh(\mathcal{T})$. We define the *degree of α, $deg^\mathcal{T}(\alpha)$*, by saying

1. $deg^\mathcal{T}(0) = k$,

[9] Of course this is only meant for $\alpha, \beta > \sup(D^\mathcal{T} \cap b)$, such that the direct limit exists.

2. if α is a successor, say $\alpha = \beta+1$, then let $deg^{\mathcal{T}}(\alpha)$ be the largest $n \leq \omega$ such that $\mathcal{M}_\alpha^{\mathcal{T}} = \mathrm{ult}_n(\mathcal{M}_\alpha^{\mathcal{T}*}, E_\beta^{\mathcal{T}})$, and
3. for λ limit, let $deg^{\mathcal{T}}(\lambda) :=$ "the eventual value of $deg^{\mathcal{T}}(\beta)$ for $\beta+1 <_{\mathcal{T}} \lambda$ sufficiently large".[10]

Definition 2.23. Let $\mathcal{T}$ be an iteration tree and b be a branch. We then say that b *drops in model (or degree)* iff $D^{\mathcal{T}} \cap b \neq \emptyset$ (or $deg^{\mathcal{T}}(b) < deg^{\mathcal{T}}(0)$).

Now for any ordinal α we introduce the *iteration game* $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$ for a k-sound A-premouse. $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$ is an elaboration of $\mathcal{G}_k(\mathcal{M}, \theta)$. Its output is a linear stack of iteration trees.

The game has α rounds. The β^{th} round is played as follows: Let $\mathcal{N}$ be the last model in the linear iteration produced so far, i. e. if $\beta = 0$ then $\mathcal{N} = \mathcal{M}$, if $\beta = \gamma + 1$ then $\mathcal{N}$ is the last model of the tree produced in round γ, and if β is a limit then $\mathcal{N}$ is the direct limit along the unique cofinal branch in the linear composition of the trees produced before β, provided this branch is wellfounded. If this branch is illfounded, I wins. Let q be the degree of $\mathcal{N}$, i. e. $q = k$ if $\beta = 0$, if β is a successor then q is the degree of $\mathcal{N}$ as defined in Definition 2.22 and if β is a limit then q is the eventual value of the degrees of the previous rounds.

Player I begins round β by choosing an initial segment $\mathcal{P} \trianglelefteq \mathcal{N}$ and some $i \leq \omega$. If $\mathcal{P} = \mathcal{N}$ then $i \leq q$ has to hold. The rest of round β is according to the rules of $\mathcal{G}_i(\mathcal{P}, \theta)$, except that I can stop playing the round before θ steps and I has to break before the end of $\mathcal{G}_i(\mathcal{P}, \theta)$ if θ is a limit ordinal (otherwise he would lose). So in any case there is a last model which serves as $\mathcal{N}$ for round $\beta + 1$.

Player II wins $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$ if he does not lose any of the component games and if for each limit $\beta \leq \alpha$ the unique cofinal branch in the composition of trees previously built is wellfounded.

Iteration strategies

Definition 2.24. A (k, θ)*-iteration strategy* (resp. (k, α, θ)*-iteration strategy*) for an A-premouse $\mathcal{M}$ is a winning strategy for II in $\mathcal{G}_k(\mathcal{M}, \theta)$ (resp. $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$).

[10] This can be done by [Steb, Theorem 3.8].

We say $\mathcal{M}$ is *(k,θ)-iterable* (resp. *(k,α,θ)-iterable*) iff there is such an iteration strategy. $\mathcal{M}$ is *(k,θ)-iterable above δ* iff there is a (k,θ)-iteration strategy with respect to iteration trees whose extenders have all critical points above δ.

Finally $\mathcal{M}$ is *θ-iterable (above δ)* iff it is (ω,θ)-iterable (above δ).

The iterability we use mostly is the so-called "countable iterability".

Definition 2.25. An A-premouse $\mathcal{M}$ is called *countably k-iterable above δ* iff $\mathfrak{C}_k(\mathcal{M})$ exists[11] and for all $\mathcal{N}$, if $\mathcal{N}$ is an $\bar{A}$-premouse $\mathcal{N}$ with $\overline{\overline{\mathcal{N}}} = \omega$ such that there is a weak k-embedding $\pi\colon \mathcal{N} \to \mathfrak{C}_k(\mathcal{M})$ with $\delta \in \operatorname{ran}(\pi)$, then $\mathcal{N}$ is $(k, \omega_1, \omega_1+1)$-iterable above $\pi^{-1}(\delta)$.

$\mathcal{M}$ is called *countably iterable above δ* iff for all $k \leq \omega$, $\mathcal{M}$ is countably k-iterable above δ.

An A-premouse $\mathcal{M}$ is called *countably iterable* or an *A-mouse* iff $\mathcal{M}$ is countably iterable above 0.

Remark. In the following when we say "$\mathcal{N}$ is elementarily embeddable into $\mathcal{M}$" in the context of countable k-iterability, we mean that $\mathcal{N}$ is a premouse and there is weak k-embedding $\pi\colon \mathcal{N} \to \mathfrak{C}_k(\mathcal{M})$.

For example if $\mathcal{M}$ is a premouse and Σ is an (k,ω_1,ω_1+1)-iteration strategy for $\mathcal{M}$, then one can show that $\mathcal{M}$ is countably k-iterable. In fact, each premouse $\mathcal{N}$ which is weakly k-elementarily embeddable into some (k,θ)-iterable (resp. (k,α,θ)-iterable) premouse $\mathcal{M}$ is also (k,θ)-iterable (resp. (k,α,θ)-iterable) via the so-called "pullback iteration strategy".

Lemma 2.26. *Suppose $\pi\colon \mathcal{N} \to \mathcal{M}$ is a weak k-embedding and Σ is a (k,θ)-iteration strategy (resp. (k,α,θ)-iteration strategy) for $\mathcal{M}$.*

Then there is an iteration strategy Σ^π in the corresponding game for $\mathcal{N}$, the pullback iteration strategy*.*

For the proof we need the Shift Lemma.

Lemma 2.27 (Shift Lemma). *Let $\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ be premice. Suppose there is a weak 0-embedding $\psi\colon \bar{\mathcal{N}} \to \mathcal{N}$ and a weak k-embedding $\pi\colon \bar{\mathcal{M}} \to \mathcal{M}$. Let*

[11] Here $\mathfrak{C}_k(\mathcal{M})$ is the k^{th} core of $\mathcal{M}$ as defined in [MS94b, Definition 2.8.1].

$\bar{F}$ be the top-extender of $\bar{\mathcal{N}}$, F the top-extender of $\mathcal{N}$ and let $\bar{\kappa} := cr(\bar{F})$, $\kappa := cr(F)$. Suppose

$$\bar{\mathcal{M}}|(\bar{\kappa}^+)^{\bar{\mathcal{M}}} = \bar{\mathcal{N}}|(\bar{\kappa}^+)^{\bar{\mathcal{M}}} \text{ and } (\bar{\kappa}^+)^{\bar{\mathcal{M}}} \leq (\bar{\kappa}^+)^{\bar{\mathcal{N}}}$$

and

$$\mathcal{M}|(\kappa^+)^{\mathcal{M}} = \mathcal{N}|(\kappa^+)^{\mathcal{M}} \text{ and } (\kappa^+)^{\mathcal{M}} \leq (\kappa^+)^{\mathcal{N}}$$

Suppose further that $\pi\restriction(\bar{\kappa}^+)^{\bar{\mathcal{M}}} = \psi\restriction(\bar{\kappa}^+)^{\bar{\mathcal{N}}}$, $\kappa < \rho_k(\bar{\mathcal{M}})$ (so that $\mathrm{ult}_k(\bar{\mathcal{M}}, \bar{F})$ and $\mathrm{ult}_k(\mathcal{M}, F)$ make sense), and that $\mathrm{ult}_k(\mathcal{M}, F)$ is wellfounded.

Then the ultrapower $\mathrm{ult}_k(\bar{\mathcal{M}}, \bar{F})$ is also wellfounded and there is a unique embedding $\sigma\colon \mathrm{ult}_k(\bar{\mathcal{M}}, \bar{F}) \to \mathrm{ult}_k(\mathcal{M}, F)$ satisfying the following conditions:

1. *σ is a weak k-embedding,*
2. *$\mathrm{ult}_k(\bar{\mathcal{M}}, \bar{F})|lh(\bar{F}) = \bar{\mathcal{N}}|lh(\bar{F})$ and $\mathrm{ult}_k(\mathcal{M}, F)|lh(F) = \mathcal{N}|lh(F)$,*
3. *$\sigma\restriction lh(\bar{F})+1 = \psi\restriction lh(\bar{F})+1$, and*
4. *the diagram*

$$\begin{tikzcd}
\mathrm{ult}_k(\bar{\mathcal{M}}, \bar{F}) \arrow[r, "\sigma"] & \mathrm{ult}_k(\mathcal{M}, F) \\
\bar{\mathcal{M}} \arrow[u, "i"] \arrow[r, "\pi"] & \mathcal{M} \arrow[u, "j"']
\end{tikzcd}$$

commutes where i and j are the canonical ultrapower embeddings.

A proof of the Shift Lemma can be found in [MS94b, Lemma 5.2]. In the representative case $k = 0$, the desired map σ is given by:

$$\sigma([a, f]^{\bar{\mathcal{M}}}_{\bar{F}}) := [\psi(a), \pi(f)]^{\mathcal{M}}_{F}$$

It is clear that if σ shall satisfy conditions 3 and 4, then it has to be defined in this way.

For the proof of Lemma 2.26 let $\pi\colon \mathcal{N} \to \mathcal{M}$ be a weak k-embedding. If $\mathcal{T}$ is an iteration tree on $\mathcal{N}$, we can use the embedding π to construct a tree $\pi\mathcal{T}$ on $\mathcal{M}$ which has the same tree order, drop structure, and degree structure as $\mathcal{T}$. We define the models of $\pi\mathcal{T}$ on $\mathcal{M}$ by induction, together with embeddings $\pi_\alpha\colon \mathcal{N}^{\mathcal{T}}_\alpha \to \mathcal{M}_\alpha$, where $\mathcal{M}_\alpha$ will be the α^{th} model of $\pi\mathcal{T}$.

Suppose we inductively have

1. π_α is a weak $deg^{\mathcal{T}}(\alpha)$-embedding,

2. if $\beta < \alpha$ and F is the last extender of $\mathcal{N}^{\mathcal{T}}_\beta$, then $\pi_\beta \restriction lh(F) = \pi_\alpha \restriction lh(F)$, and

3. if $\beta <_{\mathcal{T}} \alpha$ and $D^{\mathcal{T}} \cap (\beta, \alpha]_{\mathcal{T}} = \emptyset$, then the diagram

$$\begin{tikzcd}
\mathcal{M}_\beta \arrow[r, "e^{\pi\mathcal{T}}_{\beta,\alpha}"] & \mathcal{M}_\alpha \\
\mathcal{N}^{\mathcal{T}}_\beta \arrow[u, "\pi_\beta"] \arrow[r, "e^{\mathcal{T}}_{\beta,\alpha}"] & \mathcal{N}^{\mathcal{T}}_\alpha \arrow[u, "\pi_\alpha"']
\end{tikzcd}$$

   commutes.

We define $\mathcal{M}_{\alpha+1}$ and $\pi_{\alpha+1}$ by applying the Shift Lemma. For this we set $\bar{\mathcal{N}} := \mathcal{N}^{\mathcal{T}}_\alpha \| lh(E^{\mathcal{T}}_\alpha)$, $\psi := \pi_\alpha \restriction \bar{\mathcal{N}}$, $\bar{\mathcal{M}} := \mathcal{N}^{\mathcal{T}*}_{\alpha+1}$, and $\pi := \pi_\beta \restriction \bar{\mathcal{M}}$, where $\beta := pred_{\mathcal{T}}(\alpha+1)$ is the $\mathcal{T}$-predecessor of $\alpha + 1$. If the ultrapower giving rise to $\mathcal{M}^{\pi\mathcal{T}}_{\alpha+1}$ is illfounded, we stop the construction. Otherwise it is easy to verify the induction hypotheses so we can continue.

Now let $\lambda < lh(\mathcal{T})$ be a limit ordinal, and let $\mathcal{M}^{\pi\mathcal{T}}_\lambda$ be the transitive collapse of the direct limit of the system $(\mathcal{M}^{\pi\mathcal{T}}_\alpha : \alpha \in [0, \lambda)_{\mathcal{T}}$ sufficiently large$)$ if the direct limit is wellfounded. Otherwise we stop the construction.

Proof of 2.26. Let Σ be the iteration strategy of $\mathcal{M}$. We define Σ^π by saying when an iteration tree $\mathcal{T}$ on $\mathcal{N}$ is built according to Σ^π:

$$\mathcal{T} \text{ is by } \Sigma^\pi \quad \Leftrightarrow \quad \pi\mathcal{T} \text{ is built according to } \Sigma$$

The construction of $\pi\mathcal{T}$ ensures that each initial segment of $\pi\mathcal{T}$ is built according to the rules of the iteration game, so II does not lose because of illfoundedness of some successor model $\mathcal{N}^{\mathcal{T}}_{\alpha+1}$. At limit steps λ we inductively have that $\pi\mathcal{T} \restriction \lambda$ is built according to Σ. So let $b \subseteq \lambda$ be the branch in $\pi\mathcal{T} \restriction \lambda$ chosen by Σ. Since the tree order, drop structure, and degree structure of $\mathcal{T} \restriction \lambda$ and $\pi\mathcal{T} \restriction \lambda$ are the same we can also set $[0, \lambda)_{\mathcal{T}} := b$. We have that $\mathcal{N}^{\mathcal{T}}_\lambda$ is wellfounded, because $\mathcal{M}^{\pi\mathcal{T}}_\lambda$ is wellfounded and $\pi_\lambda : \mathcal{N}^{\mathcal{T}}_\lambda \to \mathcal{M}^{\mathcal{T}}_\lambda$ is an elementary embedding. $\square$

Lemma 2.28 (The Comparison Lemma). *Let $\mathcal{M}$ and $\mathcal{N}$ be k-sound premice of size $\leq \theta$, and suppose Σ and Γ are $(k, \theta^+ + 1)$-iteration strategies for $\mathcal{M}$ and $\mathcal{N}$, respectively.*

Then there are iteration trees $\mathcal{T}$ and $\mathcal{S}$, played according to Σ and Γ respectively, with last models $\mathcal{M}^{\mathcal{T}}_\alpha$ and $\mathcal{N}^{\mathcal{S}}_\eta$ such that either

1. *$[0, \alpha]_{\mathcal{T}}$ does not drop in model or degree, and $\mathcal{M}^{\mathcal{T}}_\alpha$ is an initial segment of $\mathcal{N}^{\mathcal{S}}_\eta$, or*
2. *$[0, \eta]_{\mathcal{S}}$ does not drop in model or degree, and $\mathcal{N}^{\mathcal{S}}_\eta$ is an initial segment of $\mathcal{M}^{\mathcal{T}}_\alpha$.*

The Comparison Lemma can be used to show the following useful result:

Lemma 2.29. *Suppose* ZF *holds. Let $\mathcal{M}$ and $\mathcal{N}$ be ω-sound A-mice, and let $\alpha \leq \min\{ht(\mathcal{M}), ht(\mathcal{N})\}$ be such that $\mathcal{M}\|\alpha = \mathcal{N}\|\alpha$, α is a cutpoint*[12] *in both $\mathcal{M}$ and $\mathcal{N}$, and $\rho_\omega(\mathcal{M}) \leq \alpha$, $\rho_\omega(\mathcal{N}) \leq \alpha$.*

Then $\mathcal{M}$ and $\mathcal{N}$ are compatible, i. e. either $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{M} \trianglerighteq \mathcal{N}$.

In particular this holds true if $\mathcal{M}$ and $\mathcal{N}$ are ω-sound A-mice such that $\rho_\omega(\mathcal{M}) = \rho_\omega(\mathcal{N}) = \sup(A)$.

Proof Sketch. This lemma in ZFC is essentially a corollary of the proof of the Comparison Lemma and can be found in [Steb, Corollary 3.12]. In the proof of the Comparison Lemma one uses a reflection argument for which it has to be possible to build elementary substructures of large initial segments of the universe. Since without choice this is impossible, we have to go into some inner model of ZFC. We just sketch the difference to the ZFC case.

So let $\mathcal{M}$ and $\mathcal{N}$ be ω-sound A-mice as stated in the lemma. Then consider $\mathsf{L}[\mathcal{M}, \mathcal{N}]$ which is a model of choice. Let Ω be large, let $\mathcal{X} \prec V_\Omega^{\mathsf{L}[\mathcal{M},\mathcal{N}]}$ be a countable substructure containing $\mathcal{M}$ and $\mathcal{N}$, and

$$\pi \colon H \to \mathsf{L}[\mathcal{M}, \mathcal{N}], \quad \pi(\mathcal{M}', \mathcal{N}') = (\mathcal{M}, \mathcal{N}),$$

where π is the uncollapsing map.

[12] α is a cutpoint in a premouse $\mathcal{M}$ if there is no extender E on the $\mathcal{M}$-sequence such that $cr(E) < \alpha \leq lh(E)$.

Since $\mathcal{M}'$ and $\mathcal{N}'$ are elementarily embeddable into $\mathcal{M}$ and $\mathcal{N}$ respectively, there are $\omega_1 + 1$-iteration strategies Σ and Γ for $\mathcal{M}'$ and $\mathcal{N}'$, respectively. So $\mathcal{M}'$ and $\mathcal{N}'$ are coiterable in $\mathsf{L}[\mathcal{M}, \mathcal{N}, \Sigma, \Gamma]$ which is a model of ZFC. Now we can show that $\mathcal{M}$ and $\mathcal{N}$ are compatible as in the proof of [Steb, Corollary 3.12] □

Definition 2.30. Let $\mathcal{M}$ be a premouse. We say $\mathcal{M}$ is *tame* iff no local Woodin cardinal is overlapped, i. e. whenever E is an extender from the $\mathcal{M}$-sequence and $\lambda = lh(E)$, then

$$\mathcal{M}\|\lambda \models \forall\delta \geq cr(E)\ \delta \text{ is not Woodin.}$$

Suppose there is no non-tame premouse. Then we can describe the iteration strategies for many premice by means of so-called $\mathcal{Q}$-structures.

Definition 2.31. Let $\mathcal{T}$ be an iteration tree of limit length on a premouse $\mathcal{M}$. A *$\mathcal{Q}$-structure for $\mathcal{T}$* is a premouse $\mathcal{Q}$, such that

1. $\mathcal{M}(\mathcal{T}) \trianglelefteq \mathcal{Q}$, such that $\delta(\mathcal{T})$ is a cutpoint of $\mathcal{Q}$.
2. $\mathcal{Q}$ is countably iterable above $\delta(\mathcal{T})$.
3. $\mathcal{Q}$ kills the Woodin property at some $k < \omega$[13], i. e.
   (a) $\mathcal{Q}$ is $k+1$-sound, and
   (b) either $\rho_{k+1}(\mathcal{Q}) < \delta(\mathcal{T})$, or there is an $f\colon \delta(\mathcal{T}) \to \delta(\mathcal{T})$ which is $r\Sigma_{k+1}^{\mathcal{Q}}$ such that for no extender E on the $\mathcal{Q}$-sequence we have $i_E(f)(cr(E)) \geq \nu(E)$, but there is no such f which is $r\Sigma_k^{\mathcal{Q}}$, i. e. $k+1$ is least such that there is a $r\Sigma_{k+1}^{\mathcal{Q}}$-definable counterexample for $\delta(\mathcal{T})$ to be Woodin.

If there is no non-tame premouse, then a comparison argument shows that there is at most one $\mathcal{Q}$-structure for $\mathcal{T}$. If the $\mathcal{Q}$-structure for $\mathcal{T}$ exists we denote it by $\mathcal{Q}(\mathcal{T})$. Otherwise we leave $\mathcal{Q}(\mathcal{T})$ undefined.

Definition 2.32. Suppose $\mathcal{T}$ is a k-maximal iteration tree of limit length and b is a wellfounded cofinal branch of $\mathcal{T}$. Let $\mathcal{Q}(b, \mathcal{T})$ be the least initial segment of $\mathcal{M}_b^{\mathcal{T}}$ such that either $\rho_\omega(\mathcal{Q}(b, \mathcal{T})) < \delta(\mathcal{T})$ or $\mathcal{Q}(b, \mathcal{T})$ defines a failure of $\delta(\mathcal{T})$ to be Woodin as in the definition of the $\mathcal{Q}$-structure for $\mathcal{T}$. If there is no such initial segment let $\mathcal{Q}(b, \mathcal{T})$ undefined.

[13] Cf. [Ste02, Definition 2.1]

Definition 2.33. The *$\mathcal{Q}$-structure iteration strategy* is the partial iteration strategy picking the unique branch b through $\mathcal{T}$ coming with a $\mathcal{Q}$-structure, i. e. $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$.

One can show

Lemma 2.34. *Let $\mathcal{M}$ be a tame premouse such that*

$$\mathcal{M} \models \text{“there are no Woodin cardinals” or } \rho_\omega(\mathcal{M}) < \delta,$$

where δ is $\mathcal{M}$'s least Woodin cardinal if there is one. If θ is an infinite cardinal and $\mathcal{M}$ is $\theta^+ + 1$-iterable, then the $\mathcal{Q}$-structure iteration strategy is the unique $\theta^+ + 1$-iteration strategy.

In particular we get:

Lemma 2.35. *Let $\mathcal{M}$ be an A-mouse such that $\rho_\omega(\mathcal{M}) = \sup(A)$, and let $\mathcal{N}$ be countable and elementarily embeddable into $\mathcal{M}$.*

Then $\mathcal{N}$ is $\omega_1 + 1$-iterable via the $\mathcal{Q}$-structure iteration strategy.

The lower part model

Definition 2.36. We define the *lower part closure of a set A* inductively by

$$\begin{aligned}
\mathcal{M}_1(A) &= \bigtriangledown\{\mathcal{M} : \mathcal{M} \text{ is an } \omega\text{-sound } A\text{-mouse such that} \\
&\qquad\qquad \rho_\omega(\mathcal{M}) = \sup(A)\} \\
\mathcal{M}_{\alpha+1}(A) &= \bigtriangledown\{\mathcal{M} : \mathcal{M} \text{ is an } \omega\text{-sound } A\text{-mouse such that} \\
&\qquad\qquad \mathcal{M}_\alpha(A) \lhd \mathcal{M},\ \rho_\omega(\mathcal{M}) \leq ht(\mathcal{M}_\alpha(A)), \\
&\qquad\qquad \text{and } ht(\mathcal{M}_\alpha(A)) \text{ is a cutpoint in } \mathcal{M}\} \\
\mathcal{M}_\lambda(A) &= \bigtriangledown\{\mathcal{M}_\alpha(A) : \alpha < \lambda\} \text{ for } \lambda \text{ limit} \\
\mathsf{Lp}(A) &= \bigtriangledown\{\mathcal{M}_\alpha(A) : \alpha \in \mathsf{On}\}
\end{aligned}$$

Note that this is also well defined in a choiceless world, because if $\mathcal{M}$ and $\mathcal{N}$ are two candidates for being an initial segment of some $\mathcal{M}_{\alpha+1}(A)$, we don't need choice to prove that $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$; cf. the proof of Lemma 2.29.

Lemma 2.37. *1. For all α, we have that $\mathcal{M}_\alpha(A)$ is an ω-sound A-mouse.*

2. $\{ht(\mathcal{M}_\gamma(A)) : \gamma < \alpha\}$ *is the set of cardinals of* $\mathcal{M}_\alpha(A)$ *above* $\sup(A)$.

If W is an inner model with $A \in W$, we let $\mathsf{Lp}^W(A)$ be the lower part closure of A built in W. We drop A whenever it is possible.

Remark. Lp is a lower part model, i. e. if $\vec{E}$ is the extender sequence of Lp, then for every ordinal α, E_α is not a total extender over Lp. In other words no cardinal in Lp is measurable witnessed by an extender on $\vec{E}$. The reason is that each cardinal in Lp is a cutpoint in Lp, so if E_α were an extender which witnesses the measurability of some κ, then it appears on the sequence before $\kappa^{+\mathsf{Lp}}$, and therefore the extender is not total on Lp.

Capturing terms

Definition 2.38. Let $\mathcal{M}$ be a countable premouse, $\delta \in \mathcal{M}$ an $\mathcal{M}$-cardinal, and $A \subseteq \mathbb{R}$ a set of reals.

A term $\tau \in \mathcal{M}^{\mathrm{Col}(\omega,\delta)}$ *weakly captures* A *over* $\mathcal{M}$ iff whenever $G \in \mathsf{V}$ is $\mathrm{Col}(\omega,\delta)$-generic over $\mathcal{M}$, then $\tau^G = A \cap \mathcal{M}[G]$ holds true.

Suppose further that there is an ω_1-iteration strategy Σ for $\mathcal{M}$ such that for every countable simple Σ-iterate $\mathcal{M}^\star$ of $\mathcal{M}$[14] with iteration map π we have that $\pi(\tau)$ weakly captures A over $\mathcal{M}^\star$. We then say τ *captures* A *over* $\mathcal{M}$.

The proof of the next lemma can be found in [SSc].

Lemma 2.39. *Let* $\mathcal{M}$ *be a countable premouse,* Σ *an* $\omega_1{+}1$*-iteration strategy, and let* $\delta < \eta$ *be such that*

$$\mathcal{M} \models \textit{both } \delta \textit{ and } \eta \textit{ are Woodin cardinals}$$

Let further $B \subseteq \mathbb{R}\times\mathbb{R}$ *and suppose* $\tau \in \mathcal{M}^{\mathrm{Col}(\omega,\eta)}$ *captures* B *over* $\mathcal{M}$.

Then there is a $\sigma \in \mathcal{M}^{\mathrm{Col}(\omega,\delta)}$ *such that* σ *captures the set* $\exists^{\mathbb{R}}B$*, where* $\exists^{\mathbb{R}}B := \{x \in \mathbb{R} : \exists y \in \mathbb{R}\ (x,y) \in B\}$.

From this we get

[14] I. e. there is a countable iteration tree on $\mathcal{M}$ played according to Σ with last model $\mathcal{M}^\star$ and there are no drops on the main branch.

Lemma 2.40. *Let $[\alpha, \beta]$ be a Σ_1-gap and $n < \omega$. Suppose that either*

1. *$\alpha < \beta$, $[\alpha, \beta]$ is weak, and n is least such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$, or*
2. *$\alpha = \beta$ is $\mathbb{R}$-inadmissible and $n = 1$.*

Let $\mathcal{M}$ be a countable $\omega_1 + 1$-iterable premouse and $(\delta_i : i \leq 2k)$ be a descending sequence of Woodin cardinals in $\mathcal{M}$. Suppose further that there is a universal $\undertilde{\Sigma}_n(J_\beta(\mathbb{R}))$-set which is captured by $\tau \in \mathcal{M}^{\mathrm{Col}(\omega, \delta_0)}$.

Then there is also a universal $\undertilde{\Sigma}_{n+2k}(J_\beta(\mathbb{R}))$-set which is captured by a term $\sigma \in \mathcal{M}^{\mathrm{Col}(\omega, \delta_{2k})}$.

Proof sketch. This follows from [Ste83] and the lemma above.

If 1 holds, then there is a $\undertilde{\Sigma}_n(J_\beta(\mathbb{R}))$ partial surjection $f \colon \mathbb{R} \to J_\beta(\mathbb{R})$; condition 2 ensures the existence of a partial surjection $f \colon \mathbb{R} \to J_\alpha(\mathbb{R})$ with $f \in \Sigma_1(J_\alpha(\mathbb{R}))$. In both cases the function f can be used to show that for each $k < \omega$ the following equality holds (cf. [Ste83, Lemma 2.5, Proof of Corollary 3.9]):

$$\undertilde{\Sigma}_{n+k+1}(J_\beta(\mathbb{R})) \cap \mathrm{Pow}(\mathbb{R}) = \exists^{\mathbb{R}}(\undertilde{\Pi}_{n+k}(J_\beta(\mathbb{R}))) \cap \mathrm{Pow}(\mathbb{R})$$

It follows that if U is a universal $\undertilde{\Sigma}_{n+2(k-1)}(J_\beta(\mathbb{R}))$-set of reals, then $\exists^{\mathbb{R}}\forall^{\mathbb{R}}U$ is a universal set for $\undertilde{\Sigma}_{n+2k}(J_\beta(\mathbb{R}))$. So if $\tau \in \mathcal{M}^{\mathrm{Col}(\omega, \delta_{n+2(k-1)})}$ is a term capturing U, then we can apply the lemma above twice to get a capturing term $\sigma \in \mathcal{M}^{\mathrm{Col}(\omega, \delta_{n+2k})}$ for $\exists^{\mathbb{R}}\forall^{\mathbb{R}}U$. $\square$

Definition 2.41. Let Γ be a class of subsets of $\mathbb{R}$ and $x \in \mathbb{R}$. Then let $C_\Gamma(x)$ *be the set of reals which are Γ in a countable ordinal and the parameter x*, i. e. $a \in C_\Gamma(x) \Leftrightarrow$

$$\exists \xi < \omega_1 \, \exists A \in \Gamma \, \forall z \in \mathit{Code}(\xi) \; a \text{ is unique such that } (a, z, x) \in A$$ [15]

We further set $C_\Gamma := C_\Gamma(\emptyset)$.

For example, if $x \in \mathbb{R}$ then the sets of reals which are Σ^1_2 in a countable ordinal and parameter x are exactly the reals in $\mathsf{L}[x]$, $C_{\Sigma^1_2}(x) = \mathbb{R} \cap \mathsf{L}[x]$.

[15] For a countable ordinal ξ, $\mathit{Code}(\xi)$ is the set of reals which code ξ.

More generally $C_{\Sigma^1_{2n+2}}(x)$ is the set of reals of the least inner model over x containing $2n$ Woodin cardinal, i. e. $C_{\Sigma^1_{2n+2}}(x) = \mathbb{R} \cap M_{2n}(x)$.

For "good" pointclasses Γ the existence of a capturing term already implies the closure under C_Γ.

Definition 2.42. Let Γ be some pointclass. We call Γ *good* iff it is ω-parameterized, closed under recursive substitution, and closed under $\exists^{\mathbb{R}}$, i. e. $\exists^{\mathbb{R}}\Gamma := \{\exists^{\mathbb{R}}B : B \in \Gamma\} \subseteq \Gamma$.

For example, each pointclass of the form $\Sigma_n(J_\alpha(\mathbb{R}))$ is good.

Lemma 2.43. *Let Γ be a good pointclass. Suppose $\mathcal{M}$ is a countable, ω_1-iterable premouse which contains a capturing term for a universal Γ-set.*

Then $\mathcal{M}$ is closed under C_Γ, i. e. for all $x \in \mathcal{M} \cap \mathbb{R}$ we have $C_\Gamma(x) \subseteq \mathcal{M}$.

Proof. Let $x \in \mathcal{M} \cap \mathbb{R}$ and let $a \in C_\Gamma(x)$ be determined by $\xi < \omega_1$ and $A \in \Gamma$. So for each $z \in \mathit{Code}(\xi)$ we have that $a \in \mathbb{R}$ is the unique real such that $(a, z, x) \in A$. If we define a set A' by

$$(n, m, z, x) \in A' \quad \Leftrightarrow \quad \exists y \in \mathbb{R}\ \big((y, z, x) \in A \wedge y(n) = m\big),$$

then $A' \in \Gamma$. So $a(n) = m \Leftrightarrow (n, m, z, x) \in A'$ for all $z \in \mathit{Code}(\xi)$. Let $\tau \in \mathcal{M}^{\mathrm{Col}(\omega,\delta)}$ be a capturing term for a universal Γ-set U. Since $A' \in \Gamma$ and U is universal, we can find an integer k such that for each $y \in \mathbb{R}$: $y \in A' \Leftrightarrow (k, y) \in U$.

Now we can iterate $\mathcal{M}$ until we reach some premouse $\mathcal{N}$ such that for the iteration map $\pi\colon \mathcal{M} \to \mathcal{N}$ the inequality $\xi \leq \pi(\delta)$ holds. Let $G \in \mathsf{V}$ be $\mathrm{Col}(\omega, \pi(\delta))$-generic over $\mathcal{N}$. So we have $\pi(\tau)^G = U \cap \mathcal{N}[G]$. ξ is countable in $\mathcal{N}[G]$, so let $z \in \mathcal{N}[G]$ be a real coding ξ. Now we can define a in $\mathcal{N}[G]$ by

$$a(n) = m \quad \Leftrightarrow \quad (k, (n, m, z, x)) \in U \quad \Leftrightarrow \quad (k, (n, m, z, x)) \in \pi(\tau)^G.$$

But if G and G' are mutually generic over $\mathcal{N}$, then $a \in \mathcal{N}[G] \cap \mathcal{N}[G']$. It follows that $a \in \mathcal{N}$ and therefore $a \in \mathcal{M}$, because iterations don't add reals. $\square$

Weakly compact cardinals

Definition 2.44. The symbol $\delta \to (\alpha)^{<\omega}$ (resp. $\delta \to (\alpha)^k$) denotes the property that for every partition F of the set $[\delta]^{<\omega}$ (resp. $[\delta]^k$)[16] into two pieces, there exists a set $H \subseteq \delta$ of order type α such that F is constant on $[H]^n$ for each $n < \omega$ (resp. F is constant on $[H]^k$).

A cardinal $\delta > \omega$ is called *weakly compact* iff $\delta \to (\delta)^2$. $\delta > \omega$ is called a *Ramsey cardinal* iff $\delta \to (\delta)^{<\omega}$.

We want to prove Theorem 1.4 using Theorem 1.3. The latter one ensures the existence of Ramsey cardinals, which are of course weakly compact.

We need the following lemmata whose proofs can be found in [Sch99].

Lemma 2.45. *Suppose* ZF. *Let δ be weakly compact. Then the following hold.*

1. *δ is regular.*
2. *δ has the tree property, i. e. there is no δ-Aronszajn tree.*[17]
3. *For no $\alpha < \delta$ there is an injection $f\colon \delta \to {}^{\alpha}2$.*
4. *δ is inaccessible in any inner model of* ZFC.

Lemma 2.46. *Let δ be weakly compact. For every inner model W such that $\overline{\overline{\mathrm{Pow}(\delta) \cap W}} = \delta$ there is a countably complete ultrafilter U with:*

$$(W; \in, U) \models U \text{ is a } \delta\text{-complete normal ultrafilter on } \delta.$$

Using this lemma we get the following:

Lemma 2.47. *Let δ and δ^+ be weakly compact cardinals, and let $\mathcal{T}$ be an iteration tree of length δ.*

Then there is a unique cofinal wellfounded branch through $\mathcal{T}$.

[16] $[\delta]^k = \{A \subseteq \delta : \overline{\overline{A}} = k\}$, $[\delta]^{<\omega} = \bigcup_{k\in\omega} [\delta]^k$

[17] A δ-tree T for which $\overline{\overline{T_\alpha}} < \delta$ for all $\alpha < \delta$ holds, and which has no cofinal branches is called a δ-Aronszajn tree.

Proof. The uniqueness and the wellfoundedness of such a branch are clear, since δ is regular of uncountable cofinality.

For the existence consider the model $W := \mathsf{HOD}[\mathcal{T}]$. Since δ^+ is inaccessible in W, it satisfies the assumption of Lemma 2.46. Let U be an ultrafilter given by the lemma. We can form the ultrapower $\mathrm{ult}(W, U)$ (which is wellfounded by countable completeness) and identify it with the transitive collapse. Now consider the usual ultrapower embedding:

$$\pi\colon W \to \mathrm{ult}(W, U), \quad cr(\pi) = \delta$$

Then $\pi(\mathcal{T})$ is an iteration tree of length $\pi(\delta)$ such that $\pi(\mathcal{T})\restriction\delta = \mathcal{T}$. But then $[0, \delta)_{\pi(\mathcal{T})}$ is a branch according to the rules of the iteration game, so $[0, \delta)_{\pi(\mathcal{T})}$ is $<_{\mathcal{T}}$-cofinal in δ and therefore a wellfounded cofinal branch through $\mathcal{T}$. $\square$

2.2 Defining the model

In this section we define the model for which we prove $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$. We need to specify the choice of μ and X from the Theorems 1.4 and 1.5. So for the rest of the section suppose that V is a model of ZF such that either

1. every uncountable successor cardinal is weakly compact and every uncountable limit cardinal is singular, or
2. every uncountable cardinal is singular.

Note that in both cases we have that each inner model of ZFC does not compute the cardinal successor of an ordinal $\geq \omega$ correctly. In the case where each uncountable successor cardinal is weakly compact, this follows from Lemma 2.45, since weakly compact cardinals are inaccessible in inner models of ZFC. If in contrast every $\delta > \omega$ is singular, then we cannot show that δ is inaccessible in any inner model of ZFC, but we can show that any singular δ is a limit cardinal in any inner model of ZFC.

Lemma 2.48. *Suppose* ZF *holds and* δ *is an uncountable singular cardinal.*

Then δ *is a limit cardinal in each inner* ZFC *model* W*.*

Proof. Let B be a set of ordinals coding V_δ^W and $A \subseteq \delta$ be cofinal of order type $\gamma := \mathrm{cof}(\delta)$. Then A is Vop_δ-generic over HOD_B. But δ has cofinality

$\gamma < \delta$ also in $\mathsf{HOD}_B[A]$ and therefore δ is singular in $\mathsf{HOD}_B[A]$. Hence, since it is a cardinal in V and therefore in $\mathsf{HOD}_B[A]$, δ is a limit cardinal in $\mathsf{HOD}_B[A]$.[18]

So there are cofinally many $\mathsf{HOD}_B[A]$-cardinals less than δ, which are of course also cardinals in $V_\delta^W \subseteq \mathsf{HOD}_B[A]$. But if $\xi < \delta$ were a cardinal in V_δ^W but not in W, then there would be a function in $V_{\xi+1}^W \subseteq V_\delta^W$ witnessing this fact. This is a contradiction, so δ is a limit cardinal in W. □

Some closure cardinals

We define some closure cardinals which we need in the following.

Definition 2.49. 1. First let ε be a V-cardinal larger than $\Theta(\omega_1)$. Suppose B is a set of ordinals and A is a subset of ω_1. Then A is Vopěnka-generic over HOD_B. But since Vop_{ω_1} has an ordinal $< \Theta(\omega_1)$ as support, we have that each dense subset of Vop_{ω_1} in HOD_B is already in $H_\varepsilon^{\mathsf{HOD}_B}$ and therefore A is Vop_{ω_1}-generic also over $H_\varepsilon^{\mathsf{HOD}_B}$.

Moreover $H_\varepsilon^{\mathsf{HOD}_B}$ contains the partial functions $\Sigma\colon \mathrm{Pow}(\omega_1) \to \mathrm{Pow}(\omega_1)$ which are in HOD_B. In particular every $\omega_1 + 1$-iteration strategy for a countable premouse in HOD_B is already in $H_\varepsilon^{\mathsf{HOD}_B}$. So if we consider a premouse over a set which codes $H_\varepsilon^{\mathsf{HOD}_B}$ (and more) for some specific B, then all countable iteration trees and all countable elementary substructures of any inner model are generic over that premouse by Vopěnka's theorem.

2. Let ζ be a Θ-closed V-cardinal larger than $\Theta(\gamma)$, where $\gamma := \varepsilon^+$. So ζ is larger than $\Theta(\gamma)$ computed in any HOD_B, and therefore $H_\zeta^{\mathsf{HOD}_B}$ contains in particular each dense subset of $\mathrm{Col}(\omega, \varepsilon)$ which is in HOD_B. So for each g the following is true:

   (a) g is $\mathrm{Col}(\omega, \varepsilon)$-generic over $H_\zeta^{\mathsf{HOD}_B}$ iff g is $\mathrm{Col}(\omega, \varepsilon)$-generic over HOD_B, and

   (b) if g is $\mathrm{Col}(\omega, \varepsilon)$-generic over HOD_B then for $\tilde{\omega}_1 := \omega_1^{\mathsf{HOD}_B[g]}$ we have

$$\mathrm{Pow}(\tilde{\omega}_1) \cap \mathsf{HOD}_B[g] = \mathrm{Pow}(\tilde{\omega}_1) \cap H_\zeta^{\mathsf{HOD}_B}[g]$$

[18] Successor cardinals are regular since $\mathsf{HOD}_B[A]$ is a model of ZFC.

and therefore each $\tilde{\omega}_1 + 1$-iteration strategy for a countable premouse which is in $\mathsf{HOD}_B[g]$ is already in $H_\zeta^{\mathsf{HOD}_B}[g]$.

3. In V there exists a cardinal $\tilde{\kappa}$ which is closed under the Θ-function with the following property: Let $B \in \mathsf{V}$ be a set of ordinals and $W \subseteq \mathsf{V}$ be a Vopěnka-generic extension of HOD_B. Then there exists a Θ-closed $\mu \in (\zeta, \tilde{\kappa})$ such that $\mu^\varepsilon = \mu$ in W.

   Proof. We consider the monotone enumeration $\mathfrak{e}$ of the cardinals larger than ζ which are closed under the Θ-function. Define $\tilde{\kappa} := \mathfrak{e}(\varepsilon^+)$. Now we can set $\mu := \mathfrak{e}(\varepsilon^{+W})$. Since W computes the cardinal successor of ε incorrectly we have $\varepsilon^{+W} < \varepsilon^+$, so $\mu < \tilde{\kappa}$. The function $\mathfrak{e}$ is ordinal definable, so it is an element of $\mathsf{HOD}_B \subseteq W$ and therefore we have that $W \models \operatorname{cof}(\mu) = \varepsilon^+$. Then the following holds in W:

$$\mu^\varepsilon = \overline{\overline{{}^\varepsilon\mu}} = \overline{\overline{\bigcup_{\alpha<\varepsilon} {}^\varepsilon\alpha}} \leq \sum_{\alpha<\mu} \alpha^\varepsilon = \mu \cdot \sup_{\alpha<\mu} \alpha^\varepsilon \leq \mu \cdot \sup_{\alpha<\mu} \Theta(\alpha) = \mu \quad [19]$$

   So $\tilde{\kappa}$ and μ are as desired. □

   The properties of μ enable us to build in W elementary submodels of V_Ω^W, for Ω large enough, which have size μ and which are closed under ε-sequences.

4. Now define inductively an ascending sequence $(\kappa_i : i < \omega)$ such that each κ_i fulfills condition 3. Let $\kappa := \sup\{\kappa_i : i < \omega\}$. In particular, for each Vopěnka-generic extension W of some HOD_B there are cofinally many cardinals $\mu < \kappa$ such that $\mu^\varepsilon = \mu$ holds in W.

The following definitions are relevant for defining the models in which we want to prove $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$.

[19] Here the second equality holds because μ has cofinality larger than ε, and the fifth holds since $\alpha^\varepsilon \leq \alpha^\alpha = 2^\alpha < \Theta(\alpha)$ for α large. Finally the last equality holds since μ is closed under Θ in V and therefore in each inner model.

The model for Theorem 1.4

Definition 2.50. Suppose V is a model of ZF in which every uncountable successor cardinal is weakly compact and each limit cardinal is singular.

Let κ be as in Definition 2.49. Suppose $A_0 \in \mathsf{HOD} \cap \mathrm{Pow}(\kappa)$ codes H_κ^{HOD} in some simple way. Since κ is closed under Θ it is a strong limit cardinal in HOD, and therefore such an $A_0 \subseteq \kappa$ exists. We build in V the lower part model $\mathsf{Lp}(A_0) = \mathsf{Lp}^{\mathsf{V}}(A_0)$ and consider

$$\lambda := \kappa^{+\mathsf{Lp}(A_0)}.$$

By Lemma 2.45 we have that κ^+ is inaccessible in $\mathsf{Lp}(A_0)$ so $\lambda < \kappa^+$. But then $\mathrm{cof}(\lambda) < \kappa$ since κ is singular ($\mathrm{cof}(\kappa) = \omega$). Let

$$X \subseteq \lambda \text{ be cofinal of order type } \mathrm{cof}(\lambda) < \kappa.$$

Now by the choice of κ we can fix a

$$\mu < \kappa \text{ such that } \mu > \mathrm{cof}(\lambda) \text{ and } \mu^{\varepsilon} = \mu \text{ holds in } \mathsf{HOD}_X.$$

So our Theorem 1.4 is:

Theorem 1.4. *Suppose V is a model of ZF such that each uncountable successor cardinal is weakly compact and each uncountable limit cardinal is singular.*

Then $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ holds in $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu^{+\mathsf{V}})}$.

The model for Theorem 1.5

Definition 2.51. Now let V be as in Theorem 1.5, i. e. V is a model of ZF in which every uncountable cardinal is singular.

As before let κ be as in Definition 2.49 and let $A_0 \in \mathsf{HOD} \cap \mathrm{Pow}(\kappa)$ be a set of ordinals which codes H_κ^{HOD} in a simple way. Then we build the lower part model $\mathsf{Lp}(A_0) = \mathsf{Lp}^{\mathsf{V}}(A_0)$ and consider

$$\lambda := \kappa^{+\mathsf{Lp}(A_0)}.$$

This time we have $\lambda < \kappa^+$ by Lemma 2.48. Since every limit ordinal has cofinality ω we can choose an

$X \subseteq \lambda$ cofinal of order type ω.

Again by the choice of κ we can fix a

$\mu < \kappa$ such that $\mu^\varepsilon = \mu$ holds in HOD_X.

In our core model induction the witnesses that AD holds at stage α are countable mice with $\omega_1 + 1$-iteration strategies. So we need to control the generic extension at least up to its first uncountable cardinal. If we would work again with the forcing $\mathrm{Col}(\omega, < \mu^{+\mathsf{V}})$ we would have to know that $\mu^{+\mathsf{V}}$ is regular in HOD_X, but since $\mu^{+\mathsf{V}}$ is singular in V it could also be singular in HOD_X. But then $\mu^{+\mathsf{V}}$ would be also singular in $\mathsf{HOD}_X^{\mathrm{Col}(\omega, < \mu^{+\mathsf{V}})}$, i. e. $\mu^{+\mathsf{V}}$ would be no cardinal in the generic extension and therefore we would have $\omega_1^{\mathsf{HOD}_X^{\mathrm{Col}(\omega, < \mu^{+\mathsf{V}})}} > \mu^{+\mathsf{V}}$. So we cannot control where $\omega_1^{\mathsf{HOD}_X^{\mathrm{Col}(\omega, < \mu^{+\mathsf{V}})}}$ lies. Since $\mu^{+\mathsf{V}}$ is singular in V, we see no way to guarantee the regularity of it in HOD_X. Instead of working with $\mathrm{Col}(\omega, < \mu^{+\mathsf{V}})$ we therefore work with $\mathrm{Col}(\omega, < \mu^{+\mathsf{HOD}_X})$.

Then our Theorem 1.5 is:

Theorem 1.5. *Let V be a model of ZF in which each uncountable cardinal is singular.*

Then $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ holds in $\mathsf{HOD}_X^{\mathrm{Col}(\omega, < \mu^{+\mathsf{HOD}_X})}$.

2.3 The core model induction

The induction hypothesis

This section introduces the concept of so-called coarse Woodin mice. The existence of these mice is what we are looking for, because if the iteration strategy for such a mouse is an element of $J_\alpha(\mathbb{R})$, then the mouse ensures that AD holds at that stage of the $\mathsf{L}(\mathbb{R})$-hierarchy. Moreover we introduce Σ_1-witnesses. The existence of Σ_1-witnesses follows from the existence of coarse Woodin mice if we are at limit stages of the $\mathsf{L}(\mathbb{R})$-hierarchy. These mice are used to go one step higher in our core model induction.

We use the following concepts in $\mathsf{HOD}_X[g]$. All the definitions in this section are due to John R. Steel and are taken from his paper [Ste05].

Definition 2.52. Suppose $\mathcal{N}$ is countable and transitive, $U \subseteq \mathbb{R}$, and $k < \omega$. We say $\mathcal{N}$ is a *coarse* (k, U)*-Woodin mouse* iff there are $\delta_1, \ldots, \delta_k, S, T \in \mathcal{N}$ such that:

1. $\mathcal{N} \models \mathsf{ZFC} + \delta_1 < \cdots < \delta_k$ are Woodin cardinals
2. S, T are trees such that in $\mathcal{N}^{\mathrm{Col}(\omega, \delta_k)}$ the projections $\mathsf{p}[S]$ and $\mathsf{p}[T]$ are complements of each other.
3. there exists an $\omega_1{+}1$-iteration strategy Σ such that whenever $i \colon \mathcal{N} \to \mathcal{P}$ is an iteration map by Σ and $\mathcal{P}$ is countable, then $\mathsf{p}[i(S)] \subseteq U$ and $\mathsf{p}[i(T)] \subseteq \mathbb{R} \setminus U$.

Our induction hypothesis will be

($W^\star_\alpha$) Let $U \subseteq \mathbb{R}$ and suppose there are scales $\vec{\varphi}$ and $\vec{\psi}$ on U and $\mathbb{R} \setminus U$ respectively such that the associated sequences of prewellorderings $\leq_{\vec{\varphi}}$, $\leq_{\vec{\psi}}$ are elements of $J_\alpha(\mathbb{R})$. Then for all $k < \omega$ and all $x \in \mathbb{R}$ there are $\mathcal{N}, \Sigma$ such that

1. $\mathcal{N}$ is a coarse (k, U)-Woodin mouse with $x \in \mathcal{N}$ and
2. Σ is an $\omega_1{+}1$-iteration strategy of $\mathcal{N}$ and Σ, restricted to countable iteration trees, is an element of $J_\alpha(\mathbb{R})$.

If we have ($W^\star_\alpha$) for all α then we are done, because:

Lemma 2.53. *If* ($W^\star_\alpha$) *holds, then* $J_\alpha(\mathbb{R}) \models \mathsf{AD}$.

Proof. See [Ste05, Lemma 1.6]. We sketch the main idea.

By the reflecting arguments of Kechris-Solovay or Kechris-Woodin, it suffices to show that U is determined whenever U and $\mathbb{R} \setminus U$ admit scales in $J_\alpha(\mathbb{R})$. So fix a U, and let $\mathcal{N}$ be a $(1, U)$-Woodin mouse given by ($W^\star_\alpha$). We then have that $\mathcal{N} \models$ "$\mathsf{p}[S]$ is homogeneously Suslin", and hence $\mathsf{p}[S]$ is determined in $\mathcal{N}$ by a result in [Stea]. We can assume that $\mathcal{N}$ belives that τ is an iteration strategy for I in the game with payoff $\mathsf{p}[S]$. Now one can show that τ is also a winning strategy for U. For if y were a play for II such that II wins against I, then we could iterate $\mathcal{N}$ by its iteration strategy Σ, yielding $i\colon \mathcal{N} \to \mathcal{P}$, with y generic over $\mathcal{P}$. Since $\tau * y \notin U$, and $i(S)$ and $i(T)$ are absolute complements over $\mathcal{P}$, we would have $\tau * y \in \mathsf{p}[i(T)]$. But then $\mathcal{P} \models \exists y\ \tau * y \in \mathsf{p}[i(T)]$, so by elementarity of i, τ would be not a winning strategy for I for $\mathsf{p}[S]$ in $\mathcal{N}$. $\square$

We also want to have a fine structural version of $(W^\star_\alpha)$. For this we need the following lemma:

Lemma 2.54. *Let φ be a Σ_1-formula. We can associate formulae $\varphi^k, k < \omega$ to φ such that $\varphi^k \in \Sigma_k$, and*

$$J_{\gamma+1}(\mathbb{R}) \models \varphi(x) \quad \Leftrightarrow \quad \exists k\ J_\gamma(\mathbb{R}) \models \varphi^k(x)$$

for any γ and any $x \in \mathbb{R}$.

Proof. First note that $J_{\gamma+1}(\mathbb{R}) = \mathrm{rud}(J_\gamma(\mathbb{R}) \cup \{J_\gamma(\mathbb{R})\})$. Let $(f_k : k < \omega)$ be an enumeration of the rudimentary functions. So each element in $J_{\gamma+1}(\mathbb{R})$ is the image of an element $\vec{x} \in J_\gamma(\mathbb{R}) \cup \{J_\gamma(\mathbb{R})\}$ under some f_k.

Claim 1. Let $\varphi(v, \vec{w})$ be a Σ_0-formula with free variables among v and $\vec{w}$. Then there is a formula $\varphi^*(\vec{w})$ such that for each transitive U and for each $\vec{x} \in U$

$$U \cup \{U\} \models \varphi(U, \vec{x}) \quad \Leftrightarrow \quad U \models \varphi^*(\vec{x})$$

Proof. We show this by induction on φ.

If v does not occur in $\varphi(v, \vec{w})$, i.e. if $\varphi \equiv w_i = w_j$ or $\varphi \equiv w_i \in w_j$, then set $\varphi^* \equiv \varphi$. Otherwise if $\varphi \equiv w_i \in v$, we set $\varphi^* := w_i = w_i$; $\varphi \equiv v = v$ gives rise to $\varphi^* := \forall z\ z = z$. In all other atomic cases, i.e. if $\varphi \equiv v = w_i$, $\varphi \equiv v \in w_i$, or $\varphi \equiv v \in v$, set $\varphi^* := \exists z\ z \neq z$.

Now as usual let $\varphi^* := \psi^* \wedge \chi^*$ and $\varphi^* := \neg\psi^*$ for $\varphi \equiv \psi \wedge \chi$ and $\varphi \equiv \neg\psi$, respectively.

Finally if $\varphi \equiv \exists z \in w_i\ \psi(z, v, \vec{w})$, then let $\varphi^* := \exists z \in w_i\ \psi^*(z, \vec{w})$. For the formula $\varphi \equiv \exists z \in v\ \psi(z, v, \vec{w})$ let $\varphi^* := \exists z\ \psi^*(z, \vec{w})$. This step increases the complexity of φ^*. $\square$(*Claim* 1)

Now suppose $\varphi(x) \in \Sigma_1$, say $\varphi(x) \equiv \exists z\ \psi(z,x)$. For the rudimentary function $f_k\colon \mathsf{V}^{i+1} \to \mathsf{V}$ let $\psi_{f_k}(y_0,\dots,y_i,x)$ be the Σ_0-formula given by Lemma 2.7 associated to f_k and ψ[20], and let $\psi^*_{f_k}(y_1,\dots,y_i,x)$ be the formula given by the claim above. We set for $k<\omega$:

$$\chi_k(x) := \exists \vec{y}\ \psi^*_{f_k}(\vec{y},x)$$

Then we have for each transitive U and each $x \in U$ (especially for $U := J_\gamma(\mathbb{R})$ and $x \in \mathbb{R}$):

$$\begin{aligned}
& \operatorname{rud}(U \cup \{U\}) \models \varphi(x) \\
\Leftrightarrow\ & \exists z \in \operatorname{rud}(U \cup \{U\}):\ \psi(z,x) \\
\Leftrightarrow\ & \exists k\ \exists \vec{y} \in U \cup \{U\}:\ \psi(f_k(\vec{y}),x) \\
\Leftrightarrow\ & \exists k\ \exists \vec{y} \in U:\ \psi(f_k(U,\vec{y}),x) \\
\Leftrightarrow\ & \exists k\ \exists \vec{y} \in U:\ U \cup \{U\} \models \psi_{f_k}(U,\vec{y},x) \\
\Leftrightarrow\ & \exists k\ \exists \vec{y} \in U:\ U \models \psi^*_{f_k}(\vec{y},x) \\
\Leftrightarrow\ & \exists k:\ U \models \chi_k(x)
\end{aligned}$$

$(\chi_k : k<\omega)$ would witness the lemma except possibly the condition $\chi_k \in \Sigma_k$, but we can stretch the sequence $(\chi_k : k<\omega)$ by inserting the Σ_1-formula φ to get a sequence $(\varphi^k : k<\omega)$ such that $\varphi^k \in \Sigma_k$. $\square$

Definition 2.55. Suppose $\varphi(v) \in \Sigma_1$ and $z \in \mathbb{R}$ is a real. A $\langle\varphi,z\rangle$-*witness* is an ω-sound, $(\omega,\omega_1,\omega_1+1)$-iterable z-premouse $\mathcal{N}$ such that there are $\delta_0,\dots,\delta_9,S,T \in \mathcal{N}$ and the following holds:

1. $\mathcal{N} \models \mathsf{ZFC} + \delta_0 < \dots < \delta_9$ are Woodin cardinals.
2. S and T are trees which project to complements of each other in $\mathcal{N}^{\mathrm{Col}(\omega,\delta_9)}$.
3. For some $k<\omega$, $\mathsf{p}[T]$ is the Σ_{k+3}-theory of $J_\gamma(\mathbb{R})$ in the language with names for each real, where γ is least, such that $J_\gamma(\mathbb{R}) \models \varphi^k(z)$.

If $\mathcal{N}$ is a z-premouse which satisfies all the conditions of a $\langle\varphi,z\rangle$-witness except for the iterability condition, then we call $\mathcal{N}$ a $\langle\varphi,z\rangle$-*pre-witness*.

[20] I. e. $\psi_{f_k}(\vec{y},x) \Leftrightarrow \psi(f_k(\vec{y}),x)$.

Remark. Condition 3 can be expressed in $\mathcal{N}$ by a first order formula (cf. proof of [Ste05, Lemma 1.10]).

Lemma 2.56. *If there is a $\langle\varphi, z\rangle$-witness, then $\mathsf{L}(\mathbb{R}) \models \varphi(z)$.*

Proof. See [Ste05, Lemma 1.10]. $\square$

So our fine structural hypothesis will be

(W_α) Suppose $\varphi \in \Sigma_1$ and $z \in \mathbb{R}$ are such that $J_\alpha(\mathbb{R}) \models \varphi(z)$. Then there exists a least γ such that $\mathsf{Lp}(z)\|\gamma$ is a $\langle\varphi, z\rangle$-witness, whose iteration strategy, if restricted to countable iteration trees, is in $J_\alpha(\mathbb{R})$.

Moreover, if $\alpha \geq \omega_1$ there are cofinally many $\gamma < \omega_1$ such that $\mathsf{Lp}(z)\|\gamma$ is such a $\langle\varphi, z\rangle$-witness.

Remark. The condition in (W_α) that there are cofinally in ω_1 many witnesses is not required in Steel's original definition in [Ste05]. It is a technical requirement we need in some steps of the core model induction to ensure that certain A-premice project to $\sup(A)$ cofinally often.

Lemma 2.57. *If α is a limit ordinal, then $(W_\alpha^\star) \Rightarrow (W_\alpha)$.*

Proof. This can be proved as in [Ste05, Lemma 1.11]. The difference here is that if $\alpha \geq \omega_1$ we then need to guarantee that for arbitrary large $\gamma < \omega_1$ there is a $\langle\varphi, z\rangle$-witness of height larger than γ which is an initial segment of $\mathsf{Lp}(z)$. In order to get this we apply the argument of the proof of [Ste05, Lemma 1.11] with (a real coding) $\mathsf{Lp}(z)\|\gamma$ instead of z. The resulting witness has of course height $> \gamma$. Furthermore the argument shows that the witness is an initial segment of $\mathsf{Lp}(z)$. $\square$

Critical ordinals

If we can show that $(W_\alpha^\star)$ holds for all ordinals α, then we are done by Lemma 2.53.[21] Since $(W_{\alpha+1}^\star)$ only mentions sets of reals U such that both U and $\mathbb{R} \setminus U$ have scales in $J_{\alpha+1}(\mathbb{R})$, we just need to show $(W_{\alpha+1}^\star)$ if there are new scales in $J_{\alpha+1}(\mathbb{R})$.

[21] Of course we need this only for $\alpha < \theta^{\mathsf{L}(\mathbb{R})}$ since all sets of reals in $\mathsf{L}(\mathbb{R})$ are already in $J_{\theta^{\mathsf{L}(\mathbb{R})}}(\mathbb{R})$.

This motivates the definition of critical ordinals given by John R. Steel in [Ste05].

Definition 2.58. An ordinal α is called *critical* iff there is some set $U \subseteq \mathbb{R}$ such that U and $\mathbb{R} \setminus U$ admit scales in $J_{\alpha+1}(\mathbb{R})$, but there is no scale on U in $J_\alpha(\mathbb{R})$.

So suppose there is a new scale on some set U in $J_{\alpha+1}(\mathbb{R})$. Once again we identify a scale with its associated sequence of prewellorderings. Since a countable sequence of prewellorderings is essentially a subset of $\mathbb{R}$, we get that the sequence is $\boldsymbol{\Sigma}_n(J_\alpha(\mathbb{R}))$ for some $n < \omega$. It follows from [Ste83] that if there is a new $\boldsymbol{\Sigma}_n(J_\alpha(\mathbb{R}))$-scale then $\boldsymbol{\Sigma}_n(J_\alpha(\mathbb{R}))$ or $\boldsymbol{\Sigma}_{n+1}(J_\alpha(\mathbb{R}))$ have the scale property.

In [Ste83] it is analyzed at which stages of the $\mathsf{L}(\mathbb{R})$-hierarchy a pointclass Γ can have the scale property. It turns out that there are the following possibilities:

1. α begins a Σ_1-gap and $\Gamma = \boldsymbol{\Sigma}_1(J_\alpha(\mathbb{R}))$. Moreover, if α is $\mathbb{R}$-inadmissible then Γ can also be $\boldsymbol{\Sigma}_{2n+1}(J_\alpha(\mathbb{R}))$ or $\boldsymbol{\Pi}_{2n}(J_\alpha(\mathbb{R}))$ for some $n < \omega$, or
2. β ends a weak Σ_1-gap and Γ is either $\boldsymbol{\Sigma}_{n+2k}(J_\beta(\mathbb{R}))$ or $\boldsymbol{\Pi}_{n+2k+1}(J_\beta(\mathbb{R}))$ where $k < \omega$ and n is least with $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$.

In our induction we will only consider ordinals which have one of the properties above. But we can limit the possibilities, by ruling out that α is $\mathbb{R}$-admissible and begins a Σ_1-gap.

Claim 1. If α begins a Σ_1-gap such that both U and $\mathbb{R} \setminus U$ have new scales in $J_{\alpha+1}(\mathbb{R})$, then α is $\mathbb{R}$-inadmissible.

Proof. Since each successor gap begins with an $\mathbb{R}$-inadmissible ordinal[22], we only have to consider limit gaps $[\alpha, \cdot]$. Suppose U is not only definable over $J_\alpha(\mathbb{R})$ but also an element of $J_\alpha(\mathbb{R})$.[23] Then $U \in J_\gamma(\mathbb{R})$ for some $\gamma < \alpha$, and since $[\alpha, \cdot]$ is a limit gap, there is a previous Σ_1-gap which guarantees that there is also a scale for U in $J_\alpha(\mathbb{R})$. So we can conclude that $U \notin J_\alpha(\mathbb{R})$ and $\mathbb{R} \setminus U \notin J_\alpha(\mathbb{R})$, respectively. The scales are both $\boldsymbol{\Sigma}_1(J_\alpha(\mathbb{R}))$ and

[22] If α begins a successor gap, then $\alpha = \gamma + 1$ and the function $n \mapsto \omega\gamma + n$ witnesses the $\mathbb{R}$-inadmissibility of α.

[23] This is of course equivalent to $\mathbb{R} \setminus U \in J_\alpha(\mathbb{R})$.

therefore also $U, \mathbb{R} \setminus U \in \underset{\sim}{\Sigma}_1(J_\alpha(\mathbb{R}))$. Now if α were $\mathbb{R}$-admissible, this would yield $U \in \underset{\sim}{\Delta}_1(J_\alpha(\mathbb{R})) \subseteq J_\alpha(\mathbb{R})$ due to the $\mathbb{R}$-admissibility of α, which is a contradiction. $\square$(*Claim* 1)

This yields the following lemma.

Lemma 2.59. *An ordinal α is critical iff one of the following conditions holds:*

1. *α is $\mathbb{R}$-inadmissible and begins a Σ_1-gap or*
2. *α ends a proper weak gap. (If the gap is not proper, then α also begins the gap and the first case applies. So we can w. l. o. g. suppose that the gap is proper.)*

At limit points λ we trivially have $(W^\star_\lambda)$, since if there is a set $U \subseteq \mathbb{R}$, $U \in J_\lambda(\mathbb{R})$ such that there are scales $\vec{\varphi}$ and $\vec{\psi}$ on U and $\mathbb{R} \setminus U$ respectively with $\leq_{\vec{\varphi}}, \leq_{\vec{\psi}} \in J_\lambda(\mathbb{R})$, then there is an $\alpha < \lambda$ with $U, \mathbb{R} \setminus U, \leq_{\vec{\varphi}}, \leq_{\vec{\psi}} \in J_\alpha(\mathbb{R})$. So $(W^\star_\alpha)$, which we have by induction hypothesis, ensures that the desired mice and iteration strategies exist in $J_\alpha(\mathbb{R}) \subseteq J_\lambda(\mathbb{R})$.

The proof of $(W^\star_\alpha) \Rightarrow (W^\star_{\alpha+1})$ for non-critical α is also trivial, since then there are no new scales in $J_{\alpha+1}(\mathbb{R})$. The first step in our induction is to show $(W^\star_0) \Rightarrow (W^\star_1)$ and hence PD holds (the condition $(W^\star_0)$ is trivially fulfilled, since there are no sequences of prewellorderings coming from a scale in $J_0(\mathbb{R})$).

The proof of $(W^\star_\alpha) \Rightarrow (W^\star_{\alpha+1})$ for $\alpha > 0$ critical breaks into five cases.

1. α begins a Σ_1-gap, is $\mathbb{R}$-inadmissible and successor of a critical ordinal.
2. α begins a Σ_1-gap, is $\mathbb{R}$-inadmissible and has uncountable cofinality.
3. α begins a Σ_1-gap, is $\mathbb{R}$-inadmissible and has countable cofinality.
4. α begins a Σ_1-gap, is $\mathbb{R}$-inadmissible and successor of a *non*-critical ordinal.
5. α ends a proper weak gap.

In case 4 the Σ_1-gap $[\alpha, \beta]$ is a successor gap and the predecessor gap $[\alpha', \beta']$ is a strong gap. Moreover, since $J_\alpha(\mathbb{R}) \models \mathsf{AD}$ by $(W^\star_\alpha)$, we have $\alpha = \beta$. This follows from [Ste83].[24]

Suppose that $[\alpha', \beta']$ is the predecessor gap of $[\alpha, \cdot]$ in case 4 and that $[\alpha', \alpha]$ is the weak gap in case 5. In both cases it follows again from [Ste83] and the definition of "critical" that

$$\alpha' = \sup\{\gamma < \alpha : \gamma \text{ critical}\}.$$

In cases 2 and 3 α itself is the supremum of critical ordinals less than α.

We call the cases 1 – 3 the *inadmissible cases* and the cases 4 and 5 the *end-of-gap cases*. This has to do with the method of proving $(W^\star_\alpha) \Rightarrow (W^\star_{\alpha+1})$. In the inadmissible cases we build ordinary premice, but in the end-of-gap cases the premice we construct additionally have a predicate for an iteration strategy. These premice are so-called "hybrid premice".

[24] If $\beta > \alpha$ then we would have $J_{\alpha+1}(\mathbb{R}) \models \mathsf{AD}$. Then by [Ste83, Lemma 2.9] there is a $\Pi_1(J_\alpha(\mathbb{R}))$-subset of $\mathbb{R} \times \mathbb{R}$ with no uniformization in $\undertilde{\Sigma}_1(J_\beta(\mathbb{R}))$, which contradicts the fact that each class of the form $\undertilde{\Sigma}_{2n+1}(J_\alpha(\mathbb{R}))$ has the scale property.

3. EVERY UNCOUNTABLE SUCCESSOR CARDINAL IS WEAKLY COMPACT

In this chapter we present a proof of Theorem 1.4

Theorem 1.4. *Suppose* V *is a model of* ZF *such that each uncountable successor cardinals is weakly compact and each uncountable limit cardinal is singular.*

Then $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ *holds in* $\mathsf{HOD}_X^{\mathrm{Col}(\omega,<\mu^{+\mathsf{V}})}$.

From now on let $\mu_+ := \mu^{+\mathsf{V}}$, let $\mathrm{Col}(\omega, < \mu_+)$ be the Lévy collapse, g a $\mathrm{Col}(\omega, < \mu_+)$-generic object over V (and therefore generic over HOD_X), and let $\mathbb{R}^g$ denote the reals of $\mathsf{HOD}_X[g]$.

Remark. Since our induction takes place in $\mathsf{HOD}_X[g]$, we use the notions $(W^\star_\alpha)$ and (W_α) from Section 2.3 for the according notions inside $\mathsf{HOD}_X[g]$.

In the first section we present the first step of the induction, i. e. we show $J_1(\mathbb{R}^g) \models \mathsf{AD}$. The second section is concerned with the inadmissible cases. The largest part of that section is the uncountable cofinality case. One gets easily the countable cofinality case and the successor of a critical ordinal case from the induction hypothesis. In the third section we will prove $(W^\star_\alpha) \Rightarrow (W^\star_{\alpha+1})$ where α ends a weak gap or α is the successor of a non-critical ordinal.

3.1 The projective case

In this section we do the first step in the core model induction. We show that $J_1(\mathbb{R}^g) \models \mathsf{AD}$, i. e. projective determinacy holds.

To show this, we use the following definitions and theorem (see [Ste95, MS89]):

Definition 3.1. Let A be a set of ordinals or a transitive set, and let $\mathcal{M}$ be an A-premouse and $\xi \geq \sup(A \cap \mathsf{On})$. We call $\mathcal{M}$ *n-small above ξ* iff whenever κ is the critical point of an extender on the $\mathcal{M}$-sequence and $\xi < \kappa$, then

$$\mathcal{M}\|\kappa \not\models \text{ there are } n \text{ Woodin cardinals } > \xi.$$

Definition 3.2. Let A be a set of ordinals or a transitive set, and define $\xi := \sup(A \cap \mathsf{On})$.

Then for $1 \leq n < \omega$, $M_n^\sharp(A)$ denotes the least active, ξ-sound A-mouse with $\rho_\omega(\mathcal{M}) \leq \xi$ which is not n-small above ξ. $M_0^\sharp(A)$ is just $A^\sharp$.

Note that by "least" we mean the mouse of least height. This choice is possible since by Lemma 2.29 any two such mice are compatible.

Theorem 3.3 (Martin, Steel). *Suppose for each $n < \omega$ and each real x we have that $M_n^\sharp(x)$ exists.*

Then PD *holds.*

So we first have to show that $M_n^\sharp(x)$ exists for each real x. But then also both HOD_X and $\mathsf{HOD}_X^{\mathrm{Col}(\omega,<\mu_+)}$ satisfy "$M_n^\sharp(x)$ exists for each real x". This will enable us to prove PD in $\mathsf{HOD}_X^{\mathrm{Col}(\omega,<\mu_+)}$ and hence $J_1(\mathbb{R}^g) \models \mathsf{AD}$. What we actually show is:

Theorem 3.4. *Suppose each uncountable successor cardinal is weakly compact.*

Then $M_n^\sharp(A)$ exists for each set of ordinals A.

First we show the iterability of $M_n^\sharp(A)$ under the assumption that V is closed under $M_{n-1}^\sharp$. This is an application of [FMS01, Lemma 2.3].

For this we need the following lemma, which is an application of [Steb, Corollary 6.14].

Lemma 3.5. *Suppose $\mathcal{M}$ is a tame, k-sound A-premouse which projects to $\xi = \sup(A \cap \mathsf{On})$. Let $\mathcal{T}$ be a k-maximal iteration tree of limit length above ξ on $\mathcal{M}$ which is built according to the $\mathcal{Q}$-structure iteration strategy.*

Then there is at most one cofinal, wellfounded branch b through $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$.

Lemma 3.6. *Let W be an inner model of* ZFC *which is closed under $M_{n-1}^\sharp$. If $W \models$ "$M_n^\sharp(A)$ exists", then $W \models$ "$M_n^\sharp(A)$ is fully iterable via the $\mathcal{Q}$-structure iteration strategy".*[1]

Proof. We work in W. Suppose the converse holds and $\mathcal{T}$ is an iteration tree on $M_n^\sharp(A)$ built according to the $\mathcal{Q}$-structure iteration strategy. First, by Lemma 3.5, we can rule out that $\mathcal{T}$ has two different branches b and c such that $\mathcal{Q}(b, \mathcal{T})$ and $\mathcal{Q}(c, \mathcal{T})$ exist, and both are equal to $\mathcal{Q}(\mathcal{T})$. So suppose there is no cofinal branch b with $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$.

Suppose Ω is large enough such that V_Ω is closed under $M_{n-1}^\sharp$, $\mathcal{X} \prec V_\Omega$ is a countable elementary substructure with $M_n^\sharp(A), \mathcal{T}, M_{n-1}^\sharp(\mathcal{M}(\mathcal{T})) \in \mathcal{X}$, and $\pi\colon H \cong \mathcal{X}$ is the uncollapsing map.[2] If $\pi(\bar{\mathcal{T}}) = \mathcal{T}$ and $\pi(M) = M_n^\sharp(A)$, then H is a model of:

1. $\bar{\mathcal{T}}$ is an iteration tree on M,
2. there is no cofinal branch $\bar{b}$ with $\mathcal{Q}(\bar{b}, \bar{\mathcal{T}}) = \mathcal{Q}(\bar{\mathcal{T}})$.

$\bar{\mathcal{T}}$ is built according to the $\mathcal{Q}$-structure strategy, because the structures which H believes to be $\mathcal{Q}$-structures are real $\mathcal{Q}$-structures.[3]

M is countable and elementarily embeddable into $M_n^\sharp(A)$, so in W there exists a cofinal branch $\bar{b}$ with $\mathcal{Q}(\bar{b}, \bar{\mathcal{T}}) = \mathcal{Q}(\bar{\mathcal{T}})$. But now we can coiterate $\mathcal{Q} := \mathcal{Q}(\bar{\mathcal{T}})$ and $\mathcal{M} := \pi^{-1}(M_{n-1}^\sharp(\mathcal{M}(\mathcal{T})))$ and we get $\mathcal{M} \trianglelefteq \mathcal{Q}$ or $\mathcal{Q} \trianglelefteq \mathcal{M}$.

Suppose that $\mathcal{M} \triangleleft \mathcal{Q}$. $\delta(\bar{\mathcal{T}})$ is Woodin in $\mathcal{Q}$, so it is also Woodin in $\mathcal{M}$. Hence there exists an initial segment of $M_{\bar{b}}^{\bar{\mathcal{T}}}$ which is not n-small. But this is impossible, because M and therefore $M_{\bar{b}}^{\bar{\mathcal{T}}}$ does not contain such an initial segment. It follows that $\mathcal{Q} \trianglelefteq \mathcal{M}$, which implies $\mathcal{Q} \in H$ since $\mathcal{M} \in H$.

Now let $g \in W$ be $\mathrm{Col}(\omega, \mathcal{Q})$-generic over H. Since $\mathcal{Q}$ is countable in $H[g]$, we can form a tree searching for a cofinal wellfounded branch b' through $\bar{\mathcal{T}}$

[1] This means that each iteration tree $\mathcal{T}$ of limit length which is played according to the $\mathcal{Q}$-structure iteration strategy, has a unique branch b such that $\mathcal{Q}(b, \mathcal{T})$ exists and $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$ holds.

[2] For the construction of $\mathcal{X}$ we need AC.

[3] Let $\lambda < lh(\bar{\mathcal{T}})$ and $H \models$ "$\mathcal{Q}$ is the $\mathcal{Q}$-structure for $[0, \lambda)_{\bar{\mathcal{T}}}$". Then we have that $\pi(\mathcal{Q})$ is the $\mathcal{Q}$-structure for the branch $[0, \pi(\lambda))_{\mathcal{T}}$ and therefore it is countably iterable. So if $\mathcal{N}$ is countable and $\sigma\colon \mathcal{N} \to \mathcal{Q}$ sufficiently elementary, then $\pi \circ \sigma\colon \mathcal{N} \to \pi(\mathcal{Q})$ guarantees that $\mathcal{N}$ is $\omega_1 + 1$-iterable.

and an initial segment $\mathcal{P} \trianglelefteq M_{b'}^{\bar{\mathcal{T}}}$ which is isomorphic to $\mathcal{Q}$. Since $(\bar{b}, \mathcal{Q})$ is such a pair in W and since wellfoundedness is absolute, there is also a cofinal branch through this tree in $H[g]$. But this is clearly $\bar{b}$, because the branch is unique by Lemma 3.5. It follows that also $\bar{b} \in H$ holds, by the homogeneity of $\mathrm{Col}(\omega, \mathcal{Q})$.

But then $b := \pi(\bar{b})$ is a wellfounded branch through $\mathcal{T}$ coming with a $\mathcal{Q}$-structure, namely $\pi(\mathcal{Q})$. This is a contradiction. $\square$

Lemma 3.7. *Let $n < \omega$. Suppose $W \models$ "$\mathsf{ZFC} + M_n^\sharp(A)$ exists for each set of ordinals A". Let $\mathbb{P}$ be a forcing in W and let G be $\mathbb{P}$-generic over W.*

$(1)_n$ *Let $A \subseteq \mathsf{On}$, $A \in W$ and $W \models \mathcal{P} = M_n^\sharp(A)$.*
Then also $W[G] \models \mathcal{P} = M_n^\sharp(A)$.

$(2)_n$ *For all sets of ordinals $A \in W[G]$, $W[G] \models M_n^\sharp(A)$ exists.*

$(3)_n$ *Let $W \models$ "H is countable and elementarily embeddable into V_Ω", where Ω is a large limit ordinal. Let further $\mathbb{Q}$ be a forcing in H and $h \in W$ $\mathbb{Q}$-generic over H.*
Then $H[h]$ is closed under $M_n^\sharp$.

Proof. The proof is by induction on n.

$(1)_n$ Let $W \models \mathcal{P} = M_n^\sharp(A)$. We only have to show that $\mathcal{P}$ is countably iterable in $W[G]$. In fact we show that $\mathcal{P}$ is fully iterable in $W[G]$ via the $\mathcal{Q}$-structure iteration strategy. So suppose $\mathcal{T}$ is an iteration tree on $\mathcal{P}$ according to the $\mathcal{Q}$-structure iteration strategy, which doesn't have a unique cofinal branch b such that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$. As before we can suppose that there is no such cofinal branch in $W[G]$:

$$W[G] \models \mathcal{T} \text{ witnesses that } \mathcal{P} \text{ is not iterable.}$$

So there is a $p \in G$ such that

$$p \Vdash^{\mathbb{P}}_{W} \dot{\mathcal{T}} \text{ witnesses that } \check{\mathcal{P}} \text{ is not iterable.}$$

Now we work in W.

We can take $\pi\colon H \to V_\Omega$ elementary such that H is countable and transitive with $A, \dot{\mathcal{T}}, \check{\mathcal{P}}$ elements in $\mathrm{ran}(\pi)$. Let further $\bar{A} \mapsto A$, $\bar{\mathcal{P}} \mapsto \mathcal{P}$, and $\dot{T} \mapsto \dot{\mathcal{T}}$. Then H not only thinks that it is closed $M_{n-1}^\sharp$ but it is really closed under $M_{n-1}^\sharp$ as well as under $M_n^\sharp$: For example, if $x \in H$ then $\pi^{-1}(M_n^\sharp(\pi(x))) \in H$ witnesses the first order properties of a potential $M_n^\sharp(x)$. But then this is the true $M_n^\sharp(x)$, because any countable model which is elementarily embeddable into it is also elementarily embeddable into $M_n^\sharp(\pi(x))$ and therefore ω_1+1-iterable. This argument also yields $\bar{\mathcal{P}} = M_n^\sharp(\bar{A})$.

Now let $h \in W$ be $\pi^{-1}(\mathbb{P})$-generic over H with $\pi^{-1}(p) \in h$. Then

$$H[h] \models \bar{\mathcal{P}} \text{ is not iterable, witnessed by } \dot{T}^h.$$

$\dot{T}^h$ is an iteration tree according to the $\mathcal{Q}$-structure strategy in $H[h]$. $H[h]$ is closed under $M_{n-1}^\sharp$ due to $(3)_{n-1}$. Since the $\mathcal{Q}$-structures are given by initial segments of $M_{n-1}^\sharp$ built over the common part model, it follows that $\dot{T}^h$ is built according to the $\mathcal{Q}$-structure strategy also in W. Since $\bar{\mathcal{P}}$ is iterable in W, let b be the unique cofinal branch coming with a $\mathcal{Q}$-structure. As in the proof of Lemma 3.6 one can see that the cofinal branch in W is already in $H[h]$. This is a contradiction.

$(2)_n$ Let $A \in W[G]$ be a set of ordinals, say $A = \tau^G$. Let Ω be large enough such that $\mathrm{Pow}(\mathbb{P}), \tau \in V_\Omega^W$. Let $\mathcal{P} := (M_n^\sharp(V_\Omega))^W$. Due to $(1)_n$ we have $W[G] \models \mathcal{P} = M_n^\sharp(V_\Omega^W)$. Since $\mathcal{P}$ is fully iterable and the forcing is small relative to the critical points of the extenders of $\mathcal{P}$, $\mathcal{P}[G]$ is also fully iterable[4]. Of course we also have $A \in \mathcal{P}[G]$. Now run the $L[E]$-construction over A in $\mathcal{P}[G]$. The resulting model inherits the Woodin cardinals and the iterability of $\mathcal{P}[G]$, and therefore we have found our desired $M_n^\sharp(A)$ in $W[G]$.

$(3)_n$ We work in W.

We know that H is closed under the $M_n^\sharp$-operator. We want to show $M_n^\sharp(A) \in H[h]$ for every $A \in H[h]$. Fix a name $\tau \in H$ such that $A = \tau^h$. Let further Ω be large enough such that $\mathrm{Pow}(\mathbb{Q})$, $\tau \in V_\Omega^H$. Consider $\mathcal{M} := M_n^\sharp(V_\Omega^H)$. Note that $\mathcal{M} \in H$ and $H \models \mathcal{M} = M_n^\sharp(V_\Omega^H)$. Then we have that $\mathcal{M}[h] \in H[h]$ can be rearranged as $M_n^\sharp(V_\Omega^H[h])$ and

[4] Note that we can rearrange $\mathcal{P}[G]$ as a mouse over $V_\Omega^W[G]$.

is fully iterable in W as well as in $H[h]$. Of course $A \in M_n^\sharp(V_\Omega^H[h])$, and if we run the $L[E]$-construction over A in $M_n^\sharp(V_\Omega^H[h])$ we get a structure whereof $H[h]$ thinks it is $M_n^\sharp(A)$. But this is the real $M_n^\sharp(A)$ because it inherits the iterability of $M_n^\sharp(V_\Omega^H[h])$.

$\square$

Lemma 3.8. *Let $n < \omega$ and suppose $\mathsf{V} \models$ "$\mathsf{ZF} + M_{n-1}^\sharp(A)$ exists for each set of ordinals A".*

Then the following are equivalent for each set of ordinals B:

$(1)_n$ HOD_B *is closed under $M_n^\sharp$.*

$(2)_n$ *For all sets of ordinals $Z \in \mathsf{V}$, $\mathsf{HOD}_B[Z]$ is closed under $M_n^\sharp$.*

$(3)_n$ *For all sets of ordinals $A \in \mathsf{V}$, $M_n^\sharp(A)$ exists.*

Moreover, if $(1)_n$ – $(3)_n$ hold and $\mathcal{P} := M_n^\sharp(A)$ for some $A \in \mathsf{HOD}_B[Z]$, then $\mathsf{HOD}_B[Z] \models \mathcal{P} = M_n^\sharp(A)$.

Remark. In case $n = 0$ this means ZF proves that if $A^\sharp$ exist, then it also exists in each HOD_B and $\mathsf{HOD}_B[Z]$.

Proof. We first show $(1)_n \Rightarrow (2)_n$. Let $A \in \mathsf{HOD}_B[Z]$. Then $A = \tau^Z$ for some $\tau \in \mathsf{HOD}_B$. Let Ω be large such that τ and the Vopěnka forcing $\mathbb{P}$ which adds Z are elements of $V_\Omega^{\mathsf{HOD}_B}$. Then we have $M_n^\sharp(V_\Omega^{\mathsf{HOD}_B}) \in \mathsf{HOD}_B$ since HOD_B is closed under $M_n^\sharp$. This of course implies $M_n^\sharp(V_\Omega^{\mathsf{HOD}_B})[Z] \in \mathsf{HOD}_B[Z]$.

Now we can rearrange $M_n^\sharp(V_\Omega^{\mathsf{HOD}_B})[Z]$ to $M_n^\sharp(V_\Omega^{\mathsf{HOD}_B}[Z])$, which contains A. If we now run the $L[E]$-construction over A in $M_n^\sharp(V_\Omega^{\mathsf{HOD}_B}[Z])$ we get $M_n^\sharp(A)$, and therefore $M_n^\sharp(A) \in \mathsf{HOD}_B[Z]$.

Now consider $(2)_n \Rightarrow (3)_n$. Let A be a set of ordinals. Then A is generic over HOD_B by Vopěnka's theorem.

Suppose $\mathsf{HOD}_B[A] \models \mathcal{M} = M_n^\sharp(A)$. The only thing we need to prove is that $\mathcal{M}$ is countably iterable above $\sup(A)$ in V. What we actually show is that $\mathcal{M}$ is fully iterable above $\sup(A)$ via the $\mathcal{Q}$-structure iteration strategy.

So let $\mathcal{T}$ be an iteration tree in $\mathcal{M}$, according to the $\mathcal{Q}$-structure iteration strategy. We show that there is a cofinal branch b with $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$.

$\mathcal{T}$ is essentially a set of ordinals, so there is a forcing $\mathbb{P} \in \mathsf{HOD}_B[A]$ and a $\mathbb{P}$-generic filter $G \in \mathsf{V}$ over $\mathsf{HOD}_B[A]$ such that $\mathcal{T} \in \mathsf{HOD}_B[A][G]$. But then

$$\mathsf{HOD}_B[A][G] \models \mathcal{T} \text{ is built according to the } \mathcal{Q}\text{-structure strategy}$$

This is true since $\mathcal{T}$ is built according to the $\mathcal{Q}$-structure iteration strategy in V and the $\mathcal{Q}$-structure of $\mathcal{T}\restriction\lambda$ for $\lambda < lh(\mathcal{T})$ is given by an initial segment of $M^{\sharp}_{n-1}(\mathcal{M}(\mathcal{T}\restriction\lambda))$.[5]

Since $\mathsf{HOD}_B[A][G] \models \mathcal{M} = M^{\sharp}_n(A)$, there is a $b \in \mathsf{HOD}_B[A][G]$ and an initial segment $\mathcal{Q} \trianglelefteq \mathcal{M}^{\mathcal{T}}_b$ such that $\mathsf{HOD}_B[A][G]$ thinks $\mathcal{Q}$ is a $\mathcal{Q}$-structure. Now we can coiterate $\mathcal{Q}$ with $M^{\sharp}_{n-1}(\mathcal{M}(\mathcal{T}\restriction\lambda))$ (which is in $\mathsf{HOD}_B[A][G]$ due to $(3)_{n-1} \Rightarrow (2)_{n-1}$) in $\mathsf{HOD}_B[A][G]$ and we see that $\mathcal{Q} \trianglelefteq M^{\sharp}_{n-1}(\mathcal{M}(\mathcal{T}\restriction\lambda))$, which guarantees that $\mathcal{Q}$ is also countably iterable in V and therefore a real $\mathcal{Q}$-structure.

For $(3)_n \Rightarrow (1)_n$ let A be in HOD_B. We assume w. l. o. g. $A \subseteq \mathsf{On}$. $M^{\sharp}_n(A) \in \mathsf{HOD}_B$ because it is ordinal definable and the extender sequence is hereditarily ordinal definable, and therefore $M^{\sharp}_n(A)$ is hereditarily ordinal definable. Now we can show that $M^{\sharp}_n(A)$ is also fully iterable in HOD_B. This is quite easy, because its iteration strategy Σ is the strategy picking the unique branch coming with a $\mathcal{Q}$-structure. So Σ is ordinal definable, and therefore $\Sigma\restriction\mathsf{HOD}_B \in \mathsf{HOD}_B$ witnesses that $M^{\sharp}_n(A)$ is also iterable in HOD_B.

Now we can show that if $\mathcal{P} := M^{\sharp}_n(A)$ for some $A \in \mathsf{HOD}_B[Z]$, then $\mathsf{HOD}_B[Z] \models \mathcal{P} = M^{\sharp}_n(A)$.

First we show this for $A \in \mathsf{HOD}_B$, i. e. we show $\mathsf{HOD}_B \models \mathcal{P} = M^{\sharp}_n(A)$. For this we prove that $\mathcal{P}$ is fully iterable in HOD_B via the $\mathcal{Q}$-structure iteration strategy. So let $\mathcal{T} \in \mathsf{HOD}_B$ be an iteration tree built in HOD_B according to the $\mathcal{Q}$-structure iteration strategy. Since the $\mathcal{Q}$-structures are given by initial segments of some $M^{\sharp}_{n-1}$, we have by induction that $\mathcal{T}$ is also built according to the $\mathcal{Q}$-structure iteration strategy in V. Let b be the unique cofinal branch in V such that $\mathcal{Q}(b,\mathcal{T}) = \mathcal{Q}(\mathcal{T})$. So b is ordinal definable from elements in HOD_B, hence $b \in \mathsf{HOD}_B$. Again, since the $M^{\sharp}_{n-1}$-operator can identify the $\mathcal{Q}$-structure, we have that b is also according to the $\mathcal{Q}$-structure iteration strategy in HOD_B.

[5] Here we use that by induction $M^{\sharp}_{n-1}(A)^{\mathsf{V}}$ and $M^{\sharp}_{n-1}(A)^{\mathsf{HOD}_B[Z]}$ are the same for each $A \in \mathsf{HOD}_B[Z]$. Notice that $M^{\sharp}_{n-1}(\mathcal{M}(\mathcal{T} \restriction \lambda)) \in \mathsf{HOD}_B[A,G]$ by Lemma 3.7.

Now let $A = \tau^Z$ be in $\mathsf{HOD}_B[Z]$. We use the proof of $(1)_n \Rightarrow (2)_n$ and the proof of Lemma 3.7 $(2)_n$. Consider $M_n^\sharp(V_\Omega^{\mathsf{HOD}_B})$ where Ω is large enough. By the last paragraph we have that $M_n^\sharp(V_\Omega^{\mathsf{HOD}_B})$ is the same whether computed in V or in HOD_B. In both cases, if we show $\mathsf{HOD}_B[Z] \models$ "$M_n^\sharp(A)$ exists" and if we show $M_n^\sharp(A)^{\mathsf{V}} \in \mathsf{HOD}_B[Z]$, we produce the corresponding $M_n^\sharp(A)$ by running the $L[E]$-construction in $M_n^\sharp(V_\Omega^{\mathsf{HOD}_B}[Z])$. So $M_n^\sharp(A)^{\mathsf{V}} = M_n^\sharp(A)^{\mathsf{HOD}_B}$ as desired. □

Now we can prove Theorem 3.4. Suppose that the assumptions of Theorem 3.4 hold, i. e. all uncountable cardinals are either successor cardinals and weakly compact or limit cardinals and singular. A consequence of this fact is that the cofinality of any infinite cardinal is either ω or weakly compact.

Proof of 3.4. Note that the proof is essentially the same as the proof of [Sch99, Theorem 2]. We show by induction that $M_n^\sharp(A)$ exists for every n and every $A \subseteq \mathsf{On}$.

We first consider the case $n = 0$. Let $A \subseteq \mathsf{On}$ and $\nu := \sup(A)^+$. By Lemma 2.45, ν is inaccessible in $\mathsf{L}[A]$. Now if either $\mathrm{cof}(\nu^{+\mathsf{L}[A]}) < \nu$ or $\mathrm{cof}(\nu^{++\mathsf{L}[A]}) < \nu$, then the existence of $A^\sharp$ follows from [DJ75]. Otherwise [FMS01, Lemma 2.1] gives us a countably complete $\sigma\colon \mathsf{L}[A] \to \mathsf{L}[A]$ with critical point ν, and therefore $A^\sharp$ exists. By Lemma 3.8 we also have that every $\mathsf{HOD}_B[Z]$ is closed under sharps.

Now let $n > 1$ and suppose some $M_n^\sharp(A)$ does not exist. First suppose $A = \emptyset$. By the induction hypothesis and Lemma 3.8 we have that HOD is closed under $M_{n-1}^\sharp$. So let $\Omega := \nu^+$ where ν is large enough and closed under Θ. Since Ω^+ is weakly compact, it is inaccessible in HOD by Lemma 2.45. So we have

$$\overline{\overline{\mathrm{Pow}(\Omega) \cap \mathsf{HOD}}} = \Omega.$$

Since Ω is also weakly compact, we can use Lemma 2.46 to get a countably complete ultrafilter U on $\mathrm{Pow}(\Omega) \cap \mathsf{HOD}$. Let

$$\mathcal{U} := (\mathsf{HOD}; \in, U).$$

Now we can build K^c in $\mathcal{U}$ up to height Ω, and under the assumption that $M_n^\sharp$ does not exist in HOD, or equivalently in V, we can use [FMS01, Lemma 2.3] to get that $(K^c)^{\mathcal{U}}$ is $\Omega + 1$-iterable in $\mathcal{U}$. So we can isolate the true core model, $\mathsf{K}^{\mathcal{U}}$, of height Ω.

Now suppose δ, δ^+ are weakly compact cardinals less than ν. We have $\delta^{+\mathsf{K}^{\mathcal{U}}} < \delta^+$ since δ^+ is inaccessible in $\mathsf{K}^{\mathcal{U}}$, so we can pick some bijection

$$f\colon \delta \to \delta^{+\mathsf{K}^{\mathcal{U}}}, \quad f \in \mathsf{V}.$$

Since ν is Θ-closed, f is Vopěnka-generic over HOD for a forcing of size $< \nu$ and therefore generic over $\mathcal{U}$. But δ^+ is also inaccessible in $\mathcal{U}[f]$, so we can use Lemma 2.46 a second time to get a countably complete ultrafilter $\tilde{U}$ on $\mathrm{Pow}(\delta) \cap \mathcal{U}[f]$.

Again we have that $\mathcal{U}[f][\tilde{U}]$ is a Vopěnka-generic extension of $\mathcal{U}[f]$. In $\mathcal{U}[f][\tilde{U}]$ we can now build the ultrapower of $\mathcal{U}[f]$ by $\tilde{U}$:

$$\pi\colon \mathcal{U}[f] \to \mathrm{ult}(\mathcal{U}[f], \tilde{U}), \quad cr(\pi) = \delta.^6$$

By the countable completeness of $\tilde{U}$ we have that $\mathrm{ult}(\mathcal{U}[f], \tilde{U})$ is wellfounded, and therefore we can identify it with its transitive collapse $\tilde{H}$. In $\tilde{H}$ we can now build the core model $\tilde{\mathsf{K}} := \mathsf{K}^{\tilde{H}}$, so

$$\pi{\restriction}\mathsf{K}^{\mathcal{U}[f]}\colon \mathsf{K}^{\mathcal{U}[f]} \to \tilde{\mathsf{K}}$$

Claim 1. $\tilde{\mathsf{K}}$ is $\Omega + 1$-iterable in $\mathcal{U}[f][\tilde{U}]$.

Proof. Since Ω is measurable in $\mathcal{U}$, it is also measurable in $\mathcal{U}[f][\tilde{U}]$ and therefore weakly compact in $\mathcal{U}[f][\tilde{U}]$. Now we work in $\mathcal{U}[f][\tilde{U}]$. As Ω is weakly compact it suffices to show that $\tilde{\mathsf{K}}$ is Ω-iterable, since if we have an iteration tree of length Ω we get a unique wellfounded cofinal branch by Lemma 2.47.

So suppose $\tilde{\mathsf{K}}$ is not Ω-iterable and let $\mathcal{T}$ be an illbehaved iteration tree on an initial segment $\mathcal{M}$ of $\tilde{\mathsf{K}}$. Pick some $\sigma, H, \bar{\mathcal{T}}, \bar{\mathcal{M}}$ with

$$\sigma\colon H \to V_\Omega \text{ elementary}, \ \overline{\overline{H}} = \omega, \ \mathcal{T} = \sigma(\bar{\mathcal{T}}), \text{ and } \mathcal{M} = \sigma(\bar{\mathcal{M}}).$$

We may suppose that $\mathcal{M} = J_\alpha^{\tilde{\mathsf{K}}}$ such that $\alpha > \delta$ is inaccessible, so we have $\pi(\alpha) = \alpha$ and for $\mathcal{M}' := J_\alpha^{\mathsf{K}^{\mathcal{U}[f]}}$ we have $\pi(\mathcal{M}') = \mathcal{M}$. Now let $\Gamma \subseteq V_\Omega^{\mathcal{U}[f]}$ be a

[6] Note that all we need is $\pi{\restriction}V_{\Omega+1}^{\mathcal{U}[f]} \in \mathcal{U}[f][\tilde{U}]$. It could be impossible to define all of π in $\mathcal{U}[f][\tilde{U}]$.

countable set of functions $g\colon \delta \to \mathcal{U}[f]$ such that $\sigma''\bar{\mathcal{M}} = \{\pi(g)(\delta) : g \in \Gamma\}$. Let further Ξ consists of all $x \in \tilde{U}$ such that there is a formula φ and some $g_0, \ldots, g_n \in \Gamma$ with $x = \{\xi < \delta : \mathcal{M}' \models \varphi(g_0(\xi), \ldots, g_n(\xi))\}$. Since $\tilde{U}$ is countably complete we can find some $\xi_0 \in \bigcap \Xi$. Now define an elementary embedding

$$\tau\colon \bar{\mathcal{M}} \to \mathcal{M}' \quad \text{via} \quad \tau(\sigma^{-1}(\pi(g)(\delta))) := g(\xi_0).$$

τ can be used to copy $\bar{\mathcal{T}}$ to an iteration tree on $\mathcal{M}'$. So the existence of τ witnesses that $\bar{\mathcal{T}}$ is wellbehaved in reality. Hence there is a cofinal branch b through $\bar{\mathcal{T}}$, which a priori need not be in H. But now we can argue as in 3.6 to verify that the branch is indeed in H. So $\bar{\mathcal{T}}$ is wellbehaved in H and therefore, by elementarity, $\mathcal{T}$ is wellbehaved in $\mathcal{U}[f][\tilde{U}]$. This is a contradiction. $\square$(*Claim* 1)

Let $\delta' := \pi(\delta)$. Then $\delta' > \delta^{+\mathsf{K}^{\mathcal{U}[f]}}$. First note that $\mathsf{K}^{\mathcal{U}} = \mathsf{K}^{\mathcal{U}[f]} = \mathsf{K}^{\mathcal{U}[f][\tilde{U}]}$. Since $\tilde{\mathsf{K}}$ is iterable in $\mathcal{U}[f][\tilde{U}]$ and $\pi\restriction\mathsf{K}^{\mathcal{U}[f][\tilde{U}]}\colon \mathsf{K}^{\mathcal{U}[f][\tilde{U}]} \to \tilde{\mathsf{K}}$, we can now use [Sch99, Lemma 4(b)] inside $\mathcal{U}[f][\tilde{U}]$ and get that $\pi''\delta^{+\mathsf{K}^{\mathcal{U}[f]}}$ is cofinal in $\pi(\delta^{+\mathsf{K}^{\mathcal{U}[f]}}) = \delta'^{+\tilde{\mathsf{K}}}$. But for $\beta < \delta^{+\mathsf{K}^{\mathcal{U}[f]}}$

$$\pi(\beta) = \gamma \quad \Leftrightarrow \quad \exists \xi < \delta \ (f(\xi) = \beta \wedge \pi(f)(\xi) = \gamma)$$

We have $f, \pi(f) \in \tilde{H}$[7] and therefore $\pi\restriction\delta^{+\mathsf{K}^{\mathcal{U}[f]}} \in \tilde{H}$. So $\pi\restriction\delta^{+\mathsf{K}^{\mathcal{U}[f]}}$ witnesses that the cofinality of $\delta'^{+\tilde{\mathsf{K}}}$ in $\tilde{H}$ is small:

$$\tilde{H} \models \operatorname{cof}(\delta'^{+\tilde{\mathsf{K}}}) \leq \delta^{+\mathsf{K}^{\mathcal{U}[f]}} < \delta'$$

This contradicts weak covering for $\tilde{\mathsf{K}}$ which yields $\operatorname{cof}(\delta'^{+\tilde{\mathsf{K}}}) \geq \delta'$ in $\tilde{H}$.

The proof for $A \neq \emptyset$ is essentially the same. In this case we must work with HOD_A, Ω needs to be the successor of a Θ-closed cardinal ν larger than $\sup(A)$, and we have to choose δ larger than $\sup(A)$. Moreover, we build $K^c(A)$ and $\mathsf{K}(A)$ instead of K^c and K. $\square$

So we have shown that $M_n^\sharp(A)$ exists for every set of ordinals A. To do the first step in the core model induction, i. e. to prove PD in $\mathsf{HOD}_X[g]$, we have to show that $\mathsf{HOD}_X[g]$ contains all $M_n^\sharp$. But this is now easy because HOD_X is closed under $M_n^\sharp$ by Lemma 3.8, and therefore $\mathsf{HOD}_X[g]$ is also closed under $M_n^\sharp$ by Lemma 3.7.

[7] f can be coded as a subset of δ in $\mathcal{U}[f]$ and $\operatorname{Pow}(\delta) \cap \mathcal{U}[f] \subseteq \tilde{H}$.

3.2 The inadmissible cases

Now suppose α begins a Σ_1-gap and α is $\mathbb{R}$-inadmissible and critical. Then we have the three subcases

1. α is the successor of a critical ordinal, or
2. α is a limit ordinal and has countable cofinality, or
3. α is a limit ordinal and has uncountable cofinality.

Remark. The case "α is the successor of a non-critical ordinal" is handled in the same way as the end-of-gap case.

It doesn't matter whether we consider "countable cofinality" computed in $\mathsf{HOD}_X[g]$ or in $\mathsf{L}(\mathbb{R}^g)$, because $\alpha < \theta^{\mathsf{L}(\mathbb{R}^g)}$.[8]

The uncountable-cofinality case

Suppose α is a limit ordinal which begins a Σ_1-gap and has uncountable cofinality. To prove $(W^\star_{\alpha+1})$ we use (W_α) which we get from the induction hypothesis $(W^\star_\alpha)$.

Let $\varphi(v_0, v_1) \in \Sigma_1$ and $x \in \mathbb{R}^g$ determine a failure of $\mathbb{R}$-admissibility[9] in $\mathsf{HOD}_X[g]$. So we have

$$\forall y \in \mathbb{R}^g\ \exists \gamma < \alpha\ J_\gamma(\mathbb{R}^g) \models \varphi(x, y)$$

and φ is true cofinally often, i. e.

$$\forall \gamma < \alpha\ \exists y \in \mathbb{R}^g\ \ \min\{\eta : J_\eta(\mathbb{R}^g) \models \varphi(x, y)\} > \gamma$$

So if we let $\beta(y) := \min\{\eta : J_\eta(\mathbb{R}^g) \models \varphi(x, y)\}$, then $\beta''\mathbb{R}^g$ is cofinal in α.

Since $\mu_+ := \mu^{+\mathsf{V}}$ is inaccessible in HOD_X, we can assume $x = \tau^{g\restriction\iota}$ for some $\iota < \mu_+, \tau \in \mathsf{HOD}_X$. Let $p_0 \in g\restriction\iota$ force the properties listed so far; in

[8] If $\alpha < \theta^{\mathsf{L}(\mathbb{R}^g)}$, then α is the surjective image of $\mathbb{R}^g$ in $\mathsf{L}(\mathbb{R}^g)$, so we can code any countable sequence of ordinals $(\alpha_i : i < \omega) \in \mathsf{HOD}_X[g]$ cofinal in α by a real.

[9] We can suppose that a real witnesses the failure of admissibility. This is because if α begins a Σ_1-gap then there is a partial $\Sigma_1(J_\alpha(\mathbb{R}))$-surjection from $\mathbb{R}^g$ onto $J_\alpha(\mathbb{R}^g)$; cf. [Ste83, Lemma 1.11(a)].

particular for each $\mathrm{Col}(\omega, <\iota)$-generic filter h over HOD_X which contains p_0 we have

$$\mathsf{HOD}_X[h] \models \exists q \in \mathrm{Col}(\omega, <\mu_+)\ q \Vdash \forall y \in \mathbb{R}\ J_{\check{\alpha}}(\mathbb{R}) \models \varphi(\check{\tau^h}, y).$$

In HOD_X, let $A \in \mathrm{Pow}(<\mu_+)$ be such that A codes

$$\mathfrak{c} := \tau \oplus H_\zeta^{\mathsf{HOD}_X}$$

in a simple fashion.[10] Then there is a term σ_A such that whenever $G \times H$ is $\mathrm{Col}(\omega, <\iota) \times \mathrm{Col}(\omega, A)$-generic over HOD_X, then

1. $\sigma_A^{G\times H} \in \mathbb{R}^G$
2. $\left(\sigma_A^{G\times H}\right)_0 = \tau^G$
3. $\left\{\left(\sigma_A^{G\times H}\right)_i : i \in \omega\right\} = \{\rho^{G\times H} : \rho \text{ is simply coded into } A,\ \rho^{G\times H} \in \mathbb{R}^G\}$

Here $\mathbb{R}^G$ is the set of reals of $\mathsf{HOD}_X[G]$.

The term σ_A is absolute enough so that it exists in each A-premouse $\mathcal{M}$. Moreover, if $G\times H$ is $\mathrm{Col}(\omega, <\iota)\times\mathrm{Col}(\omega, A)$-generic over $\mathcal{M}$, then $\mathcal{M}[G\times H]$ can be considered as a z-premouse, for a real $z = z_{G,H}$, obtained in a simple fashion from A, G, and H (see [Stec]).

For $n < \omega$ let φ_n be the Σ_1-formula[11]

$$\varphi_n(v) \equiv \exists\gamma\big(\gamma + \omega n \text{ exists} \wedge J_\gamma(\mathbb{R}) \models \forall i > 0\ \varphi((v)_0, (v)_i)\big),$$

and let ψ be the natural sentence, such that for all A-premice $\mathcal{M}$:

> $\mathcal{M} \models \psi$ iff whenever $G \times H$ is $\mathrm{Col}(\omega, <\iota) \times \mathrm{Col}(\omega, A)$-generic over $\mathcal{M}$ with $p_0 \in G$, then for all n there is a strictly increasing sequence $(\gamma_i : i \leq n)$ such that for all $i \in [1, n)$
>
> 1. $\mathcal{M}[G \times H]\|\gamma_i$ is a $\langle\varphi_{i+1}, \sigma_A^{G\times H}\rangle$-pre-witness and
> 2. there is a $\delta \in (\gamma_i, \gamma_{i+1}]$ such that $\rho_\omega(\mathcal{M}[G\times H]\|\delta) = \sup(A)$.

[10] Here ζ is as in Definition 2.49, and $a \oplus b$ is a set which simply codes a and b.

[11] We need the various φ_n to prove that the mouse operator we are going to define relativizes well at μ_+ (see Lemma 3.15 below).

Definition 3.9. For any set of ordinals A coding $\mathfrak{c}$, let $\mathcal{M}(A)$ be the shortest initial segment of $\mathsf{Lp}^{\mathsf{V}}(A)$ which satisfies ψ, if it exists, and let $\mathcal{M}(A)$ be undefined otherwise.

Remark. If $\mathcal{M}(A)$ is defined, it is countably iterable in V, so it follows that for each n there is an initial segment of $\mathcal{M}(A)$ which is a $\langle\varphi_n, \sigma_A^{G\times H}\rangle$-witness.

The definition of ψ is different to that of [Ste05]. The reason is that we want to ensure that a premouse $\mathcal{M}$ which satisfies ψ contains cofinally many $\gamma < ht(\mathcal{M})$ with $\rho_\omega(\mathcal{M}\|\gamma) = \sup(A)$, and over each such γ there shall be a λ such that $\mathcal{M}\|\lambda \models \mathsf{ZFC}$.

Proof. There are cofinally many $\gamma < ht(\mathcal{M}(A))$ such that $\mathcal{M}(A)\|\gamma$ projects to $\sup(A)$: This is because if $\lambda := \mathrm{lub}\{\gamma : \rho_\omega(\mathcal{M}(A)\|\gamma) = \sup(A)\}$ were less than the height of $\mathcal{M}(A)$ then each element of the finite sequence $(\gamma_i : i \leq n)$, required by ψ would be less than λ[12] and therefore $\mathcal{M}(A)\|\lambda \models \psi$.

For arbitrary γ with $\rho_\omega(\mathcal{M}(A)\|\gamma) = \sup(A)$, let $\delta_\gamma > \gamma$ be least with $\mathcal{M}(A)\|\delta_\gamma \models \mathsf{ZFC}$. We can prove the existence of δ_γ by showing that there are cofinally many $\delta < ht(\mathcal{M}(A))$ with $\mathcal{M}(A)\|\delta \models \mathsf{ZFC}$. Suppose not and let $\lambda := \mathrm{lub}\{\delta : \mathcal{M}(A)\|\delta \models \mathsf{ZFC}\} < ht(\mathcal{M}(A))$. Since each $\langle\varphi_n, \sigma_A^{G\times H}\rangle$-witness is a model of ZFC we have that if $\mathcal{M}(A)[G\times H]\|\delta$ is a $\langle\varphi_n, \sigma_A^{G\times H}\rangle$-witness, then $\delta < \lambda$. But then already $\mathcal{M}(A)\|\lambda \models \psi$, which contradicts the minimality of $\mathcal{M}(A)$. $\square$

Since $\mathcal{M}(A)$ projects to $\sup(A)$ cofinally often, it is therefore an initial segment of $\mathcal{M}_1(A)$ as defined in 2.36.

It makes sense to define the sentence ψ not only for bounded subsets of μ_+, but for all sets of ordinals A which code $\mathfrak{c}$. Of course for the definition of $\mathcal{M}(A)$ for $A \in \mathrm{Pow}(< \mu_+)$ we would only have needed that A simply codes τ, but if we want to extend the definition to larger sets of ordinals we need that A also codes $H_\zeta^{\mathsf{HOD}_X}$.

We will show that $\mathcal{M}(A)$ exists for all bounded subsets A of κ^+ which code $\mathfrak{c}$.

1. First we show that $\mathcal{M}(A)$ exists for all bounded subsets of μ_+ which are in HOD_X and code $\mathfrak{c}$.

[12] Except possibly for γ_n, but then we can slightly change the sequence to ensure this.

2. By a lift-up argument and the fact that A_0 codes H_κ^{HOD} we can show that $\mathcal{M}(A \oplus \mathfrak{c})$ exists for all $A \in \mathrm{Pow}(< \kappa) \cap \mathsf{HOD}$.

3. Then we use a forcing argument to show that $\mathcal{M}(A)$ is defined for all bounded subsets of κ in V which code $\mathfrak{c}$.

4. Finally another lift-up argument ensures that $\mathcal{M}(A)$ is defined for all $A \in \mathrm{Pow}(< \kappa^+)$ coding $\mathfrak{c}$.

Lemma 3.10. *$\mathcal{M}(A)$ exists for any $A \in \mathrm{Pow}(< \mu_+) \cap \mathsf{HOD}_X$ which simply codes $\mathfrak{c}$.*

The basis for the proof of Lemma 3.10 is the next lemma, which is the analogue of [Ste05, Lemma 1.28]. It gives us a premouse $\mathcal{M}(A)^*$ of minimal height which satisfies ψ and which is countably closed in HOD_X. Then we have to show that $\mathcal{M}(A)^*$ is countably iterable in V to get that it is an initial segment of $\mathsf{Lp}^{\mathsf{V}}(A)$. If we have shown this we have found the desired $\mathcal{M}(A)$.

In the following if we use the notation $\mathcal{M}(A)^*$ for a premouse with the same first order properties as $\mathcal{M}(A)$, i.e., $\mathcal{M}(A)^* \models \psi$, and we shall then prove that $\mathcal{M}(A)^*$ is the desired $\mathcal{M}(A)$. That is, the * indicates that we have already shown $\mathcal{M}(A)^* \models \psi$, but it remains to prove that $\mathcal{M}(A)^*$ is countably iterable in V.

Lemma 3.11. *$\mathcal{M}(A)^*$ exists for any $A \in \mathrm{Pow}(< \mu_+) \cap \mathsf{HOD}_X$ which simply codes $\mathfrak{c}$ and is countably iterable in HOD_X.*

Note that for this argument it suffices that A codes τ. The proof is essentially that of [Ste05, Lemma 1.28].

Proof. Working in $\mathsf{HOD}_X[g]$, let $h \times H$ be $\mathrm{Col}(\omega, < \iota) \times \mathrm{Col}(\omega, A)$-generic over HOD_X with $p_0 \in h$. Since g is $\mathrm{Col}(\omega, < \mu_+)$-generic such an object exists.

For all $q \leq p_0$ and all r let $h_q \times H_r$ be the finite variant of $h \times H$ as defined on page 2. But then $z_{h_q,H_r} \equiv_T z_{h,H} =: z$.

Claim 1. For all $n < \omega$ $J_\alpha(\mathbb{R}^g) \models \varphi_n(\sigma_A^{h_q \times H_r})$, so we can use the induction hypothesis (W_α) to get a $\langle \varphi_n, \sigma_A^{h_q \times H_r} \rangle$-witness for each $n < \omega$.

Proof. We us the fact that p_0 forces the properties of x and φ. Since $p_0 \in h_q$ we have

$$\mathsf{HOD}_X[h_q] \models \exists \tilde{q} \in \mathrm{Col}(\omega, < \mu_+)\ \tilde{q} \Vdash \forall y \in \mathbb{R}\ J_{\check{\alpha}}(\mathbb{R}) \models \varphi(\tau^{\check{h}_q}, y).$$

Fix such a $\tilde{q}$ and let $k \in \mathsf{HOD}_X[g]$ be a $\mathrm{Col}(\omega, < \mu_+)$-generic object over $\mathsf{HOD}_X[h_q]$ such that $\mathsf{HOD}_X[h_q][k_{\tilde{q}}] = \mathsf{HOD}_X[g]$.[13]

$$\mathsf{HOD}_X[g] = \mathsf{HOD}_X[h_q][k_{\tilde{q}}] \models \forall y \in \mathbb{R}\ J_\alpha(\mathbb{R}) \models \varphi(\tau^{h_q}, y).$$

So by the definition of σ_A:

$$J_\alpha(\mathbb{R}^g) \models \forall i > 0\big(\varphi((\sigma_A^{h_q \times H_r})_0, (\sigma_A^{h_q \times H_r})_i)\big).$$

Now we use the uncountable cofinality of α which yields $J_\alpha(\mathbb{R}^g) \models \varphi_n(\sigma_A^{h_q \times H_r})$ for all n. $\square$(*Claim* 1)

First let

$\mathcal{N}_{0,q,r}$ be the $\langle \varphi_0, \sigma_A^{h_q \times H_r}\rangle$-witness of least height, which exists by (W_α).

The definition of (W_α) implies that $\mathcal{N}_{0,q,r} \trianglelefteq \mathsf{Lp}(\sigma_A^{h_q \times H_r})$, so $\mathcal{N}_{0,q,r}$ is a $\sigma_A^{h_q \times H_r}$-mouse. But $\sigma_A^{h_q \times H_r}$ and z_{h_q, H_r} are easily computable from one another, so $\mathcal{N}_{0,q,r}$ can be considered as a z_{h_q,H_r}-mouse. Moreover, z_{h_q,H_r} is Turing-equivalent to z, so we can suppose that each $\mathcal{N}_{0,q,r}$ is a z-mouse, whose iteration strategy, if restricted to countable trees, is in $J_\alpha(\mathbb{R}^g)$. Therefore we have $\mathcal{N}_{0,q,r} \trianglelefteq \mathsf{Lp}(z)$ for any q and r, so that we can build the union

$$\mathcal{N}_0 = \bigvee\{\mathcal{N}_{0,q,r} : q \in \mathrm{Col}(\omega, < \iota), q \leq p_0, r \in \mathrm{Col}(\omega, A)\}.$$

$\mathcal{N}_0$ is an initial segment of $\mathsf{Lp}(z)$ up to its ω_1. Since each of the countably many $\mathcal{N}_{0,q,r}$ has an iteration strategy whose restriction to countable trees is in $J_\alpha(\mathbb{R}^g)$, and since α has uncountable cofinality, also $\mathcal{N}_0$ has an iteration strategy which, if restricted to countable trees, is in $J_\alpha(\mathbb{R}^g)$. Finally let $\gamma_0 \geq ht(\mathcal{N}_0)$ be least such that $\rho_\omega(\mathsf{Lp}(z)\|\gamma_0) = \omega$.

Now suppose we have built $\mathcal{N}_{n-1}$ of height γ_{n-1}. Then as before let

$\mathcal{N}_{n,q,r}$ be a $\langle \varphi_n, \sigma_A^{h_q \times H_r}\rangle$-witness of height $> \gamma_{n-1}$, which exists by (W_α).[14]

A witness of height larger than γ_{n-1} exists, since by definition of (W_α) there are cofinally many $\gamma < \omega_1$ such that $\mathsf{Lp}(\sigma_A^{h_q \times H_r})\|\gamma$ is a $\langle \varphi_n, \sigma_A^{h_q \times H_r}\rangle$-witness.

[13] Such a k exists, because there is an dense embedding from a dense subset of $\mathrm{Col}(\omega, < \mu_+)$ into $\mathrm{Col}(\omega, < \iota) \times \mathrm{Col}(\omega, < \mu_+)$.

[14] Here is the point where we use the additional condition in (W_α) that we have cofinally in $\omega_1^{\mathsf{HOD}_X[g]}$ many witnesses.

Again we consider each $\mathcal{N}_{n,q,r}$ as a z-mouse so each $\mathcal{N}_{n,q,r}$ end-extends $\mathcal{N}_{n-1}$. We may let $\mathcal{N}_n$ be the union of these $\mathcal{N}_{n,q,r}$ which implies $\mathcal{N}_n \trianglerighteq \mathcal{N}_{n-1}$. As before we use the uncountable cofinality of α to show that $\mathcal{N}_n$ has an iteration strategy which, if restricted to countable iteration trees, is an element of $J_\alpha(\mathbb{R}^g)$. Let $\gamma_n \geq ht(\mathcal{N}_n)$ be least such that $\rho_\omega(\mathsf{Lp}(z)\|\gamma_n) = \omega$.

Finally set

$$\mathcal{N} := \bigtriangledown_{n\in\omega} \mathcal{N}_n$$

Now let $\mathcal{P}$ be the premouse constructed over A from the extender sequence of $\mathcal{N}$. Then one can show that $\mathcal{P}$ is a mouse over A such that $\mathcal{P}[h \times H] = \mathcal{N}$. This is an induction on η. One shows inductively that $\mathcal{P}|\eta \in \mathsf{HOD}_X$ and $(\mathcal{P}|\eta)[h \times H] = \mathcal{N}|\eta$ (see [Stec, Theorem 3.9] or [SSb, Lemma 1.5] for such an argument). Moreover, $\mathcal{P}$ is an iterable mouse in $\mathsf{HOD}_X[g]$ and the canonical iteration strategy, when restricted to HOD_X, is in HOD_X. So $\mathcal{P}$ is also iterable in HOD_X.

Now we have that $\mathcal{P}$ is the desired $\mathcal{M}(A)^*$: Of course $\mathcal{N} = \mathcal{P}[h \times H]$ fulfills the properties which are required of the generic extension of a model which satisfies ψ. But then, by the homogeneity of $\mathrm{Col}(\omega, < \iota) \times \mathrm{Col}(\omega, A)$, we have $\mathcal{P} \models \psi$. □

Lemma 3.11 ensures that $\mathcal{M}(A)^*$ is countably iterable in HOD_X. For the countable iterability in V we use the following lemmata:

Lemma 3.12. *For all $A \in \mathrm{Pow}(< \mu_+) \cap \mathsf{HOD}_X$ which simply code $\mathfrak{c}$ and for all γ such that $\rho_\omega(\mathcal{M}(A)^*\|\gamma) \leq \sup(A)$, we have*

$$\Vdash^{\mathrm{Col}(\omega,\varepsilon)}_{\mathcal{M}(A)^*\|\delta_\gamma} (\mathcal{M}(A)^*\|\gamma)\check{} \text{ is countably iterable.}$$

Remember from Definition 2.49 that ε is chosen such that $H_\varepsilon^{\mathsf{HOD}_X}$ contains every subset of $\omega_1^{\mathsf{HOD}_X}$.

Proof. Let h be $\mathrm{Col}(\omega, \varepsilon)$-generic over $\mathcal{M}(A)^*\|\delta_\gamma$. Since A codes $H_\zeta^{\mathsf{HOD}_X}$, and $H_\zeta^{\mathsf{HOD}_X}$ contains all subsets of $\mathrm{Col}(\omega, \varepsilon)$ which lie in HOD_X, it follows that h is $\mathrm{Col}(\omega, \varepsilon)$-generic over HOD_X. $(\mathcal{M}(A)^*\|\delta_\gamma)[h]$ and $\mathsf{HOD}_X[h]$ contain the same sets of subsets of their ω_1, so we have:

$$(\mathcal{M}(A)^*\|\delta_\gamma)[h] \models \mathcal{M}(A)^*\|\gamma \text{ is countably iterable} \quad \Leftrightarrow$$
$$\mathsf{HOD}_X[h] \models \mathcal{M}(A)^*\|\gamma \text{ is countably iterable}$$

But $\mathcal{M}(A)^*$ is iterable via the canonical iteration strategy in $\mathsf{HOD}_X[g]$ and since $\rho_\omega(\mathcal{M}(A)^*\|\gamma) \leq \sup(A)$ also $\mathcal{M}(A)^*\|\gamma$ is iterable. So the following is forced: There is a $q \in g, q \leq p_0$ such that

$$q \Vdash^{\mathrm{Col}(\omega,<\mu_+)}_{\mathsf{HOD}_X} (\mathcal{M}(A)^*\|\gamma)\check{} \text{ is iterable via the unique iteration strategy.}$$

Since we have that a dense subset of $\mathrm{Col}(\omega,\varepsilon) \times \mathrm{Col}(\omega,<\mu_+)$ is isomorphic to a dense subset of $\mathrm{Col}(\omega,<\mu_+)$, we also have that $\mathcal{M}(A)^*\|\gamma$ is iterable in $\mathsf{HOD}_X[h][g']$ for any $\mathrm{Col}(\omega,<\mu_+)$-generic g'.[15] The ω_1+1-iteration strategy is unique, so the restriction to $\mathsf{HOD}_X[h]$ is in $\mathsf{HOD}_X[h]$ by the homogeneity of $\mathrm{Col}(\omega,<\mu_+)$. But then $\mathcal{M}(A)^*\|\gamma$ is countably iterable in $\mathsf{HOD}_X[h]$ and therefore

$$(\mathcal{M}(A)^*\|\delta_\gamma)[h] \models \mathcal{M}(A)^*\|\gamma \text{ is countably iterable.} \qquad \square$$

Now we can show that $\mathcal{M}(A)^*$ is countably iterable in V.

Lemma 3.13. *Let A code $\mathfrak{c}$ such that $\mathcal{M}(A)^*$ exists and suppose that for all γ with $\rho_\omega(\mathcal{M}(A)^*\|\gamma) \leq \sup(A)$ we have $\Vdash^{\mathrm{Col}(\omega,\varepsilon)}_{\mathcal{M}(A)^*\|\delta_\gamma}$ "$(\mathcal{M}(A)^*\|\gamma)\check{}$ is countably iterable".*

Then $\mathcal{M}(A)^*$ is countably iterable in V as well as in any inner model $W \supseteq \mathsf{HOD}_{X,A}$.

Proof. Since the set of all γ with $\rho_\omega(\mathcal{M}(A)^*\|\gamma) \leq \sup(A)$ is cofinal in $ht(\mathcal{M}(A)^*)$, it suffices to only consider the premice which are elementarily embeddable into a "good" initial segment of $\mathcal{M}(A)^*$, i. e. the premice which are embeddable into $\mathcal{M}(A)^*\|\gamma$ where γ is such that $\rho_\omega(\mathcal{M}(A)^*\|\gamma) \leq \sup(A)$. We show that these "good" initial segments of $\mathcal{M}(A)^*$ are countably iterable.

Set $\mathcal{M} := \mathcal{M}(A)^*\|\delta_\gamma$ and suppose $\pi\colon \mathcal{N} \to \mathcal{M}(A)^*\|\gamma$ is elementary where $\mathcal{N} \in \mathsf{V}$ is countable. We define an ω_1+1-iteration strategy Σ for $\mathcal{N}$. Suppose $\Sigma\restriction\xi$ is already defined for a limit ordinal $\xi \leq \omega_1$ and suppose $\mathcal{T} \in \mathsf{V}$ is

[15] Consider $\tilde{g}$, the $\mathrm{Col}(\omega,<\mu_+)$-generic object over HOD_X, associated to $h \times g'$. Then $\mathcal{M}(A)^*\|\gamma$ is iterable via the unique iteration strategy in $\mathsf{HOD}_X[\tilde{g}_q]$. The equations $\mathsf{HOD}_X[\tilde{g}_q] = \mathsf{HOD}_X[\tilde{g}] = \mathsf{HOD}_X[h][g']$ give the desired conclusion.

an iteration tree on $\mathcal{N}$ of length ξ according to $\Sigma\restriction\xi$. We assume that $\Sigma\restriction\xi$ was defined according to the very same recipe which we are about to use for the definition of $\Sigma(\mathcal{T})$. [16] As usual we identify $\mathcal{N}$ and $\mathcal{T}$ with countable sets of ordinals coding them, so $\mathcal{N}$ and $\mathcal{T}$ are Vopěnka-generic over $H_\varepsilon^{\mathsf{HOD}_X}$ and therefore, since $\mathcal{M}$ is an A-premouse and A codes $H_\varepsilon^{\mathsf{HOD}_X}$, they are also Vopěnka-generic over $\mathcal{M}$. In $\mathcal{M}[\mathcal{N},\mathcal{T}]$ we can construct a tree searching for an elementary embedding from $\mathcal{N}$ to $\mathcal{M}(A)^*\|\gamma$. By absoluteness of well-foundedness we can find a $\sigma \in \mathcal{M}[\mathcal{N},\mathcal{T}]$ such that $\sigma\colon \mathcal{N} \to \mathcal{M}(A)^*\|\gamma$ (cf. [Sch01, Lemma 0.2]).

Since $\mathcal{N},\mathcal{T}$ are both Vop_{ω_1}-generic and since ε is larger than the size of Vop_{ω_1}, we can find for each $\mathrm{Col}(\omega,\varepsilon)$-generic object h over $\mathcal{M}[\mathcal{N},\mathcal{T}]$ an h' which is $\mathrm{Col}(\omega,\varepsilon)$-generic over $\mathcal{M}$ such that $\mathcal{M}[h'] = \mathcal{M}[\mathcal{N},\mathcal{T}][h]$. Because of $\Vdash^{\mathrm{Col}(\omega,\varepsilon)}_{\mathcal{M}(A)^*\|\delta_\gamma}$ "$(\mathcal{M}(A)^*\|\gamma)\check{}$ is countably iterable", we get that $\mathcal{M}(A)^*\|\gamma$ is countably iterable in $\mathcal{M}[h']$. Hence $\mathcal{N}$ is ω_1+1-iterable via the $\mathcal{Q}$-structure iteration strategy in $\mathcal{M}[h']$. So there is a $p \in h$ and a term $\dot\Sigma \in \mathcal{M}[\mathcal{N},\mathcal{T}]$ with:

$$p \Vdash^{\mathrm{Col}(\omega,\varepsilon)}_{\mathcal{M}[\mathcal{N},\mathcal{T}]} \check{\mathcal{N}} \text{ is } \omega_1+1\text{-iterable via the } \mathcal{Q}\text{-structure iteration strategy } \dot\Sigma$$

Claim 1. Suppose k is another $\mathrm{Col}(\omega,\varepsilon)$-generic object over $\mathcal{M}[\mathcal{N},\mathcal{T}]$, and $q \in k$, $\dot\Gamma \in \mathcal{M}[\mathcal{N},\mathcal{T}]$ satisfy

$$q \Vdash^{\mathrm{Col}(\omega,\varepsilon)}_{\mathcal{M}[\mathcal{N},\mathcal{T}]} \check{\mathcal{N}} \text{ is } \omega_1+1\text{-iterable via the } \mathcal{Q}\text{-structure iteration strategy } \dot\Gamma$$

Then $\dot\Gamma^k\restriction\mathcal{M}[\mathcal{N},\mathcal{T}] = \dot\Sigma^h\restriction\mathcal{M}[\mathcal{N},\mathcal{T}]$.

Proof. Suppose not and let $\mathcal{S} \in \mathcal{M}[\mathcal{N},\mathcal{T}]$ be an iteration tree of minimal length such that $\mathcal{S}$ is built according to $\dot\Gamma^k$ as well as $\dot\Sigma^h$, but $\dot\Gamma^k(\mathcal{S}) \neq \dot\Sigma^h(\mathcal{S})$. W. l. o. g. there is some $\xi \in \dot\Sigma^h(\mathcal{S}) \setminus \dot\Gamma^k(\mathcal{S})$. Let $r \in h$, $r \leq p$ and $s \in k$, $s \leq q$ be such that

$$r \Vdash^{\mathrm{Col}(\omega,\varepsilon)}_{\mathcal{M}[\mathcal{N},\mathcal{T}]} \check\xi \in \dot\Sigma(\check{\mathcal{S}}) \qquad \text{and} \qquad s \Vdash^{\mathrm{Col}(\omega,\varepsilon)}_{\mathcal{M}[\mathcal{N},\mathcal{T}]} \check\xi \notin \dot\Gamma(\check{\mathcal{S}})$$

But now we have for all g which are $\mathrm{Col}(\omega,\varepsilon)$-generic over $\mathcal{M}[\mathcal{N},\mathcal{T}]$:

$$\mathcal{M}[\mathcal{N},\mathcal{T}][g] = \mathcal{M}[\mathcal{N},\mathcal{T}][g_r] = \mathcal{M}[\mathcal{N},\mathcal{T}][g_s]$$

Then $\dot\Sigma^{g_r} = \dot\Gamma^{g_s}$, since both are identical with the $\mathcal{Q}$-structure iteration strategy. That is contradictory. $\square$(*Claim* 1)

[16] The proof to come even works for iteration trees of size $< \varepsilon$.

So we want to define $\Sigma(\mathcal{T}) := \dot{\Sigma}^h(\mathcal{T})$. If we can show that $\dot{\Sigma}^h(\mathcal{T})$ is defined, then by Claim 1, this is independent from h, i. e. we have to show the following.

Claim 2. $\mathcal{T}$ is built according to $\dot{\Sigma}^h$.

Proof. Here we use that $\Sigma\restriction\xi$ was defined according to the very same recipe which we are about to use for the definition of $\Sigma(\mathcal{T})$. Suppose not. Let $\lambda < \xi$ be least such that $\dot{\Sigma}^h(\mathcal{T}\restriction\lambda) \neq [0,\lambda)_{\mathcal{T}}$. We can split the set of ordinals coding $\mathcal{T}$ into two parts: The first part codes $\mathcal{T}\restriction\lambda$ and the second part codes the rest, call it $\mathcal{T}^\lambda$. So $\mathcal{M}[\mathcal{N},\mathcal{T}] = \mathcal{M}[\mathcal{N},\mathcal{T}\restriction\lambda,\mathcal{T}^\lambda]$. But now there is a forcing $\mathbb{Q} \in \mathcal{M}[\mathcal{N},\mathcal{T}\restriction\lambda]$ of size $< \varepsilon$ such that $\mathcal{M}[\mathcal{N},\mathcal{T}]$ is a $\mathbb{Q}$-generic extension of $\mathcal{M}[\mathcal{N},\mathcal{T}\restriction\lambda]$. So there is an h' $\mathrm{Col}(\omega,\varepsilon)$-generic over $\mathcal{M}[\mathcal{N},\mathcal{T}\restriction\lambda]$ with

$$\mathcal{M}[\mathcal{N},\mathcal{T}][h] = \mathcal{M}[\mathcal{N},\mathcal{T}\restriction\lambda][h'].$$

By the definition of $\Sigma\restriction\xi$ we have $[0,\lambda)_{\mathcal{T}} = \dot{\Gamma}^{h'}$ for a name $\dot{\Gamma} \in \mathcal{M}[\mathcal{N},\mathcal{T}\restriction\lambda]$ for the unique ω_1+1-iteration strategy in $\mathcal{M}[\mathcal{N},\mathcal{T}\restriction\lambda][h']$. But since $\mathcal{T}\restriction\lambda$ is built according to the iteration strategy $\dot{\Sigma}^h$, and $\dot{\Sigma}^h$ is the unique ω_1+1-iteration strategy in $\mathcal{M}[\mathcal{N},\mathcal{T}][h] = \mathcal{M}[\mathcal{N},\mathcal{T}\restriction\lambda][h']$, we get

$$\dot{\Sigma}^h(\mathcal{T}\restriction\lambda) = \dot{\Gamma}^{h'}(\mathcal{T}\restriction\lambda) = [0,\lambda)_{\mathcal{T}}$$

by the first claim, which is a contradiction. So we can set $\Sigma(\mathcal{T}\restriction\lambda) := \dot{\Sigma}^h(\mathcal{T}\restriction\lambda)$. □(*Claim* 2)

Since the branch $\Sigma(\mathcal{T})$ can be defined in any $\mathrm{Col}(\omega,\varepsilon)$-generic extension of $\mathcal{M}(A)^*[\mathcal{N},\mathcal{T}]$ we also have $\Sigma(\mathcal{T}) \in \mathcal{M}(A)^*[\mathcal{N},\mathcal{T}] \subseteq \mathsf{V}$ by homogeneity, and therefore $\mathcal{N}$ is ω_1+1-iterable in V. □

The last two lemmata ensure that $\mathcal{M}(A)^*$ is countably iterable not only in HOD_X but also in V and therefore $\mathcal{M}(A)^* \trianglelefteq \mathsf{Lp}(A)$. Hence $\mathcal{M}(A) = \mathcal{M}(A)^*$. This completes the proof of Lemma 3.10.

To show that the $\mathcal{M}$-operator is defined for all bounded subsets of κ in HOD we want to use a reflection argument, so we need the $\mathcal{M}$-operator to behave correctly. For this purpose we use the following definition.

Definition 3.14. An operator $\mathcal{O}$ *relativizes well at* μ_+ iff there is a formula $\Phi(v_0,v_1,v_2)$ such that the following condition is fulfilled:

Suppose $A \in \mathrm{Pow}(< \mu_+)$ is coded into some $B \in \mathrm{Pow}(< \mu_+)$, $\mathcal{O}(A)$ exists, and W is a transitive model of ZFC^- such that $\mathcal{O}(B) \in W$. Then $\mathcal{O}(A)$ is the unique $x \in W$ such that $W \models \Phi(x, A, \mathcal{O}(B))$.

So $\mathcal{O}$ relativizes well if $\mathcal{O}(A)$ is uniformly computable from A and $\mathcal{O}(B)$.

Lemma 3.15. *The $\mathcal{M}$-operator relativizes well at μ_+ in HOD_X.*

For the proof, see [Ste05, Lemma 1.29].

Lemma 3.16. *$\mathcal{M}(A \oplus \mathfrak{c})$ exists for all $A \in \mathrm{Pow}(< \kappa) \cap \mathsf{HOD}$, i. e. $\mathcal{M}(A \oplus \mathfrak{c})$ satisfies ψ and is countably iterable in V.*

Proof. Let Ω be a large ordinal. In HOD_X we can build elementary substructures of V_Ω of size μ, closed under ε-sequences and cofinal in λ.[17] So let $\mathcal{X}$ be an elementary substructure of $V_\Omega^{\mathsf{HOD}_X}$ with these properties such that additionally A_0, A, $\mathfrak{c}$, X, κ, and λ are elements of $\mathcal{X}$. Moreover, we demand that $\mu + 1$ and $\mathrm{Pow}(\omega_1)^{\mathsf{HOD}_X}$ are subsets of $\mathcal{X}$. Then we can collapse $\mathcal{X}$ to a transitive structure H with uncollapsing map π:

$$\pi \colon H \to \mathsf{HOD}_X$$

Since $\mathfrak{c}$ is small it is not moved by π, so let $\bar{A}_0 \oplus \mathfrak{c}$, $\bar{A} \oplus \mathfrak{c}$, $\bar{X}$, $\bar{\kappa}$, and $\bar{\lambda}$ be the preimages of $A_0 \oplus \mathfrak{c}$, $A \oplus \mathfrak{c}$, X, κ, and λ, respectively. It suffices to see that $\mathcal{M}(\bar{A} \oplus \mathfrak{c}) \in H$, because then $\pi(\mathcal{M}(\bar{A} \oplus \mathfrak{c}))$ is countably iterable in V by Lemmata 3.12 and 3.13 and therefore it is equal to $\mathcal{M}(A \oplus \mathfrak{c})$.

It actually suffices to prove $\mathcal{M}(\bar{A}_0 \oplus \mathfrak{c}) \in H$, because $\bar{A} \oplus \mathfrak{c}$ is simply coded into $\bar{A}_0 \oplus \mathfrak{c}$ by the elementarity of π and the $\mathcal{M}$-operator relativizes well at μ_+, so also $\mathcal{M}(\bar{A} \oplus \mathfrak{c}) \in H$.

Claim 1. $\mathcal{M}(\bar{A}_0 \oplus \mathfrak{c})$ is an element of H.

Proof. Let $\mathsf{Lp}^H(\bar{A}_0 \oplus \mathfrak{c})$ be the lower part closure of $\bar{A}_0 \oplus \mathfrak{c}$ built in H. First note

$$\mathsf{Lp}^H(\bar{A}_0 \oplus \mathfrak{c}) \| \bar{\lambda} = \mathsf{Lp}^{\mathsf{HOD}_X}(\bar{A}_0 \oplus \mathfrak{c}) \| \bar{\lambda}.$$

This is true since for $\mathcal{M} \trianglelefteq \mathsf{Lp}^H(\bar{A}_0 \oplus \mathfrak{c}) \| \bar{\lambda}$ we know that $\mathcal{M}$ is countably iterable in HOD_X[18], and therefore $\mathcal{M} \trianglelefteq \mathsf{Lp}^{\mathsf{HOD}_X}(\bar{A}_0 \oplus \mathfrak{c}) \| \bar{\lambda}$.

[17] Note that $\lambda = \kappa^{+\mathsf{Lp}(A_0)}$, $\mathrm{cof}(\lambda) < \mu$ and $\mu^\varepsilon = \mu$ hold in HOD_X; cf. Definition 2.50.

[18] We arranged that $\mathrm{Pow}(\omega_1)^{\mathsf{HOD}_X} \subseteq H$.

Suppose $\mathcal{M}(\bar{A}_0 \oplus \mathfrak{c})$ is not an element of H. Then there is a first initial segment $\mathcal{M} \trianglelefteq \mathcal{M}(\bar{A}_0 \oplus \mathfrak{c})$ which extends $\mathsf{Lp}^H(\bar{A}_0 \oplus \mathfrak{c})\|\bar{\lambda}$ and projects to $\bar{\kappa}$, $\rho_\omega(\mathcal{M}) \leq \bar{\kappa}$.

Remark. Note that in fact we have $\mathsf{Lp}^{\mathsf{HOD}_X}(\bar{A}_0 \oplus \mathfrak{c})\|\bar{\lambda} \trianglelefteq \mathsf{Lp}^{\mathsf{V}}(\bar{A}_0 \oplus \mathfrak{c})$. For this consider some $\mathcal{M} \trianglelefteq \mathsf{Lp}^{\mathsf{HOD}_X}(\bar{A}_0 \oplus \mathfrak{c})$ with $\rho_\omega(\mathcal{M}) = \bar{\kappa}$. One can show as in Lemma 3.12 that "$\check{\mathcal{M}}$ is countably iterable" is forced over $\mathsf{Lp}^{\mathsf{HOD}_X}(\bar{A}_0 \oplus \mathfrak{c})\|\bar{\lambda}$. Now Lemma 3.13 can be used to show that $\mathcal{M}$ is iterable in V, and therefore $\mathcal{M} \trianglelefteq \mathsf{Lp}^{\mathsf{V}}(\bar{A}_0 \oplus \mathfrak{c})$.

Subclaim 1.1. We can lift $\mathcal{M}$ to a premouse $\mathcal{N}$ extending $\mathsf{Lp}^{\mathsf{HOD}_X}(A_0 \oplus \mathfrak{c})\|\lambda$ such that $\rho_\omega(\mathcal{N}) \leq \kappa$.

Proof. Let E be the $(cr(\pi), \lambda)$-extender derived from $\pi \upharpoonright (\mathsf{Lp}^H(\bar{A}_0 \oplus \mathfrak{c})\|\bar{\lambda})$. Then one can see as in [SZ, Lemma 8.14] that E is $\aleph_0$-complete and therefore $\mathrm{ult}(\mathcal{M}, E)$ is wellfounded. The premouse $\mathcal{N}$ we are looking for will be the transitive collapse of $\mathrm{ult}(\mathcal{M}, E)$. Moreover, $\mathcal{N}$ extends $\mathsf{Lp}^{\mathsf{HOD}_X}(A_0 \oplus \mathfrak{c})\|\lambda$ since $\mathrm{ran}(\pi) \cap \lambda$ is cofinal in λ. $\square$(*Subclaim* 1.1)

Subclaim 1.2. $\mathcal{N}$ is countably iterable in V.

Proof. Let $\sigma \colon \mathcal{N}' \to \mathcal{N}$ be an elementary embedding where $\mathcal{N}'$ is countable and $\sigma, \mathcal{N}' \in \mathsf{V}$. We then have that $\mathcal{N}'$, or rather a set of countable ordinals simply coding $\mathcal{N}'$, is Vopěnka-generic over H, because it is Vopěnka-generic over $H_\varepsilon^{\mathsf{HOD}_X} \subseteq H$ by the choice of ε (cf. Definition 2.49). Since $cr(\pi) > \mu > \varepsilon$, we can extend π to an elementary embedding $\tilde{\pi}$ with

$$\tilde{\pi} \colon H[\mathcal{N}'] \to \mathsf{HOD}_X[\mathcal{N}'], \quad \tilde{\pi} \upharpoonright H = \pi.$$

H is closed under ε-sequences in HOD_X, so we can conclude that $H[\mathcal{N}']$ is closed under ω-sequences in $\mathsf{HOD}_X[\mathcal{N}']$.[19] By absoluteness of wellfoundedness we can find in $\mathsf{HOD}_X[\mathcal{N}']$ an embedding $\bar{\sigma} \colon \mathcal{N}' \to \mathcal{N}$. Since

$$\tilde{\pi} \upharpoonright (\mathsf{Lp}^H(\bar{A}_0 \oplus \mathfrak{c})\|\bar{\lambda}) = \pi \upharpoonright (\mathsf{Lp}^H(\bar{A}_0 \oplus \mathfrak{c})\|\bar{\lambda})$$

and $H[\mathcal{N}']$ is closed under ω-sequences it follows that the extender E from Subclaim 1.1 is also $\aleph_0$-complete in $\mathsf{HOD}_X[\mathcal{N}']$. So by the proof of [SZ, Lemma 8.12] we get an elementary embedding $\pi' \colon \mathcal{N}' \to \mathcal{M}$. Note that $\pi' \in \mathsf{V}$. Since $\mathcal{M}$ is countably iterable in V, we get that $\mathcal{N}'$ is $\omega_1 + 1$-iterable in V. This ensures that $\mathcal{N}$ is countably iterable in V. $\square$(*Subclaim* 1.2)

[19] See for example [Fuc, Lemma 2.6].

But this is a contradiction, because then $\mathcal{N}$ would be an initial segment of the lower part closure of A_0 in V of height $\geq \lambda = \kappa^{+\mathsf{Lp}(A_0)}$, which projects to κ. $\square$(*Claim* 1)

So we have $\mathcal{M}(\bar{A}_0 \oplus \mathfrak{c}) \in H$ and therefore, since the $\mathcal{M}$-operator relativizes well, we also have $\mathcal{M}(\bar{A} \oplus \mathfrak{c}) \in H$.

The restriction of the embedding π to $\mathcal{M}(\bar{A} \oplus \mathfrak{c})$ is an elementary embedding into $\mathcal{M}(A \oplus \mathfrak{c})^*$, so since countable iterability is expressible by first order formulae which are preserved by elementary embeddings we also get

$$\Vdash^{\mathrm{Col}(\omega,\varepsilon)}_{\mathcal{M}(A\oplus\mathfrak{c})^*\|\delta_\gamma} (\mathcal{M}(A \oplus \mathfrak{c})^* \| \gamma)\check{} \text{ is countably iterable}$$

for each γ such that $\rho_\omega(\mathcal{M}(A \oplus \mathfrak{c})^* \| \gamma) \leq \sup(A)$. So we have that $\mathcal{M}(A \oplus \mathfrak{c})^*$ is countably iterable in V by Lemma 3.13, hence $\mathcal{M}(A \oplus \mathfrak{c})^* = \mathcal{M}(A \oplus \mathfrak{c})$. $\square$

Now we show that $\mathcal{M}(A)$ exists for every bounded subset of κ in V.

Lemma 3.17. $\mathcal{M}(A)$ *exists for all* $A \in \mathrm{Pow}(< \kappa) \cap \mathsf{V}$ *which simply code* $\mathfrak{c}$.

Proof. Let $A \in \mathsf{V}$ be bounded in κ. Then A is Vopěnka-generic over HOD for a forcing of size $< \kappa$. Let γ be the size of the Vopěnka forcing. So we can find an $\mathrm{Col}(\omega, \gamma)$-generic object G over HOD such that $A \in \mathsf{HOD}[G]$.[20] Then there exists a name $\rho \in \mathsf{HOD}^{\mathrm{Col}(\omega,\gamma)}$ with $\rho^G = A$. But ρ is essentially a bounded subset of κ in HOD. By the previous lemmata $\mathcal{M}(\rho \oplus \mathfrak{c})$ exists and the forcing $\mathrm{Col}(\omega, \gamma)$ is an element of $\mathcal{M}(\rho \oplus \mathfrak{c})$.

Since $A = \rho^G \in \mathcal{M}(\rho \oplus \mathfrak{c})[G]$ we can now use the proof of [Jec03, Corollary 15.42] to get a complete subalgebra $\mathbb{B} \in \mathcal{M}(\rho \oplus \mathfrak{c})$ of $\mathrm{Col}(\omega, \gamma)$ and a $\mathbb{B}$-generic object H over $\mathcal{M}(\rho \oplus \mathfrak{c})$ such that $\mathcal{M}(\rho \oplus \mathfrak{c})[H] = \mathcal{M}(\rho \oplus \mathfrak{c})[A]$ and $\rho^H = \rho^G = A$. We have $\mathbb{B}, \mathfrak{c} \in \mathcal{M}(\rho \oplus \mathfrak{c})$, so $\mathcal{M}(\rho \oplus \mathfrak{c})$ can calculate the canonical name $\check{\mathfrak{c}}$ from $\mathfrak{c}$ and vice versa. Hence $\mathcal{M}(\rho \oplus \check{\mathfrak{c}})$ is also defined. But now $\mathcal{M}(\rho \oplus \check{\mathfrak{c}})[H]$ can be rearranged as a $\rho^H \oplus \check{\mathfrak{c}}^H = A \oplus \mathfrak{c}$-premouse $\mathcal{N}$.

Claim 1. $\mathcal{N}$ satisfies the sentence ψ from the definition of $\mathcal{M}(A)$.

Proof. Suppose $I \times K$ is $\mathrm{Col}(\omega, < \iota) \times \mathrm{Col}(\omega, A \oplus \mathfrak{c})$-generic over $\mathcal{N}$ such that $p_0 \in I$. Since

$$\mathcal{N} = \mathcal{M}(\rho \oplus \check{\mathfrak{c}})[H],$$

[20] Note that $G \notin \mathsf{V}$.

the following holds true:

$$\mathcal{N}[I \times K] = \mathcal{M}(\rho \oplus \check{\mathfrak{c}})[H][I][K] = \mathcal{M}(\rho \oplus \check{\mathfrak{c}})[H][K][I]$$

If $\dot{Q} \in \mathcal{M}(\rho \oplus \check{\mathfrak{c}})$ is a name for the forcing $\mathrm{Col}(\omega, A \oplus \mathfrak{c}) \in \mathcal{M}(\rho \oplus \check{\mathfrak{c}})[H]$, we have that H and K determine a $\mathbb{B} * \dot{Q}$-generic filter $H * K$ over $\mathcal{M}(\rho \oplus \check{\mathfrak{c}})$ such that

$$\mathcal{M}(\rho \oplus \check{\mathfrak{c}})[H][K] = \mathcal{M}(\rho \oplus \check{\mathfrak{c}})[H * K].$$ [21]

Let $e \colon D \to \mathbb{B} * \dot{Q}$ be an injective dense embedding, where $D \subseteq \mathrm{Col}(\omega, \rho \oplus \check{\mathfrak{c}})$ is dense and suppose K' is the $\mathrm{Col}(\omega, \rho \oplus \check{\mathfrak{c}})$-generic filter determined by e and $H * K$. So

$$\mathcal{N}[I \times K] = \mathcal{M}(\rho \oplus \check{\mathfrak{c}})[K'][I] = \mathcal{M}(\rho \oplus \check{\mathfrak{c}})[I \times K'].$$

Now find $n < \omega$ and $\gamma \leq ht(\mathcal{M}(\rho \oplus \check{\mathfrak{c}})[I \times K'])$ such that $\mathcal{M}(\rho \oplus \check{\mathfrak{c}})[I \times K'] \| \gamma$ is a $\langle \varphi_n, \sigma^{I \times K'}_{\rho \oplus \check{\mathfrak{c}}} \rangle$-pre-witness. Since the sets of reals which are coded by $\sigma^{I \times K'}_{\rho \oplus \check{\mathfrak{c}}}$ and $\sigma^{I \times K}_{\rho^H \oplus \check{\mathfrak{c}}^H}$ respectively are the same, we can conclude that such in initial segment is also a $\langle \varphi_n, \sigma^{I \times K}_{\rho^H \oplus \check{\mathfrak{c}}^H} \rangle$-pre-witness, so we are done. $\square$(*Claim* 1)

Claim 2. $\mathcal{N}$ is countably iterable in V and therefore an initial segment of $\mathsf{Lp}(A \oplus \mathfrak{c})$.

Proof. We prove this by showing that Lemma 3.12 holds for $\mathcal{M}(\rho \oplus \check{\mathfrak{c}})[H]$. Then $\mathcal{N} = \mathcal{M}(\rho \oplus \check{\mathfrak{c}})[H]$ is countably iterable in V by Lemma 3.13.

Let $\rho_\omega(\mathcal{M}(\rho \oplus \check{\mathfrak{c}})[H] \| \gamma) \leq \sup(A)$ and let h be a $\mathrm{Col}(\omega, \varepsilon)$-generic filter over $\mathcal{M}(\rho \oplus \check{\mathfrak{c}})[H] \| \delta_\gamma$[22]. Then we have the equality

$$(\mathcal{M}(\rho \oplus \check{\mathfrak{c}}) \| \delta_\gamma)[H][h] = (\mathcal{M}(\rho \oplus \check{\mathfrak{c}}) \| \delta_\gamma)[h][H]$$

Since $\mathcal{M}(\rho \oplus \check{\mathfrak{c}})$ satisfies Lemma 3.12:

$$(\mathcal{M}(\rho \oplus \check{\mathfrak{c}}) \| \delta_\gamma)[h] \models (\mathcal{M}(\rho \oplus \check{\mathfrak{c}}) \| \gamma) \text{ is countably iterable}$$

The forcing which adds H is small enough compared to $\rho \oplus \check{\mathfrak{c}}$ and hence:

$$(\mathcal{M}(\rho \oplus \check{\mathfrak{c}}) \| \delta_\gamma)[h][H] \models (\mathcal{M}(\rho \oplus \check{\mathfrak{c}}) \| \gamma)[H] \text{ is countably iterable}$$

[21] Cf. [Jec03, Definition 16.1, Theorem 16.2].

[22] Note that this model can be rearranged to $(\mathcal{M}(\rho \oplus \check{\mathfrak{c}}) \| \delta_\gamma)[H]$.

This implies

$$(\mathcal{M}(A \oplus \mathfrak{c})\|\delta_\gamma)[h] \models \mathcal{M}(A \oplus \mathfrak{c})\|\gamma \text{ is countably iterable}$$

and therefore

$$\Vdash^{\mathrm{Col}(\omega,\varepsilon)}_{\mathcal{M}(A\oplus\mathfrak{c})\|\delta_\gamma} (\mathcal{M}(A \oplus \mathfrak{c})\|\gamma)\check{} \text{ is countably iterable} \qquad \square(\textit{Claim } 2)$$

A and $A \oplus \mathfrak{c}$ are easily computable from one another since A simply codes $\mathfrak{c}$, so $\mathcal{N}$ can be considered as an A-premouse. But then $\mathcal{N}$ satisfies all conditions of $\mathcal{M}(A)$. $\square$

So $\mathcal{M}(A)$ exists for every bounded subset of κ in V which simply codes $\mathfrak{c}$. Moreover, we get that also Lemma 3.12 holds for such A.

Finally

Lemma 3.18. $\mathcal{M}(A)$ *exists for all* $A \in \mathrm{Pow}(< \kappa^+)$ *which simply code* $\mathfrak{c}$.

Remark. Note that this proof actually works for arbitrary sets of ordinals A such that $\mathrm{cof}(\sup(A)^{+\mathsf{Lp}(A)}) < \kappa$.

Proof. Consider $\mathsf{Lp}^{\mathsf{V}}(A)$. Let Y be cofinal in $\lambda_A := \sup(A)^{+\mathsf{Lp}^{\mathsf{V}}(A)}$ and of minimal order type. κ^+ is a successor cardinal, so by hypothesis it is weakly compact. Since κ^+ is inaccessible in each inner model by Lemma 2.45, we have in particular that $\lambda_A < \kappa^+$ and therefore $\mathrm{cof}(\lambda_A) < \kappa$ since κ is singular. Of course $\mathsf{Lp}^{\mathsf{V}}(A)$ is an inner model of $\mathsf{HOD}_{A,Y}$.

We work in $\mathsf{HOD}_{A,Y}$. Fix $\tilde{\mu} \in (\mathrm{cof}(\lambda_A), \kappa)$ such that $\tilde{\mu}^\varepsilon = \tilde{\mu}$ in $\mathsf{HOD}_{A,Y}$. Such a $\tilde{\mu}$ exists by Definition 2.49. Let Ω be large enough and $\mathcal{X}$ be an elementary substructure of V_Ω of size $\tilde{\mu}$ which is cofinal in λ_A, closed under ε-sequences and contains A, Y, κ and λ_A. Moreover, let $\mathcal{X}$ contain $\tilde{\mu}+1$ and $\mathrm{Pow}(\omega_1)^{\mathsf{HOD}_{A,Y}}$ as subsets. Collapse $\mathcal{X}$ to a transitive structure H.

Now we can copy the proof of Lemma 3.16 and draw the desired conclusion. Moreover, we have that Lemma 3.12 holds for $\mathcal{M}(A)$. $\square$

Definition 3.19. Let $\mathcal{P}$ be an A-premouse and $\mathcal{O}$ an operator. We then say that $\mathcal{P}$ is $\mathcal{O}$-*closed* iff for all $\mathcal{P}$-cardinals ξ such that $\xi > \sup(A)$, $\mathcal{O}(\mathcal{P}|\xi) \trianglelefteq \mathcal{P}$ holds.

Then we define

Definition 3.20. For any n and any $A \in \mathrm{Pow}(< \kappa^+)$ which simply codes $\mathfrak{c}$, let $P_n^\sharp(A)$ be the least countably iterable $\mathcal{M}$-closed active A-premouse having n Woodin cardinals, and let $P_n^{\sharp\sharp}(A)$ be the least countably iterable $P_n^\sharp$-closed active A-premouse. If such a premouse does not exist, this is undefined.

Lemma 3.21. *$P_n^\sharp(A)$ exists for all n and all $A \in \mathrm{Pow}(< \kappa^+)$ which simply code $\mathfrak{c}$.*

Proof. Let $n = 0$. First we show this for $A \in \mathrm{Pow}(< \mu_+)$. For this purpose we build the minimal $\mathcal{M}$-closed model:

$$\begin{aligned} \mathcal{N}_0 &:= J_1(A), \\ \mathcal{N}_{\gamma+1} &:= \mathcal{M}(\mathcal{N}_\gamma), \\ \mathcal{N}_\lambda &:= \bigvee_{\alpha<\lambda} \mathcal{N}_\alpha \text{ for } \lambda \leq \kappa^+ \text{ limit.} \end{aligned}$$

Set $L^{\mathcal{M}}(A) := \mathcal{N}_{\kappa^+}$. Since the $\mathcal{M}$-operator condenses to itself, we can adapt the proof for L to see that $L^{\mathcal{M}}(A)$ is a fine structural model which satisfies condensation as L does, i. e. if $\mathcal{X}$ is an elementary substructure of an initial segment of $L^{\mathcal{M}}(A)$ with $A \cup \{A\} \subseteq \mathcal{X}$ then $\mathcal{X}$ collapses to an initial segment of $L^{\mathcal{M}}(A)$. We have $L^{\mathcal{M}}(A) \subseteq \mathsf{HOD}_A$. Now let $\nu > \sup(A)$ be a Θ-closed singular cardinal less than κ. This is possible by the choice of κ. Then we have, since ν^+ is weakly compact, that $\nu^{+L^{\mathcal{M}}(A)} \leq \nu^{+\mathsf{HOD}_A} < \nu^+$, so if we add a witness $G \in \mathsf{V}$ for the singularity of $\nu^{+L^{\mathcal{M}}(A)}$ to HOD_A, we get that in $\mathsf{HOD}_A[G]$ the mouse $L^{\mathcal{M}}(A)$ does not compute ν^+ correctly.

Claim 1. $P_0^\sharp(A)$ exists.

Proof. The existence of $P_0^\sharp(A)$ follows from the standard covering argument for $L^{\mathcal{M}}(A)$. We work in $\mathsf{HOD}_A[G]$.

First we build a substructure $\mathcal{X}$ of some large V_Ω which is cofinal in $\eta := \nu^{+L^{\mathcal{M}}(A)}$, closed under ω-sequences, and which has size $< \eta$. This is possible because ν is closed under the Θ-function. Moreover, let $L^{\mathcal{M}}(A)$, A, ν and η be elements of $\mathcal{X}$ and let A be a subset of $\mathcal{X}$.

$\mathcal{X}$ collapses to a transitive structure H. Let $\pi \colon H \to V_\Omega$ be the uncollapsing map and ν', η' be the preimages of ν, η under π. Since $L^{\mathcal{M}}(A)$ satisfies condensation there is a γ with

$$\pi^{-1}(L^{\mathcal{M}}(A)) = J_\gamma^{\mathcal{M}}(A) \trianglelefteq L^{\mathcal{M}}(A).$$

Then η' is the cardinal successor of ν' in $J^{\mathcal{M}}_{\gamma}(A)$.

We also have $\eta' = \nu'^{+L^{\mathcal{M}}(A)}$: Suppose $\eta' < \nu'^{+L^{\mathcal{M}}(A)}$ and let $\beta \geq \gamma$ be least, such that $\rho_\omega(J^{\mathcal{M}}_{\beta}(A)) \leq \nu'$. Since $\mathcal{X}$ is cofinal in η, we can now lift $J^{\mathcal{M}}_{\beta}(A)$ via π to a mouse $\mathcal{P}$ which extends $L^{\mathcal{M}}(A)\|\eta$ and projects to ν, as in Subclaim 1.1 of Lemma 3.16. But one also has $\mathcal{P} \trianglelefteq L^{\mathcal{M}}(A)$, since the $\mathcal{M}$-operator relativizes well, which is a contradiction.

So $\nu'^{+J^{\mathcal{M}}_{\gamma}(A)} = \nu'^{+L^{\mathcal{M}}(A)}$. Now if $\mathcal{U}$ is the ultrafilter derived from $\pi\restriction\nu'$, then the structure $(J^{\mathcal{M}}_{\eta'}(A); \in, \mathcal{U})$ witnesses the existence of $P^\sharp_0(A)$. $\square$(*Claim* 1)

Then one can use a covering argument as in Lemma 3.16 to show that $P^\sharp_0(A)$ exists for all A bounded in κ which are in HOD. For this we use the fact that since the $\mathcal{M}$-operator condenses to itself, the $P^\sharp_0$-operator does too. Now we can use the proof of Lemma 3.17 to show that $P^\sharp_0(A)$ exists for each bounded subset of κ in V. The reason is essentially that if $A \in \mathrm{Pow}(<\kappa)$ then there is a name $\rho \in H^{\mathsf{HOD}}_{\kappa}$ with $A = \rho^G$, and we can rearrange $P^\sharp_0(\rho)[G]$ to $P^\sharp_0(\rho^G) = P^\sharp_0(A)$. Now we can argue as in Lemma 3.18 to get $P^\sharp_0(A)$ for each bounded $A \subseteq \kappa^+$.

For the case $n \geq 1$ we need the following additional ingredient.

Lemma 3.22. *Suppose $P^\sharp_n(A)$ exists for each bounded $A \subseteq \mu_+$ coding $\mathfrak{c}$. Then the $P^\sharp_n$-operator relativizes well at μ_+.*

Proof. This is [Ste05, Lemma 1.34]. We briefly sketch the argument.

Let $A, B \subseteq \mu_+$ such that A is simply coded into B. Let δ be the top Woodin cardinal of $P^\sharp_n(B)$ and W be the result of the K^c-construction over A inside $V^{P^\sharp_n(B)}_{\delta}$. One can show that W is $P^\sharp_n$-closed. Moreover, if $\mathcal{Q}$ is a proper initial segment of $P^\sharp_n(W)$ and $\delta < ht(\mathcal{Q})$, then $\rho_\omega(\mathcal{Q}) \geq \delta$. This implies that δ is Woodin not only in $P^\sharp_n(B)$, which is the background universe for the K^c-construction, but also in $P^\sharp_n(W)$. But then $P^\sharp_n(A)$ is the core of $P^\sharp_n(W)$. So $P^\sharp_n(A)$ can be recovered from $P^\sharp_n(B)$, and we are done. $\square$

Suppose $P^\sharp_{n-1}(A)$ exists for all bounded $A \subseteq \kappa^+$. A similar argument as above shows that $P^{\sharp\sharp}_{n-1}(A)$ exists for all $A \in \mathrm{Pow}(<\mu_+)$ which code $\mathfrak{c}$. For this first build the minimal $P^\sharp_{n-1}$-closed model like the minimal $\mathcal{M}$-closed model: $\mathcal{N}_0 := J_1(A)$, $\mathcal{N}_{\gamma+1} := P^\sharp_{n-1}(\mathcal{N}_\gamma)$ and $\mathcal{N}_\lambda := \bigtriangledown_{\gamma<\lambda} P^\sharp_{n-1}(\mathcal{N}_\gamma)$. Then we can prove the existence of $P^{\sharp\sharp}_{n-1}(A)$ as in Claim 1.

Now suppose $n \geq 1$.

> Let $R(A \oplus A_0)$ be the minimal model over $A \oplus A_0$ of height κ^+ which is closed under the $P^\sharp_{n-1}$-operator

and

> let Ω be the first indiscernible for $R(A \oplus A_0)$.

Now we can build K^c over A below Ω in $R(A \oplus A_0)$ via the construction of [Ste96], with result $K^c(A)$. We are done if we find some γ such that for $\mathcal{Q} := K^c(A)\|\gamma$ there is a δ with $\mathcal{Q} \models$ "δ is Woodin" and $P^\sharp_{n-1}(\mathcal{Q}|\delta) = \mathcal{Q}$, because then $\mathcal{Q} = P^\sharp_n(A)$.

We use the next claim, which can be proved as in [Ste05, Lemma 1.33].

Claim 1. In $R(A \oplus A_0)$ either $P^\sharp_n(A)$ exists or $K^c(A)$ is $\Omega + 1$-iterable.

Given the claim, assume $K^c(A)$ is $\Omega + 1$-iterable. Then we can isolate $\mathsf{K}(A)^{R(A \oplus A_0)}$, the true core model over A built in $R(A \oplus A_0)$ below Ω. Now we copy the arguments in the proof of Theorem 3.4 with $\mathcal{U}$ replaced by $R(A \oplus A_0)$ to reach a contradiction. We briefly repeat the argument.

Let δ, δ^+ be weakly compact cardinals larger than $\sup(A)$ but less than κ from Definition 2.49. Then let $f \colon \delta \to \delta^{+\mathsf{K}(A)^{R(A\oplus A_0)}}$ be a bijection. f is Vopěnka-generic over H^{HOD}_κ and therefore generic over $R(A \oplus A_0)$. So we get a countably complete ultrafilter $\tilde{U}$ on $\mathrm{Pow}(\delta) \cap R(A \oplus A_0)[f]$ by Lemma 2.46. Again we have that $R(A \oplus A_0)[f][\tilde{U}]$ is a generic extension of $R(A \oplus A_0)[f]$. Now we can build in $R(A \oplus A_0)[f][\tilde{U}]$ the ultrapower of $R(A \oplus A_0)[f]$ by $\tilde{U}$:

$$\pi \colon R(A \oplus A_0)[f] \to \mathrm{ult}(R(A \oplus A_0)[f], \tilde{U}), \quad cr(\pi) = \delta. \text{ [23]}$$

By the countable completeness of $\tilde{U}$, the ultrapower is wellfounded and we can identify it with the transitive collapse. Now consider the restriction of π to the core model. So

$$\pi \restriction \mathsf{K}(A)^{R(A\oplus A_0)[f]} \colon \mathsf{K}(A)^{R(A\oplus A_0)[f]} \to \tilde{\mathsf{K}}(A),$$

where $\tilde{\mathsf{K}}(A)$ is the core model over A up to Ω, built in the ultrapower, i. e. $\tilde{\mathsf{K}}(A) = \mathsf{K}(A)^{\mathrm{ult}(R(A\oplus A_0)[f], \tilde{U})}$. As before we get that $\tilde{\mathsf{K}}(A)$ is $\Omega + 1$-iterable in $R(A \oplus A_0)[f][\tilde{U}]$.

[23] Again in fact $\pi \restriction V_{\Omega+1}^{R(A\oplus A_0)[f]} \in R(A \oplus A_0)[f][\tilde{U}]$ and this is sufficient.

Note that $\mathsf{K}(A)^{R(A\oplus A_0)} = \mathsf{K}(A)^{R(A\oplus A_0)[f]} = \tilde{\mathsf{K}}(A)$. Let $\delta' := \pi(\delta)$. Now we use [Sch99, Lemma 4(b)] in $R(A\oplus A_0)[f][\tilde{U}]$ and get that $\pi''\delta^{+\mathsf{K}(A)^{R(A\oplus A_0)[f]}}$ is cofinal in $\pi(\delta^{+\mathsf{K}(A)^{R(A\oplus A_0)[f]}}) = \delta'^{+\tilde{\mathsf{K}}(A)}$. So

$$\pi\restriction\delta^{+\mathsf{K}(A)^{R(A\oplus A_0)[f]}} \in \mathrm{ult}(R(A\oplus A_0)[f], \tilde{U})$$

witnesses that the cofinality of $\delta'^{+\tilde{\mathsf{K}}(A)}$ in $\mathrm{ult}(R(A\oplus A_0)[f], \tilde{U})$ is small:

$$\mathrm{ult}(R(A\oplus A_0)[f], \tilde{U}) \models \mathrm{cof}(\delta'^{+\tilde{\mathsf{K}}(A)}) \leq \delta^{+\mathsf{K}(A)^{R(A\oplus A_0)[f]}} < \delta'$$

But this contradicts "weak covering for $\mathsf{K}(A)$" which implies $\mathrm{cof}(\delta'^{+\tilde{\mathsf{K}}(A)}) \geq \delta'$ in $\mathrm{ult}(R(A\oplus A_0)[f], \tilde{U})$.

So it is impossible for $K^c(A)$ to be $\Omega_1 + 1$-iterable and therefore $P_1^\sharp(A)$ exists. To show that $P_1^\sharp(A)$ is countably iterable in V we can show two Lemmata similar to 3.12 and 3.13. Then we can use our lift-up arguments to show that $P_1^\sharp(A)$ exists for each bounded subsets A of κ^+. $\square$

The premice $P_n^\sharp(A)$ collectively enable us to go to the next step in the core model induction; we can show that $(W^\star_{\alpha+1})$ is true. The proof is similar to the corresponding proof of [Ste05, Lemma 1.38].

Lemma 3.23. $(W^\star_{\alpha+1})$ *holds. Moreover, if* $\mathcal{P}$ *is the mouse witnessing* $(W^\star_{\alpha+1})$ *with respect to a* $\Sigma_n(J_\alpha(\mathbb{R}^g))$*-set of reals, then* $\mathcal{P}$ *is closed under* $C_{\Sigma_n(J_\alpha(\mathbb{R}^g))}$.

Proof. Let $U \in J_{\alpha+1}(\mathbb{R}^g)$ be a set of reals and $k < \omega$. Then $U \in \Sigma_n(J_\alpha(\mathbb{R}^g), z)$ for some real z and $n < \omega$. Again we can assume that $z = \rho^{g\restriction\bar{\iota}}$ for some $\bar{\iota} < \mu_+$. Define $\bar{g} := g\restriction\bar{\iota}$. Let

$$\mathcal{P} = P^\sharp_{k+n+2}(\langle\mathfrak{c}, \rho\rangle)$$

We can show that $\mathcal{P}[\bar{g}]$ is the desired witness.

Claim 1. $\mathcal{P}[\bar{g}]$ is ω_1-iterable in $J_{\alpha+1}(\mathbb{R}^g)$.

Proof. $\mathcal{P}$ has a unique iteration strategy Σ in HOD_X. Since the $\mathcal{Q}$-structures used to define Σ can be identified by the $P^\sharp_{k+n+1}$-operator, Σ can be extended to $\mathsf{HOD}_X[g]$. Now one can show that $\mathcal{P}[\bar{g}]$ is also ω_1-iterable in $\mathsf{HOD}_X[g]$. Moreover, the canonical ω_1-iteration strategy for $\mathcal{P}[\bar{g}]$ is definable over $J_\alpha(\mathbb{R}^g)$, so it is an element of $J_{\alpha+1}(\mathbb{R}^g)$. $\square$(*Claim* 1)

Suppose δ_0 is the top Woodin cardinal and δ_1 is the next smaller Woodin cardinal of $\mathcal{P}$. Let W be a universal $\Sigma_1(J_\alpha(\mathbb{R}^g))$-set of reals and let $\psi \in \Sigma_1$ define W over $J_\alpha(\mathbb{R}^g)$. Let Σ be the iteration strategy for both $\mathcal{P}$ and $\mathcal{P}[\bar{g}]$.

Claim 2. There is a term $\dot{W} \in \mathcal{P}[\bar{g}]^{\mathrm{Col}(\omega,\delta_1)}$ which captures W over $\mathcal{P}[\bar{g}]$, i.e. whenever $i\colon \mathcal{P}[\bar{g}] \to \mathcal{R}[\bar{g}]$ is a simple iteration map by Σ, and h is $\mathrm{Col}(\omega, i(\delta_1))$-generic over $\mathcal{R}[\bar{g}]$, then

$$i(\dot{W})^h = W \cap \mathcal{R}[\bar{g}][h].$$

Proof. We first sketch the meaning of the term $\dot{W}$.

For the construction of $\dot{W}$ first note that $\mathcal{M}(\mathcal{P}|\delta_0) \trianglelefteq \mathcal{P}$, because $\mathcal{P}$ is closed under the $\mathcal{M}$-operator, and since $\bar{g}$ is a "small" generic object we also have $\mathcal{M}(\mathcal{P}[\bar{g}]|\delta_0) \trianglelefteq \mathcal{P}[\bar{g}]$. Now if l is $\mathrm{Col}(\omega, \mathcal{P}|\delta_0)$-generic over $\mathcal{M}(\mathcal{P}|\delta_0)[\bar{g}][h]$, then, due to the smallness of δ_1 in relation to δ_0, we can find a $\mathrm{Col}(\omega, \mathcal{P}|\delta_0)$-generic object l' over $\mathcal{M}(\mathcal{P}|\delta_0)[\bar{g}]$ such that

$$\mathcal{M}(\mathcal{P}|\delta_0)[\bar{g}][h][l] = \mathcal{M}(\mathcal{P}|\delta_0)[\bar{g}][l']$$

and therefore by the definition of the $\mathcal{M}$-operator:

$$\forall n\ \exists \gamma_n\ \mathcal{M}(\mathcal{P}|\delta_0)[\bar{g}][l'] \| \gamma_n \text{ is a } \langle \varphi_n, \sigma^{\bar{g}\times l'}_{\mathcal{P}|\delta_0} \rangle\text{-witness.}$$

As usual $\mathcal{M}(\mathcal{P}|\delta_0)[\bar{g}][l']$ can be considered as a z-mouse $\mathcal{M}_z$, where z is a real coding $\bar{g}, l'$ in a simple fashion. So let $T \in \mathcal{M}_z$ be the tree such that $\mathsf{p}[T]$ is the Σ_k-theory of $J_\gamma(\mathbb{R}^g)$ where γ is least with $J_\gamma(\mathbb{R}^g) \models \varphi^k_1(\sigma^{\bar{g}\times l'}_{\mathcal{P}|\delta_0})$.[24] In particular we have $J_\gamma(\mathbb{R}^g) \models \varphi(x, \rho^{\bar{g}\times l'})$ for all ρ which are simply coded in $\mathcal{P}|\delta_0$. Now $\psi(y)$ has been verified before γ iff $(\psi, y) \in \mathsf{p}[T]$.

Given $\rho \in \mathcal{P}[\bar{g}]^{\mathrm{Col}(\omega,\delta_1)}$, let $\bar{\rho}$ be such that for all simple Σ-iterates $\mathcal{R}[\bar{g}]$ with iteration map j and for all $h \in \mathsf{HOD}_X[g]$ being $\mathrm{Col}(\omega, j(\delta_1))$-generic over $\mathcal{R}[\bar{g}]$ we have that $j(\bar{\rho})^h \in \mathcal{R}[\bar{g}][h]^{\mathrm{Col}(\omega, j(\delta_0))}$ is a name for the real ρ^h. Let $\bar{T}$ be a name such that $j(\bar{T}^h) \in \mathcal{R}[\bar{g}][h]^{\mathrm{Col}(\omega, j(\delta_0))}$ is a name for the tree $T^{\mathcal{M}(\mathcal{R}|\delta_0)[\bar{g}][h][l]} \in \mathcal{M}(\mathcal{R}|\delta_0)[\bar{g}][h][l]$, whose projection is the Σ_{k+3}-theory of $J_\gamma(\mathbb{R}^g)$ where γ is least with $J_\gamma(\mathbb{R}^g) \models \varphi^k_1(\sigma^{\bar{g}\times\{h,l\}}_{\mathcal{R}|\delta_0})$.

So let $(\rho, p) \in \dot{W}$ iff $\rho \in \mathcal{P}[\bar{g}]^{\mathrm{Col}(\omega,\delta_1)}$ is a standard name for a real and $p \in \mathrm{Col}(\omega, \delta_1)$ is such that:

$$p \Vdash \exists q \in \mathrm{Col}(\omega, \delta_0)\ q \Vdash (\psi, \bar{\rho}) \in \mathsf{p}[\bar{T}]$$

[24] φ^k_n is the Σ_k-formula associated with φ_n from Lemma 2.54.

First let $y \in W \cap \mathcal{R}[\bar{g}][h]$, $y = \rho^h$. Since there are cofinally many $\gamma < \alpha$ such that γ is minimal with $J_\gamma(\mathbb{R}^g) \models \varphi(x,t)$ for some real t, we can find a real t such that

if $J_\gamma(\mathbb{R}^g) \models \varphi(x,t)$ then already $J_\gamma(\mathbb{R}^g) \models \psi(y)$.

By Woodin's genericity theorem there is a countable simple iterate $\tilde{\mathcal{R}}[\bar{g}]$ of $\mathcal{R}[\bar{g}][h]$ with iteration map j such that t is $\mathbb{E}^{\tilde{\mathcal{R}}[\bar{g}]}_{j(i(\delta_0))}$-generic over $\tilde{\mathcal{R}}[\bar{g}]$ for the extender algebra $\mathbb{E}^{\tilde{\mathcal{R}}[\bar{g}]}_{j(i(\delta_0))}$ which has the $j(i(\delta_0))$-c. c. (cf. [Steb, Theorem 7.14]), so that in fact $\tilde{\mathcal{R}}[\bar{g}][t]$ is a $j(i(\delta_0))$-c. c. extension of a Σ-iterate $\mathcal{R}'[\bar{g}]$ of $\mathcal{P}[\bar{g}]$, and we may find a $\mathrm{Col}(\omega, j(i(\delta_0)))$-generic object l with $\tilde{\mathcal{R}}[\bar{g}][t] = \mathcal{R}'[\bar{g}][h][l]$. But now

$$\mathcal{R}'[\bar{g}][h][l] \models (\psi, y) \in \mathsf{p}[T^{\mathcal{M}(\mathcal{R}'|\delta_0)[\bar{g}][h][l]}],$$

since the real t occurs in $\sigma^{\bar{g}\times\{h,l\}}_{\mathcal{R}'|\delta_0}$ and therefore $T^{\mathcal{M}(\mathcal{R}'|\delta_0)[\bar{g}][h][l]}$ can decide whether $\psi(y)$ is true. By the definition of $\bar{\rho}$ and $\bar{T}$ we get

$$\mathcal{R}'[\bar{g}][h][l] \models (\psi, (j(\bar{\rho})^h)^l) \in \mathsf{p}[j(i(\bar{T})^h)^l],$$

so $\mathcal{R}'[\bar{g}][h] \models \exists q \in \mathrm{Col}(\omega, j(i(\delta_0)))\ q \Vdash (\psi, j(\bar{\rho})^h) \in \mathsf{p}[j(i(\bar{T})^h)]$, which yields

$$\mathcal{R}[\bar{g}][h] \models \exists q \in \mathrm{Col}(\omega, i(\delta_0))\ q \Vdash (\psi, \bar{\rho}^h) \in \mathsf{p}[i(\bar{T})^h]$$

by elementarity and therefore $\rho^h \in i(\dot{W})^h$.

For the other direction let $y \in i(\dot{W})^h$. So by the definition of $\dot{W}$ we can find a standard name $\rho \in \mathcal{R}[\bar{g}]^{\mathrm{Col}(\omega, i(\delta_1))}$ for the real y and some condition $p \in h$ such that "$p \Vdash \exists q \in \mathrm{Col}(\omega, i(\delta_0))\ q \Vdash (\psi, \bar{\rho}) \in \mathsf{p}[i(\bar{T})]$". But then we get

$$\mathcal{R}[\bar{g}][h] \models \exists q \in \mathrm{Col}(\omega, i(\delta_0))\ q \Vdash (\psi, \bar{\rho}^h) \in \mathsf{p}[i(\bar{T})^h]$$

and it follows for each $\mathrm{Col}(\omega, i(\delta_0))$-generic l over $\mathcal{R}[\bar{g}][h]$ containing q

$$\mathcal{R}[\bar{g}][h][l] \models (\psi, (\bar{\rho}^h)^l) \in \mathsf{p}[(i(\bar{T})^h)^l].$$

So $\mathcal{R}[\bar{g}][h][l] \models (\psi, y) \in \mathsf{p}[T^{\mathcal{M}(\mathcal{R}|\delta_0)[\bar{g}][h][l]}]$ by the definition of $\bar{\rho}, \bar{T}$ and therefore $J_\gamma(\mathbb{R}^g) \models \psi(y)$ for some $\gamma < \alpha$ so $y \in W$. $\square$(*Claim* 2)

Since α is inadmissible and begins a Σ_1-gap, the Σ_n-theory of $J_\alpha(\mathbb{R})$ can be computed from the Σ^1_n theory in parameters W and x.

Let δ be the k^{th} Woodin cardinal of $\mathcal{P}[\bar{g}]$. We now use the Woodin cardinals above δ and the term $\dot{W}$ to get a term $\dot{W}_n \in \mathcal{P}[\bar{g}]^{\mathrm{Col}(\omega,\delta)}$ as in Lemma 2.40 which weakly captures a universal $\Sigma_n(J_\alpha(\mathbb{R}^g))$-set W_n over $\mathcal{P}[\bar{g}]$. From $\dot{W}_n$ we get a term $\dot{U}$ which weakly captures U over $\mathcal{P}[\bar{g}]$, i. e. whenever h is $\mathrm{Col}(\omega,\delta)$-generic over $\mathcal{P}[\bar{g}]$ then

$$\dot{U}^h = U \cap \mathcal{P}[\bar{g}][h]$$

Now for all elementary substructures $\mathcal{X} \prec \mathcal{P}[\bar{g}]|\delta_0^{+\mathcal{P}[\bar{g}]}$ with $\pi\colon \mathcal{H}[\bar{g}] \cong \mathcal{X}$ and $\pi(\dot{Z}) = \dot{U}$, and for all h being $\mathrm{Col}(\omega,\pi^{-1}(\delta))$-generic over $\mathcal{H}[\bar{g}]$, we have again

$$\dot{Z}^h = U \cap \mathcal{H}[\bar{g}][h]$$

This relies on the fact that an elementary submodel of an iterable structure is still iterable. So if $\dot{W}'$ is the preimage of $\dot{W}$ under π, then by definition of $\dot{W}$ we have that $\dot{W}'$ captures W over $\mathcal{H}[\bar{g}]$. Now the construction of $\dot{W}_n$ (resp. the construction of $\dot{U}$) ensures that $\dot{Z}$ weakly captures U over $\mathcal{H}[\bar{g}]$.

In $\mathcal{P}[\bar{g}]$ we can construct the required trees S and T: T_y tries to build such π, $\mathcal{H}$ and h with $y \in \dot{Z}^h$, whereas S_y tries to build the same but with $y \notin \dot{Z}^h$ instead.

The way to show that $\mathcal{P}[\bar{g}]$ is closed under $C_{\Sigma_n(J_\alpha(\mathbb{R}^g))}$ is similar to the proof of Lemma 2.43: Let $x \in \mathcal{P}[\bar{g}]$ be a real and let $a \in C_{\Sigma_n(J_\alpha(\mathbb{R}^g))}(x)$ be determined by ξ and φ. Suppose ψ is the Σ_n-formula

$$\psi(n,m,x,c) \equiv \exists y \in \mathbb{R}\ \big(\varphi(y,x,c) \wedge y(n) = m\big),$$

so for each $c \in \mathbb{R}^g$ coding ξ:

$$a(n) = m \quad \Leftrightarrow \quad J_\alpha(\mathbb{R}^g) \models \psi(n,m,x,c)$$

Since W_n is universal, there is a k such that

$$((n,m,x,c),k) \in W_n \quad \Leftrightarrow \quad J_\alpha(\mathbb{R}^g) \models \psi(n,m,x,c).$$

Now we can iterate $\mathcal{P}[\bar{g}]$ to some premouse $\mathcal{M}$ such that $\xi \in \mathcal{M}$. If we now collapse ξ via $G \in \mathsf{HOD}_X[g]$ we have a real c_G coding ξ. So in $\mathcal{M}[G]$ we can define y by saying

$$a(n) = m \quad \Leftrightarrow \quad ((n, m, x, c_G), k) \in \dot{W}_n^G.$$

But if G, G' are mutually generic over $\mathcal{M}$, then $a \in \mathcal{M}[G] \cap \mathcal{M}[G']$. It follows that $a \in \mathcal{M}$ and therefore in $\mathcal{P}[\bar{g}]$. $\square$

The successor-of-a-critical case and countable-cofinality case

Now suppose α is critical and either the successor of a critical ordinal β' or a limit ordinal which has countable cofinality in $\mathsf{HOD}_X[g]$. In these cases we use that we have witnesses for countably many pointclasses "cofinal" in $\Sigma_1(J_\alpha(\mathbb{R}^g))$.

Let $\Gamma_k, k < \omega$ be pointclasses which are cofinal in $\Sigma_1(J_\alpha(\mathbb{R}^g))$. This means that if $\alpha = \beta' + 1$ we set $\Gamma_k := \Sigma_k(J_{\beta'}(\mathbb{R}^g))$. If α is a limit of countable cofinality, then we let $(\alpha_k : k < \omega)$ be cofinal in α such that α_k is critical. In this case we set $\Gamma_k := \Sigma_1(J_{\alpha_k}(\mathbb{R}^g))$. In both cases we use the fact that $C_{\Sigma_1(J_\alpha(\mathbb{R}^g))} \subseteq \bigcup_{k<\omega} C_{\Gamma_k}$ and the following lemma

Lemma 3.24. *Let $\mathcal{M}$ be a countable transitive model of a sufficiently large fragment of* ZFC *which is closed under $C_{\Sigma_1(J_\alpha(\mathbb{R}))}$. Let κ be a cardinal of $\mathcal{M}$ and z a real in $\mathcal{M}$.*

Then every set of reals which is in $\Sigma_1(J_\alpha(\mathbb{R}), z)$ is weakly captured by some τ over $\mathcal{M}$.

Proof. We define the capturing term τ for a $\Sigma_1(J_\alpha(\mathbb{R}), z)$-set A as in the proof of [SSc, Lemma 4.5]: $(p, \sigma) \in \tau$ iff $p \in \mathrm{Col}(\omega, \kappa)$, σ is a $\mathrm{Col}(\omega, \kappa)$-standard term for a real, and for comeager many g being $\mathrm{Col}(\omega, \kappa)$-generic over $\mathcal{M}$: $p \in g \Rightarrow \sigma^g \in A$. $\square$

If α is a limit ordinal of countable cofinality and begins a gap, then it begins a limit gap. Since each successor gap begins with a critical ordinal there are cofinally many critical $\gamma < \alpha$. So let $(\alpha_k : k < \omega)$ be critical and cofinal in α and let $\mathcal{M}_k$ be the mouse-operator witnessing $(W^\star_{\alpha_k+1})$ with respect to $\Sigma_1(J_{\alpha_k}(\mathbb{R}^g))$. If $\alpha = \beta' + 1$, let $\mathcal{M}_k$ be the mouse-operator witnessing

$(W^\star_{\beta'+1})$ with respect to $\Sigma_k(J_{\beta'}(\mathbb{R}^g))$. By induction there exists a set $\mathfrak{c}_k$ such that $\mathcal{M}_k(A)$ exists for all sets of ordinals coding $\mathfrak{c}_k$. Let $\mathfrak{c} := \oplus_{k<\omega}\mathfrak{c}_k$.

Now we can define our mouse-operator:

Definition 3.25. For any n let $P^\sharp_n(A)$ be the least countably iterable active A-premouse which has n Woodin cardinals and is closed under all $\mathcal{M}_k$. $P^{\sharp\sharp}_n(A)$ will be the least countably iterable $P^\sharp_n$-closed active A-premouse. If such a premouse does not exist, this is undefined.

Lemma 3.26. *$P^\sharp_n(A)$ exists for all n and all sets of ordinals A coding $\mathfrak{c}$.*

This is similar to the proof of 3.21.

Lemma 3.27. *$(W^\star_{\alpha+1})$ holds. Moreover, if $\mathcal{P}$ is the mouse witnessing $(W^\star_{\alpha+1})$ with respect to a $\Sigma_n(J_\alpha(\mathbb{R}^g))$-set of reals, then $\mathcal{P}$ is closed under $C_{\Sigma_n(J_\alpha(\mathbb{R}^g))}$.*

Proof. Let U be a set of reals in $J_{\alpha+1}(\mathbb{R}^g)$ and $k<\omega$. Then there are a real z and an $n<\omega$ with $U \in \Sigma_n(J_\alpha(\mathbb{R}^g), z)$. Suppose $z = \rho^{g\restriction\bar\iota}$ for some $\bar\iota < \mu_+$. Set

$$\mathcal{P} = P^\sharp_{k+n+2}(\langle\mathfrak{c},\rho\rangle)$$

and let W be a universal $\Sigma_1(J_\alpha(\mathbb{R}^g))$-set of reals.

Since $\mathcal{P}[\bar g]$ is closed under each $\mathcal{M}_l(\cdot)$ and since $\mathcal{M}_l(\cdot)$ is closed under $C_{\Sigma_1(J_{\alpha_l}(\mathbb{R}^g))}$ it follows that $\mathcal{P}[\bar g]$ is also closed under $C_{\Sigma_1(J_{\alpha_l}(\mathbb{R}^g))}$. Now we can use $C_{\Sigma_1(J_\alpha(\mathbb{R}^g))} \subseteq \bigcup_{l<\omega} C_{\Sigma_1(J_{\alpha_l}(\mathbb{R}^g))}$ to get that $\mathcal{P}[\bar g]$ is closed under $C_{\Sigma_1(J_\alpha(\mathbb{R}^g))}$.

Now we have in $\mathcal{P}[\bar g]$ a capturing term $\dot W$ by Lemma 3.24 for a universal $\Sigma_1(J_\alpha(\mathbb{R}^g))$-set. The rest in an adaption of the proof of Lemma 3.23. $\square$

3.3 The end-of-gap cases

Now suppose that either α is the end of a proper Σ_1-gap or that α is the successor of a non-critical ordinal. Let α' be the supremum of the critical ordinals $<\alpha$. If α ends a proper weak gap, then this gap is $[\alpha',\alpha]$; if α is the successor of a non-critical ordinal then the predecessor gap is $[\alpha',\alpha-1]$.

We fix for the rest of this section a self-justifying system $\mathcal{A} := (A_i : i<\omega)$ such that each $A_i \in J_\alpha(\mathbb{R}^g)$, A_0 is a universal $\Sigma_1(J_{\alpha'}(\mathbb{R}^g))$-set of reals, and

$\bigcup_{i\in\omega} A_{2i}$ is a universal $\underset{\sim}{\Sigma}_j(J_\alpha(\mathbb{R}^g))$-set of reals, where j is least such that $\rho_j(J_\alpha(\mathbb{R}^g)) = \mathbb{R}^g$.[25] Suppose x^* is such that each A_i is ordinal definable from x^* over $J_\gamma(\mathbb{R}^g)$ for some $\gamma < \alpha$. Then $x^* := \tau^{g\restriction\iota}$ for some $\iota < \mu_+, \tau \in \mathsf{HOD}_X$.

We will prove $(W^\star_{\alpha+1})$.

Definition 3.28. Let $\mathcal{S}$ and $\mathcal{T}$ be iteration trees on some premouse $\mathcal{N}$. Then we say $\mathcal{S}$ *is a hull of* $\mathcal{T}$ iff there is an order preserving map

$$\sigma\colon (lh(\mathcal{S}), <_{\mathcal{S}}) \to (lh(\mathcal{T}), <_{\mathcal{T}})$$

such that the following conditions are fulfilled:

1. $\mathrm{ran}(\sigma)$ is a support[26].
2. There are elementary embeddings $\pi_\gamma\colon \mathcal{M}^{\mathcal{S}}_\gamma \to \mathcal{M}^{\mathcal{T}}_{\sigma(\gamma)}$ which commute with the tree embeddings.
3. $\pi_{\gamma+1}$ satisfies the conclusion of the Shift Lemma 2.27, i. e. if we set $\bar{\mathcal{N}} := \mathcal{M}^{\mathcal{S}}_\gamma \| lh(E^{\mathcal{S}}_\gamma)$, $\mathcal{N} := \mathcal{M}^{\mathcal{T}}_{\sigma(\gamma)} \| lh(E^{\mathcal{T}}_{\sigma(\gamma)})$, $\psi := \pi_\gamma \restriction \bar{\mathcal{N}}$, $\bar{\mathcal{M}} := \mathcal{M}^{\mathcal{S}}_\eta$, $\mathcal{M} := \mathcal{M}^{\mathcal{T}}_{\sigma(\eta)}$, and $\varphi := \pi_\eta$, where $\eta := pred_{\mathcal{S}}(\gamma+1)$, and if we identify the ultrapowers with their transitive collapses, then $\pi_{\gamma+1}$ ensures that the diagram in Lemma 2.27(4) commutes.
4. $\pi_\gamma(E^{\mathcal{S}}_\gamma) = E^{\mathcal{T}}_{\sigma(\gamma)}$.

We say that an iteration strategy Σ *condenses well* iff whenever $\mathcal{T}$ is an iteration tree built according to Σ and $\mathcal{S}$ is a hull of $\mathcal{T}$, then $\mathcal{S}$ is also built according to Σ.

Lemma 3.29. *Suppose $\mathcal{N}$ is a premouse and Σ is an iteration strategy for $\mathcal{N}$ which condenses well. For a substructure $\mathcal{X} \prec V_\Omega$ with $\mathcal{N}, \Sigma \in \mathcal{X}$ let $\pi\colon H \to \mathcal{X}$ be the uncollapsing map and $\pi(\mathcal{M}) = \mathcal{N}, \pi(\Gamma) = \Sigma$.*

Then $\Gamma = \Sigma^\pi \restriction H$, i. e. Σ collapses to its pullback strategy (cf. 2.26).

Proof. Let $\mathcal{S}$ be an iteration tree on $\mathcal{M}$ in H built according to Γ. Set $\mathcal{T} := \pi(\mathcal{S})$. So $\mathcal{T}$ is an iteration tree on $\mathcal{N}$ according to Σ.

[25] See [SSc, Section 4] for the definition of self-justifying systems.
[26] See [Ste93, p. 198] for the concept of support.

We show inductively that for each $\xi < lh(\mathcal{S})$ the tree $\pi\mathcal{S}\restriction\xi$ is a hull of $\mathcal{T}$ witnessed by $\pi\restriction\xi$.[27] At limit points $\lambda < lh(\mathcal{S})$ we can then show that

$$\Gamma(\mathcal{S}\restriction\lambda) = [0,\lambda)_{\mathcal{S}} = [0,\lambda)_{\pi\mathcal{S}} = \Sigma(\pi\mathcal{S}\restriction\lambda),$$

so $\mathcal{S}$ is built according to the pullback strategy Σ^π.

For this let $\pi_\xi := \pi\restriction\mathcal{M}_\xi^{\mathcal{S}}$ be the canonical embedding from $\mathcal{M}_\xi^{\mathcal{S}}$ into $\mathcal{N}_{\pi(\xi)}^{\mathcal{T}}$ for $\xi < lh(\mathcal{S})$. We show by induction that for each $\xi < lh(\mathcal{S})$ there are embeddings σ_ξ, τ_ξ such that $\sigma_\xi \circ \tau_\xi = \pi_\xi$:

$$\begin{tikzcd} \mathcal{M}_\xi^{\mathcal{S}} \arrow[r, "\tau_\xi"] \arrow[rr, bend left, "\pi_\xi"] & \mathcal{N}_\xi^{\pi\mathcal{S}} \arrow[r, "\sigma_\xi"] & \mathcal{N}_{\pi(\xi)}^{\mathcal{T}} \end{tikzcd}$$

Moreover, τ_ξ should be defined by the copying process as for the proof of Lemma 2.26.

So it is clear that $\tau_0 := \pi$ and $\sigma_0 :=$ identity.

Suppose we have defined τ_β and σ_β for $\beta \leq \xi$. Since the τ_β are defined by the copying process there is a function $\tau_{\xi+1}^\star$ such that for the $\mathcal{S}$-predecessor β of $\xi + 1$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{ult}(\mathcal{M}_\beta^{\mathcal{S}*}, E_\xi^{\mathcal{S}}) \arrow[r, "\tau_{\xi+1}^\star"] & \mathrm{ult}(\mathcal{N}_\beta^{\pi\mathcal{S}*}, E_\xi^{\pi\mathcal{S}}) \\ \mathcal{M}_\beta^{\mathcal{S}*} \arrow[u, "i"] \arrow[r, "\tau_\beta"'] & \mathcal{N}_\beta^{\pi\mathcal{S}*} \arrow[u, "j"'] \end{tikzcd}$$

Note that $\mathcal{M}_\beta^{\mathcal{S}*}$ is the initial segment of $\mathcal{M}_\beta^{\mathcal{S}}$ to which $E_\xi^{\mathcal{S}}$ is applied. Suppose $E_\xi^{\mathcal{S}}$ is indexed by γ, i.e. $E_\xi^{\mathcal{S}} = E_\gamma^{\mathcal{M}_\xi^{\mathcal{S}}}$. Using the elementarity of τ_ξ and the construction of the copied tree $\pi\mathcal{S}$, it is easy to see that $E_\xi^{\pi\mathcal{S}} = E_{\tau_\xi(\gamma)}^{\mathcal{N}_\xi^{\pi\mathcal{S}}}$. Now the Shift Lemma and the embedding $\sigma_\xi \colon \mathcal{N}_\xi^{\pi\mathcal{S}} \to \mathcal{N}_{\pi(\xi)}^{\mathcal{T}}$ give us a function $\sigma_{\xi+1}^\star$ such that for the extender $F := E_{\sigma_\xi(\tau_\xi(\gamma))}^{\mathcal{N}_{\pi(\xi)}^{\mathcal{T}}}$ the following diagram also

[27] $\pi\mathcal{S}$ is the copied tree using π as done for the proof of Lemma 2.26.

commutes:

$$\begin{tikzcd} \mathrm{ult}(\mathcal{N}^{\pi\mathcal{S}*}_{\beta}, E^{\pi\mathcal{S}}_{\xi}) \arrow[r, "\sigma^{\star}_{\xi+1}"] & \mathrm{ult}(\mathcal{N}^{\mathcal{T}*}_{\pi(\beta)}, F) \\ \mathcal{N}^{\pi\mathcal{S}*}_{\beta} \arrow[u, "j"] \arrow[r, "\sigma_{\beta}"'] & \mathcal{N}^{\mathcal{T}*}_{\pi(\beta)} \arrow[u, "k"'] \end{tikzcd}$$

Since $\pi_\xi = \sigma_\xi \circ \tau_\xi$ we also have $F = E^{\mathcal{T}}_{\pi(\xi)}$, so that $\mathrm{ult}(\mathcal{N}^{\mathcal{T}*}_{\pi(\beta)}, F)$ is exactly the uncollapsed $\mathcal{N}^{\mathcal{T}}_{\pi(\xi)+1}$ and therefore wellfounded. Let

$$\mathcal{N}^{\pi\mathcal{S}}_{\xi+1} \cong \mathrm{ult}(\mathcal{N}^{\pi\mathcal{S}*}_{\beta}, E^{\pi\mathcal{S}}_{\xi})$$

be the transitive collapse and $\sigma_{\xi+1}$ and $\tau_{\xi+1}$ the collapses of $\sigma^{\star}_{\xi+1}$ and $\tau^{\star}_{\xi+1}$, respectively. So we have that the copying process does not break down at successor steps and we can go one step further. Moreover, one can see by the definition of the $\sigma_{\xi+1}$, that $\pi\mathcal{S}\restriction\xi{+}2$ is a hull of $\mathcal{T}$ witnessed by $\pi\restriction\xi{+}2$.

Now let $\lambda < lh(\mathcal{S})$ be a limit ordinal. First note that $[0,\lambda)_{\pi\mathcal{S}} = [0,\lambda)_{\mathcal{S}}$ by the construction of $\pi\mathcal{S}$.

Since for each $\xi < \lambda$ we have that $\pi\mathcal{S}\restriction\xi$ is a hull of $\mathcal{T}$ witnessed by $\pi\restriction\xi$, we also have that $\pi\mathcal{S}\restriction\lambda$ is a hull of $\mathcal{T}$ witnessed by $\pi\restriction\lambda$: The only point to show here is that $\pi''(\pi\mathcal{S}\restriction\lambda)$ is a support of $\mathcal{T}$. So let $\xi < \lambda$. We have to show that $\pi''(\pi\mathcal{S}\restriction\lambda)$ is a $\pi(\xi)$-support. But $\pi''(\pi\mathcal{S}\restriction\xi{+}1)$ is a $\pi(\xi)$-support, and therefore $\pi''(\pi\mathcal{S}\restriction\lambda)$ is a $\pi(\xi)$-support, too.

Now the elementarity of π implies that $(\pi\mathcal{S}\restriction\lambda)^\frown[0,\lambda)_{\mathcal{S}}$ is a hull of $\mathcal{T}$: We have to show that $\mathcal{X} := \pi''((\pi\mathcal{S}\restriction\lambda)^\frown[0,\lambda)_{\mathcal{S}})$ is a $\pi(\lambda)$-support. If $\mathcal{X}$ is cofinal in $\pi(\lambda)$ we are done by the definition of a $\pi(\lambda)$-support. If $\mathcal{X}$ is not cofinal let $\xi := \sup \pi''\lambda < \pi(\lambda)$. By the elementarity of π there is no drop on $[\xi, \pi(\lambda))_{\mathcal{T}}$. To show that $\mathcal{X}$ is a $\pi(\lambda)$-support, we must now prove that $\mathcal{X} \cup \{\xi\}$ is a ξ-support, but this is clear since $\mathcal{X}$ is cofinal in ξ.

Since $\mathcal{T}$ is built according to Σ, we can use the "condenses well" property to get that $(\pi\mathcal{S}\restriction\lambda)^\frown[0,\lambda)_{\mathcal{S}}$ is also built according to Σ, i. e. $[0,\lambda)_{\mathcal{S}} = \Sigma(\pi\mathcal{S}\restriction\lambda)$.

So in the end we get that $\mathcal{S}$ is built according to the pullback strategy Σ^π. $\square$

Definition 3.30. For any $A \in \mathrm{Pow}(<\mu_+)$ let $Lp_\alpha(A)$ be the union of all A-premice $\mathcal{M}$, such that $\rho_\omega(\mathcal{M}) = \sup(A)$ and

$$\Vdash^{\mathrm{Col}(\omega,<\mu_+)} J_{\check{\alpha}}(\mathbb{R}) \models \check{\mathcal{M}} \text{ is } \omega_1\text{-iterable}$$

Remark. $Lp_\alpha(A)$ is a initial segment of $\mathsf{Lp}(A)$, because the iteration strategy witnessing $\mathcal{M} \trianglelefteq Lp_\alpha(A)$ is unique, so that its restriction to V is in V.

Definition 3.31. Let $A \in \mathrm{Pow}(<\mu_+)$. An A-premouse $\mathcal{M}$ is called *suitable* iff $\overline{\overline{\mathcal{M}}} = \overline{\overline{\sup(A)}}$ and

1. $\mathcal{M} \models$ There is exactly one Woodin cardinal $\delta^{\mathcal{M}}$.
2. $\mathcal{M}$ is the Lp_α closure of $\mathcal{M}|\delta^{\mathcal{M}}$, up to its ω^{th} cardinal above $\delta^{\mathcal{M}}$, i. e. if $\mathcal{M}_0 := \mathcal{M}|\delta^{\mathcal{M}}$ and $\mathcal{M}_{i+1} := Lp_\alpha(\mathcal{M}_i)$, then we have $\mathcal{M} = \bigtriangledown_{i<\omega} \mathcal{M}_i$.
3. For each $\xi < \delta^{\mathcal{M}}$, if ξ is a cardinal of $\mathcal{M}$, then ξ is not Woodin in $Lp_\alpha(\mathcal{M}|\xi)$.

Definition 3.32. Let $\mathcal{T}$ be an iteration tree on some premouse $\mathcal{M}$. We say $\mathcal{T}$ *lives below* η iff $\mathcal{T}$ can be considered as an iteration tree on $\mathcal{M}|\eta$.

Let $A \in \mathrm{Pow}(<\mu_+)$ and $\mathcal{T}$ be an iteration tree of length $< \overline{\overline{\mathcal{M}}}^+$ which lives below $\delta^{\mathcal{M}}$ on some suitable A-premouse $\mathcal{M}$. Then we say $\mathcal{T}$ is *short* iff for all $\xi \leq lh(\mathcal{T})$ we have that $\delta(\mathcal{T}\restriction\xi)$ is not Woodin in $Lp_\alpha(\mathcal{M}(\mathcal{T}\restriction\xi))$. Otherwise we say $\mathcal{T}$ is *maximal.*

Definition 3.33. Let $A \in \mathrm{Pow}(<\mu_+)$ and Σ be an $\overline{\overline{\mathcal{M}}}^+$-iteration strategy for some suitable A-premouse $\mathcal{M}$. Then we say Σ is *fullness-preserving* iff for any iteration tree $\mathcal{T}$ on $\mathcal{M}$ which is built according to Σ and lives below $\delta^{\mathcal{M}}$ and for any $\xi < lh(\mathcal{T})$ we have either

1. $[0,\xi]_{\mathcal{T}}$ does not drop and $\mathcal{M}^{\mathcal{T}}_\xi$ is suitable, or
2. $[0,\xi]_{\mathcal{T}}$ drops and $\Vdash^{\mathrm{Col}(\omega,<\mu_+)}$ "$J_{\check{\alpha}}(\mathbb{R}) \models (\mathcal{M}^{\mathcal{T}}_\xi)\check{}$ is ω_1-iterable".

Lemma 3.34. *Suppose Σ is a fullness-preserving iteration strategy for a suitable $\mathcal{M}$, and $\mathcal{T}$ is an iteration tree living below $\delta^{\mathcal{M}}$ according to Σ. Let b be the branch through $\mathcal{T}$ chosen by Σ.*

Then we have that $\mathcal{T}\restriction\xi$ is short for all $\xi < lh(\mathcal{T})$ and

1. *if $\mathcal{T}$ is short, then $\mathcal{Q}(\mathcal{T})$ exists and is a proper initial segment of $Lp_\alpha(\mathcal{M}(\mathcal{T}))$, and*

2. *if $\mathcal{T}$ is maximal, then b does not drop and $e_b(\delta^{\mathcal{M}}) = \delta(\mathcal{T})$.*

Remark. Note that this implies that a fullness-preserving iteration strategy is guided by $\mathcal{Q}$-structures which are initial segments of $Lp_\alpha(\mathcal{M}(\mathcal{T}))$, and that the iteration game is over as soon as we reach a maximal tree.

Let b be the branch through a maximal tree determined by Σ. Although it might happen that we don't know the branch, we can identify the model $\mathcal{M}_b^{\mathcal{T}}$ using Lp_α. $\mathcal{M}_b^{\mathcal{T}}$ is suitable by definition of fullness-preserving and since $\delta(\mathcal{T})$ is Woodin in $\mathcal{M}_b^{\mathcal{T}}$ we have $\delta^{\mathcal{M}_b^{\mathcal{T}}} = \delta(\mathcal{T})$. So we get that $\mathcal{M}_b^{\mathcal{T}}$ is the Lp_α closure of $\mathcal{M}(\mathcal{T})$ up to its ω^{th} cardinal above $\delta(\mathcal{T})$.

To prove $(W^\star_{\alpha+1})$, we need a mouse closure operator which will serve as the basis for a "projective like" induction. The mice we need are "hybrid mice" which contain an additional predicate Σ for the iteration strategy of some suitable premouse $\mathcal{N}$. Σ and $\mathcal{N}$ are given by

Lemma 3.35. *There is in HOD_X a suitable premouse $\mathcal{N}$ and a fullness-preserving μ_+-iteration strategy Σ which condenses well. Moreover:*

$$\Vdash^{\mathrm{Col}(\omega,<\mu_+)}_{\mathsf{HOD}_X} \check{\mathcal{N}} \text{ is } \omega_1\text{-iterable, witnessed by the unique fullness-preserving iteration strategy } \dot{\Sigma}_0\text{, which extends } \check{\Sigma}^{28} \text{ and condenses well.}$$

This result heavily relies on the following theorem by Woodin.

Theorem 3.36 (Woodin). *Let $z \in \mathbb{R}^g$ code x^* and let $\mathcal{A}$ be a countable collection of sets of reals ordinal definable from z over some $J_\gamma(\mathbb{R}^g), \gamma < \alpha$.*

Then there is a suitable, weakly $\mathcal{A}$-iterable z-premouse.[29]

Proof sketch of Lemma 3.35. This can be proved as in [Ste05, Lemma 1.39]: One gets a suitable mouse $\mathcal{N}^\star \in \mathsf{HOD}_X[g]$ together with a fullness-preserving ω_1-iteration strategy Σ_0. This mouse already exists in $\mathsf{HOD}_X[g{\restriction}\iota]$ for some $\iota < \mu_+$. Now, by considering all finite variants of $g{\restriction}\iota$, it is possible to get such a $\mathcal{N}^\star$ as a generic extension of a mouse $\mathcal{N}$ in HOD_X: $\mathcal{N}[g{\restriction}\iota] = \mathcal{N}^\star$.

[28] This means, if G is $\mathrm{Col}(\omega,<\mu_+)$-generic over HOD_X then $\dot{\Sigma}_0^G$ and Σ coincide in HOD_X: $\dot{\Sigma}_0^G{\restriction}\mathsf{HOD}_X = \check{\Sigma}^G$.

[29] See [Ste05, Definition 1.45] for the definition of weak $\mathcal{A}$-iterability.

This mouse $\mathcal{N}$ inherits the iteration strategy from $\mathcal{N}^\star$ and is iterable in $\mathsf{HOD}_X[g]$ as well as in HOD_X. $\square$

So for the rest of this chapter let $\mathcal{N}$ be the suitable mouse in HOD_X given by the Lemma, and let Σ be the fullness-preserving μ_+-iteration strategy for $\mathcal{N}$ which condenses well in HOD_X.

Lemma 3.37. *Let $\mathcal{M}$ be a premouse with $\overline{\overline{\mathcal{M}}} = \bar{\mu} < \mu$ such that there is an elementary embedding $\pi\colon \mathcal{M} \to \mathcal{N}$.*

Then there is a $\bar{\mu}^+$-iteration strategy $\Sigma_\mathcal{M}$ for short trees on $\mathcal{M}$ which condenses well. If $Y \subseteq \mathsf{On}$ is a set of ordinals such that $\mathcal{M} \in \mathsf{HOD}_{X,Y}$, then $\Sigma_\mathcal{M}{\restriction}\mathsf{HOD}_{X,Y} \in \mathsf{HOD}_{X,Y}$.

Moreover, in each $\mathsf{HOD}_{X,Y}$ containing $\mathcal{M}$ there is an extension of $\Sigma_\mathcal{M}$ which also works for maximal trees and condenses well. We call it $\Sigma_{\mathcal{M},Y}$.

Proof. We show first that if Y is such that $\mathcal{M} \in \mathsf{HOD}_{X,Y}$, then there is such an iteration strategy $\Sigma_{\mathcal{M},Y}$. Then we prove that these iteration strategies coincide when restricted to short trees, so that for short trees $\mathcal{T}$ we can set:

$$\Sigma_\mathcal{M}(\mathcal{T}) := \bigcup\{\Sigma_{\mathcal{M},Y}(\mathcal{T}) : \mathcal{T} \in \mathsf{HOD}_{X,Y}\}$$

Let $\bar{\eta} = \bar{\mu}^+$. Suppose further that Y is given and let $H := H_{\bar{\eta}}^{\mathsf{HOD}_{X,Y}}$. Then $\mathcal{M}$ and all potential iteration trees of length $< \bar{\eta}$ in $\mathsf{HOD}_{X,Y}$ already are in $\mathsf{HOD}_X[H]$. Since the cardinality of H is small enough we can absorb $H \times g$ via a $\mathrm{Col}(\omega, < \mu_+)$-generic object h over HOD_X:

$$\mathsf{HOD}_X[H][g] = \mathsf{HOD}_X[h].$$

In this model, $\mathcal{N}$ is ω_1-iterable via a fullness-preserving iteration strategy $\Gamma := \dot{\Sigma}_0^h$ which condenses well. Moreover, since $\mathcal{M}$ is countable in $\mathsf{HOD}_X[h]$, an absoluteness argument produces an elementary embedding $\sigma\colon \mathcal{M} \to \mathcal{N}$ with $\sigma \in \mathsf{HOD}_X[h]$. So $\mathcal{M}$ is ω_1-iterable in $\mathsf{HOD}_X[h]$ via the pullback strategy Γ^σ.

Claim 1. Γ^σ restricted to short trees is the $\mathcal{Q}$-structure iteration strategy.

Proof. Assume $\Gamma^\sigma{\restriction}\xi$ is the $\mathcal{Q}$-structure iteration strategy and $\mathcal{T}$ is a short tree on $\mathcal{M}$ of length ξ according to Γ^σ. Let $b := \Gamma^\sigma(\mathcal{T})$. We have $\delta^{\mathcal{M}_b^\mathcal{T}} > \delta(\mathcal{T})$

since $\mathcal{T}$ is short, so $\delta(\mathcal{T})$ is not a Woodin cardinal in $\mathcal{M}_b^{\mathcal{T}}$ and therefore there is an initial segment $\mathcal{Q} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$ which defines a failure of $\delta(\mathcal{T})$ to be Woodin. But $\mathcal{Q}$ is embedded into an initial segment of $\mathcal{N}_b^{\sigma\mathcal{T}}$, so it is countably iterable and therefore a $\mathcal{Q}$-structure. □(*Claim* 1)

Claim 2. Γ^σ condenses well.

Proof. Suppose $\mathcal{S}$ is a hull of $\mathcal{T}$ and $\mathcal{T}$ is built according to Γ^σ, then $\sigma\mathcal{S}$ is a hull of $\sigma\mathcal{T}$ which is a tree on $\mathcal{N}$ according to Γ. So, since Γ condenses well, we have that $\sigma\mathcal{S}$ is also according to Γ. This ensures that $\mathcal{S}$ is built with the pullback strategy, i. e. $\mathcal{S}$ is according to Γ^σ. □(*Claim* 2)

But now Γ^σ induces an iteration strategy in $\mathsf{HOD}_X[H]$, i. e.

$$\Sigma_{\mathcal{M},Y} := \Gamma^\sigma \restriction \mathsf{HOD}_X[H] \in \mathsf{HOD}_X[H],$$

and $\Sigma_{\mathcal{M},Y}$ condenses well in $\mathsf{HOD}_X[H]$. As H contains the relevant trees of $\mathsf{HOD}_{X,Y}$, $\Sigma_{\mathcal{M},Y}$ is an iteration strategy in $\mathsf{HOD}_{X,Y}$ for arbitrary trees of size $< \bar{\mu}^+$. Thus, the second part of the lemma is proved.

To show that this strategy restricted to short trees is independent of the choice of Y, we have to suppose that Y' is another set of ordinals with $\mathcal{M} \in \mathsf{HOD}_{X,Y'}$. We can assume that $\mathsf{HOD}_{X,Y} \subseteq \mathsf{HOD}_{X,Y'}$. If we build H', h' in the same way as H, h, we also have $\mathsf{HOD}_X[H] \subseteq \mathsf{HOD}_X[H']$ and therefore $\mathsf{HOD}_X[h] \subseteq \mathsf{HOD}_X[h']$. But now $\mathsf{HOD}_X[h']$ is a $\mathrm{Col}(\omega, < \mu_+)$-generic extension of $\mathsf{HOD}_X[h]$ and the same arguments as before give that $\dot{\Sigma}_0^{h'} \restriction \mathsf{HOD}_X[h] \in \mathsf{HOD}_X[h]$ is fullness-preserving and condenses well. Since fullness-preserving iteration strategies can only disagree at some maximal tree, we get

$$\dot{\Sigma}_0^{h}(\mathcal{T}) = \dot{\Sigma}_0^{h'}(\mathcal{T}) \text{ for each short tree } \mathcal{T} \in \mathsf{HOD}_X[h].$$

Since not only the iteration strategies but also the pullback strategies are guided by $\mathcal{Q}$-structures if the trees are short, they coincide. □

Using that elementary substructures of $\mathcal{N}$ are iterable via the same iteration strategy (they are iterable as $\mathcal{N}$ via the $\mathcal{Q}$-structure iteration strategy as long as no maximal tree is reached), we can now extend the iteration strategy Σ to trees in V of arbitrary length. We also call this extension Σ. Again the restriction of that extended strategy to a model of the form $\mathsf{HOD}_{X,Y}$ is an element of $\mathsf{HOD}_{X,Y}$ and again they all coincide as long as no maximal tree is reached.

Lemma 3.38. *There is in* V *a partial iteration strategy* Σ *for* $\mathcal{N}$ *which works for short trees of arbitrary length and condenses well. This iteration strategy is ordinal definable from* $\mathcal{N}$ *and* X*, and therefore* $\Sigma\restriction\mathsf{HOD}_{X,Y} \in \mathsf{HOD}_{X,Y}$ *for each* $Y \subseteq \mathsf{On}$*.*

The proof heavily uses the methods from the proof of [Ste05, Lemma 1.25]. Recall that each successor cardinal in V is weakly compact and all limit cardinals are singular, so the cofinality of the length of some iteration tree is either ω or weakly compact.

Proof. Suppose we have defined $\Sigma\restriction\xi$ for short trees of length less than a limit ordinal ξ. Let $\mathcal{T}$ be an iteration tree of length ξ according to $\Sigma\restriction\xi$. We have the following cases

1. $\mathrm{cof}(\xi) = \omega$, or
2. $\mathrm{cof}(\xi) = \kappa$, where κ is weakly compact and $\xi \in [\kappa, \kappa^+)$, or
3. $\mathrm{cof}(\xi) = \kappa$, where κ is weakly compact and $\xi \geq \kappa^+$.

Suppose first that $\mathrm{cof}(\xi) = \omega$. We work in $\mathsf{HOD}_{X,Y}$, where Y codes $\mathcal{T}$ and is a witness for the fact that ξ has cofinality ω. Let further $\Omega > \xi$ be a large enough limit ordinal.

Definition 3.39. Fix a cardinal $\tilde{\mu} < \mu$ with $\overline{\overline{\tilde{\mu}^\omega}} = \tilde{\mu}$. We say that a substructure $\mathcal{X} \prec V_\Omega$ is *nice* iff $\{\mathcal{T}, \mathcal{N}\} \cup \tilde{\mu} \subseteq \mathcal{X}$, $\overline{\overline{\mathcal{X}}} = \tilde{\mu}$, and $\mathcal{X}$ is closed under ω-sequences. Moreover, we demand that $\mathcal{X}$ is cofinal in ξ.

For a nice $\mathcal{X}$, let $\pi_{\mathcal{X}} \colon H_{\mathcal{X}} \to V_\Omega$ be the uncollapsing map. Suppose $\mathcal{T}_{\mathcal{X}} := \pi_{\mathcal{X}}^{-1}(\mathcal{T})$ and $\mathcal{N}_{\mathcal{X}} := \pi^{-1}(\mathcal{N})$. Now define $\Sigma_{\mathcal{X}} := \Sigma_{\mathcal{N}_{\mathcal{X}},Y}$ as the iteration strategy in $\mathsf{HOD}_{X,Y}$ for $\mathcal{N}_{\mathcal{X}}$ given by Lemma 3.37. We have that $\mathcal{N}_{\mathcal{X}}$ has size $\overline{\overline{H_{\mathcal{X}}}}$. Since $\mathcal{T}_{\mathcal{X}} \in H_{\mathcal{X}}$ it has length $< H_{\mathcal{X}} \cap \mathsf{On}$ and therefore less then $\overline{\overline{H_{\mathcal{X}}}}^+$, so $b_{\mathcal{X}} := \Sigma_{\mathcal{X}}(\mathcal{T}_{\mathcal{X}})$ is defined. Since $lh(\mathcal{T}_{\mathcal{X}})$ has cofinality ω, the branch $b_{\mathcal{X}}$ is determined by any ω-sequence cofinal in it. Thus we have $b_{\mathcal{X}} \in H_{\mathcal{X}}$, because $\mathcal{X}$ (and therefore $H_{\mathcal{X}}$) is closed under ω-sequences.

For $\mathcal{X} \prec \mathcal{Y}$ both nice, we set $\pi_{\mathcal{X},\mathcal{Y}} := \pi_{\mathcal{Y}}^{-1} \circ \pi_{\mathcal{X}} \colon H_{\mathcal{X}} \to H_{\mathcal{Y}}$.

Definition 3.40. $\mathcal{X}$ is $\mathcal{T}$*-stable* iff $\mathcal{X}$ is nice and for each nice $\mathcal{Y}$ such that $\mathcal{X} \prec \mathcal{Y}$, we have

$$c_{\mathcal{X},\mathcal{Y}} := \text{“downward closure of } \pi_{\mathcal{X},\mathcal{Y}}{}''b_{\mathcal{X}} \text{ in } \mathcal{T}_{\mathcal{Y}}\text{”} = b_{\mathcal{Y}}.$$

Claim 1. There is a $\mathcal{T}$-stable $\mathcal{X}$.

Proof. Suppose the claim is false. Then we can build a continuous elementary chain $(\mathcal{X}_\nu : \nu < \tilde{\mu}^+)$, such that $\mathcal{X}_\nu$ is nice and $\pi_{\mathcal{X}_\nu,\mathcal{X}_{\nu+1}}(b_{\mathcal{X}_\nu}) \neq b_{\mathcal{X}_{\nu+1}}$ whenever ν is a successor ordinal or a limit ordinal of uncountable cofinality.[30]

From now on we drop the subscript $\mathcal{X}$, whenever it is possible, i. e. we abbreviate $\pi_\nu := \pi_{\mathcal{X}_\nu}$, $\pi_{\nu,\gamma} := \pi_{\mathcal{X}_\nu,\mathcal{X}_\gamma}$, $b_\nu := b_{\mathcal{X}_\nu}$, $\mathcal{T}_\nu := \mathcal{T}_{\mathcal{X}_\nu}$, and $\Sigma_\nu := \Sigma_{\mathcal{X}_\nu}$.

For the set $\mathcal{Z} := \bigcup_{\nu<\tilde{\mu}^+} \mathcal{X}_\nu$ we then have:

$$S := \{\mathcal{X}_\nu : \nu < \tilde{\mu}^+, \operatorname{cof}(\nu) > \omega\} \text{ is stationary in } [\mathcal{Z}]^{<\tilde{\mu}^+}$$

Define a regressive function $f\colon S \to \mathcal{Z}$ via $f(\mathcal{X}_\nu) := \pi_\nu(b_\nu) \in \mathcal{X}_\nu$. The usual Fodor argument, see [Jec03, Theorem 8.24], now yields a stationary $\tilde{S} \subseteq S$ and an $x \in \mathcal{Z}$ such that $f''\tilde{S} = \{x\}$. But $\tilde{S}$ gives rise to a stationary $S' \subseteq \tilde{\mu}^+$ and since $x = \pi_\iota(b_\iota)$ for some ι we have

$$\forall \nu \in S' \ \operatorname{cof}(\nu) > \omega \quad \text{and} \quad \forall \nu, \gamma \in S' \ (\nu < \gamma \Rightarrow \pi_{\nu,\gamma}(b_\nu) = b_\gamma).$$

Fix $\nu, \gamma \in S', \nu < \gamma$. Then we have the situation

$$\begin{tikzcd}
\mathcal{T}_\nu \arrow[r, "\pi_{\nu,\nu+1}"] \arrow[rr, bend left, "\pi_{\nu,\gamma}"] & \mathcal{T}_{\nu+1} \arrow[r, "\pi_{\nu+1,\gamma}"] & \mathcal{T}_\gamma \\
b_\nu \arrow[r, "\pi_{\nu,\nu+1}"] \arrow[rr, bend right, "\pi_{\nu,\gamma}"'] & \pi_{\nu,\nu+1}(b_\nu) \arrow[r, "\pi_{\nu+1,\gamma}"] & b_\gamma
\end{tikzcd}$$

Now the function $\pi_{\nu+1,\gamma}$ maps $\mathcal{T}_{\nu+1}{}^\frown\pi_{\nu,\nu+1}(b_\nu)$ to $\mathcal{T}_\gamma{}^\frown b_\gamma$. It follows that $(\pi_{\nu+1,\gamma}\mathcal{T}_{\nu+1})^\frown\pi_{\nu,\nu+1}(b_\nu)$ is a hull of $\mathcal{T}_\gamma{}^\frown b_\gamma$. Since $\mathcal{T}_\gamma{}^\frown b_\gamma$ is built according

[30] If ν has countable cofinality then $\mathcal{X}_\nu$ is not closed under ω-sequences.

to Σ_γ which condenses well, $(\pi_{\nu+1,\gamma}\mathcal{T}_{\nu+1})^\frown\pi_{\nu,\nu+1}(b_\nu)$ is also according to Σ_γ. This means that $\pi_\gamma\pi_{\nu+1,\gamma}\mathcal{T}_{\nu+1} = \pi_{\nu+1}\mathcal{T}_{\nu+1}$ is a tree on $\mathcal{N}$ according to the iteration strategy $\dot{\Sigma}_0^h$ (cf. Lemma 3.37) of which Σ_γ is the pullback strategy, and $\pi_{\nu,\nu+1}(b_\nu) = \Sigma(\pi_{\nu+1}\mathcal{T}_{\nu+1})$. But $\Sigma_{\nu+1}$ also is the pullback of $\dot{\Sigma}_0^h$ by the proof of Lemma 3.37, so $\pi_{\nu,\nu+1}(b_\nu) = \Sigma_{\nu+1}(\mathcal{T}_{\nu+1}) = b_{\nu+1}$, a contradiction. □(*Claim* 1)

The claim gives us a $\mathcal{T}$-stable $\mathcal{X}$, and if we set

$$c_\mathcal{X} := \text{“downward closure of } \pi_\mathcal{X}(b_\mathcal{X}) \text{ in } \mathcal{T}\text{”}$$

we can define

$$\Sigma_Y(\mathcal{T}) := \bigcup\{c_\mathcal{X} : \mathcal{X} \text{ is } \mathcal{T}\text{-stable}\}$$

One can easily see that $\Sigma_Y(\mathcal{T})$ is a wellfounded cofinal branch through $\mathcal{T}$. So $\Sigma_Y(\mathcal{T})$ gives us what we want, except that it depends on the choice of Y. But we can show that $\Sigma_Y(\mathcal{T}) = \Sigma_Z(\mathcal{T})$ for all adequate Z.

Claim 2. Let Z be another code for $\mathcal{T}$ and a witness for $\mathrm{cof}(\xi) = \omega$. Suppose w. l. o. g. $\mathsf{HOD}_{X,Y} \subseteq \mathsf{HOD}_{X,Z}$.

Then $\Sigma_Y(\mathcal{T}) = \Sigma_Z(\mathcal{T})$.

Proof. We have to show that $b_{\Sigma_Y} = b_{\Sigma_Z}$ where $b_{\Sigma_Y} := \Sigma_Y(\mathcal{T})$, $b_{\Sigma_Z} := \Sigma_Z(\mathcal{T})$. For this we show that Σ_Y is guided by $\mathcal{Q}$-structures in V. Let $\mathcal{X}$ be $\mathcal{T}$-stable in $\mathsf{HOD}_{X,Y}$. Since $\mathcal{T}$ is short, $\mathcal{T}_\mathcal{X}$ is short too, so the iteration strategy $\Sigma_\mathcal{X}$ is guided by $\mathcal{Q}$-structures. So let $\mathcal{Q}$ be the image of the $\mathcal{Q}$-structure from the branch $b_\mathcal{X}$ in $\mathsf{HOD}_{X,Y}$. If we can show that $\mathcal{Q}$ is a mouse in V, i. e. countably iterable, then we are done. This is because then $\mathcal{Q}$ is a $\mathcal{Q}$-structure for $\mathcal{T}$ in V and therefore b_{Σ_Y} is a cofinal branch coming with a $\mathcal{Q}$-structure. Since there can be at most one branch with a $\mathcal{Q}$-structure and since we can do the same with Z instead of Y, we get $b_{\Sigma_Y} = b_{\Sigma_Z}$.

So suppose $\sigma\colon \mathcal{R} \to \mathcal{Q}$ is elementary, $\sigma \in \mathsf{V}$, and $\mathcal{R}$ is countable. Consider $\mathsf{HOD}_{X,Y}[\mathcal{R}]$. We can suppose that $\mathcal{R}$ is also countable in $\mathsf{HOD}_{X,Y}[\mathcal{R}]$, so by absoluteness of wellfoundedness there is an elementary $\sigma'\colon \mathcal{R} \to \mathcal{Q}$ with $\sigma' \in \mathsf{HOD}_{X,Y}[\mathcal{R}]$. Since the forcing which adds $\mathcal{R}$ to $\mathsf{HOD}_{X,Y}$ has cardinality less than $\tilde{\mu}$, we can find a set $U \in \mathsf{HOD}_{X,Y}$ of $\mathsf{HOD}_{X,Y}$-cardinality $< \tilde{\mu}$ which covers $\mathrm{ran}(\sigma')$. So in $\mathsf{HOD}_{X,Y}$ we can take a $\mathcal{T}$-stable substructure $\mathcal{X} \supseteq U$.

By definition of Σ_Y we have that $\pi_{\mathcal{X}}(b_{\mathcal{X}}) = b_{\Sigma_Y}$ so $\mathcal{Q}_{b_{\mathcal{X}}} := \pi_{\mathcal{X}}^{-1}(\mathcal{Q})$ is the $\mathcal{Q}$-structure for the branch $b_{\mathcal{X}}$. But now $\pi_{\mathcal{X}}^{-1} \circ \sigma' \colon \mathcal{R} \to \mathcal{Q}_{b_{\mathcal{X}}}$ is elementary, and since $\mathcal{N}_{\mathcal{X}}$ is iterable in V (cf. Lemma 3.37), $\mathcal{R}$ is embeddable into an initial segment of an iterate of some iterable structure, so $\mathcal{R}$ is itself iterable. $\square$(*Claim* 2)

So we can set:

$$\Sigma(\mathcal{T}) := \bigcup\{\Sigma_Y(\mathcal{T}) : Y \text{ codes } \mathcal{T} \text{ and is a witness for } \mathrm{cof}(\xi) = \omega\}$$

Claim 3. Σ condenses well.

Proof. Let $c := \Sigma(\mathcal{T})$ and let $\mathcal{S}^\frown b$ be a hull of $\mathcal{T}^\frown c$, witnessed by σ. Assume w. l. o. g. that $\mathrm{cof}(lh(\mathcal{S})) = \omega$, and that $Y \subseteq \mathsf{On}$ is such that we have the same situation in $\mathsf{HOD}_{X,Y}$. Then we work in $\mathsf{HOD}_{X,Y}$.

Take some elementary substructure $\mathcal{X}$ of a large initial segment of the universe which is both $\mathcal{T}$-stable and $\mathcal{S}$-stable, and which contains $\mathcal{T}$ and $\mathcal{S}$. Let π be the uncollapsing map. Then we have:

$$\begin{array}{ccc} \mathcal{S}^\frown b & \xrightarrow{\sigma} & \mathcal{T}^\frown c \\ \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \pi} \\ \bar{\mathcal{S}}^\frown \bar{b} & \xrightarrow{\bar{\sigma}} & \bar{\mathcal{T}}^\frown \bar{c} \end{array}$$

Now we have $\pi(\bar{c}) = c$ and therefore $\bar{c} = \Sigma_Y(\bar{\mathcal{T}})$ by definition of Σ_Y. Since $\bar{\mathcal{S}}^\frown \bar{b}$ is a “small” hull of $\bar{\mathcal{T}}^\frown \bar{c}$ it is built according to Σ_Y by induction hypothesis. So we get $\bar{b} = \Sigma_Y(\bar{\mathcal{T}})$. Therefore $\Sigma_Y(\mathcal{S}) \supseteq \pi(\bar{b}) = b$ follows from the definition of Σ_Y and we are done. $\square$(*Claim* 3)

The other cases are not as complicated since there is at most one cofinal branch, due to the uncountable cofinality of ξ. This branch is automatically wellfounded and the iteration strategy condenses well since the length of each hull of $\mathcal{T}$ either has uncountable cofinality, in which case there is at most one cofinal branch, or it has countable cofinality, but then it is also a hull of an initial segment of $\mathcal{T}$ and the induction hypothesis implies the rest.

Now let $\mathrm{cof}(\xi) = \kappa > \omega$ and $\bar{\bar{\xi}} = \kappa$. We then have that κ and κ^+ are weakly compact. Suppose that no cofinal branch through $\mathcal{T}$ exists. We apply Lemma 2.46 to the model $W := \mathsf{HOD}_{X,\mathcal{T},Y}$ where Y is a witness for the cofinality of ξ.[31] Abbreviate $\tilde{W} := \mathrm{ult}(W, U)$ and let $\pi\colon W \to \tilde{W}$ be the ultrapower embedding with critical point κ. We have the following:

$$\tilde{W} \models \exists \mathcal{S}\ (\mathcal{S} \text{ is a hull of } \pi(\mathcal{T}) \text{ of length} < \pi(\kappa) \wedge \mathcal{S} \text{ has no cofinal branch})$$

This is because $\mathcal{T}$ witnesses this fact. $\mathcal{T}$ is an element of $\tilde{W}$ because it can be coded in a subset of κ and $\tilde{W}$ does not contain a cofinal branch since not even V does. Moreover, $\mathcal{T}$ is a hull of $\pi(\mathcal{T})$. This is witnessed by $\pi\restriction\xi$. But this function is also an element of $\tilde{W}$.

So this fact reflects to W:

$$W \models \exists \mathcal{S}\ (\mathcal{S} \text{ is a hull of } \mathcal{T} \text{ of length} < \kappa \wedge \mathcal{S} \text{ has no cofinal branch})$$

Now, since the restriction of Σ to trees of length $< \kappa$ in W is an element of W and condenses well there, we have that $\mathcal{S}$ is built according to Σ. So $\Sigma(\mathcal{S})$ is defined and an element of W. But this is of course a contradiction. So if $\mathrm{cof}(\xi) = \kappa > \omega$ and $\overline{\overline{\xi}} = \kappa$, then there is a cofinal branch through $\mathcal{T}$.

Now suppose $\mathrm{cof}(\xi) = \kappa > \omega$ and $\xi > \kappa^+$. Again we work in the ZFC model $W := \mathsf{HOD}_{X,\mathcal{T},Y}$, where Y witnesses the cofinality of ξ. Let $\mathcal{S}$ be a hull of $\mathcal{T}$ of size κ, witnessed by a function σ which is cofinal in ξ.[32] So $\mathcal{S}$ is built according to Σ of size $< \xi$, which ensures that $\Sigma(\mathcal{S})$ is defined. Now consider the downward closure of $\sigma''\Sigma(\mathcal{S})$ in $\mathcal{T}$. But this is of course a cofinal branch since the function σ is cofinal in ξ.

So all possible cases for the cofinality of ξ have been discussed. Thus, we get an iteration strategy whose restriction to $\mathsf{HOD}_{X,Y}$ is in $\mathsf{HOD}_{X,Y}$. □(Lemma 3.38)

Hence we have an iteration strategy Σ in V which works for short trees, such that $\Sigma\restriction\mathsf{HOD}_{X,Y} \in \mathsf{HOD}_{X,Y}$ for each set of ordinals Y. Using the proof

[31] Note that the assumptions of Lemma 2.46 are satisfied since κ^+ is weakly compact and therefore inaccessible in W.

[32] For this we first take a subset of ξ which is cofinal and has cardinality κ. Then let $\mathcal{X} \prec V_\Omega^W$ be an elementary substructure containing this cofinal set and the tree $\mathcal{T}$. Now if we consider the transitive collapse H of $\mathcal{X}$ with uncollapsing map σ, we have that $\mathcal{S} := \sigma^{-1}(\mathcal{T})$ is the desired hull of $\mathcal{T}$ and $\sigma''lh(\mathcal{S})$ is the support cofinal in ξ.

of Lemma 3.38 we can extend Σ in each $\mathsf{HOD}_{X,Y}$ to an iteration strategy Σ_Y which also works for maximal trees. Unfortunately, if we have two sets of ordinals Y and Z we see no reason why $\Sigma_Y(\mathcal{T}) = \Sigma_Z(\mathcal{T})$ should hold for maximal trees.

Lemma 3.41. *For each Y there is in $\mathsf{HOD}_{X,Y}$ an extension of Σ which also works for maximal trees. This strategy condenses well in $\mathsf{HOD}_{X,Y}$.*

Proof. We use the notation of Lemma 3.38 and work in $\mathsf{HOD}_{X,Y}$.

Let $\mathcal{T}$ be a maximal tree built according to Σ. If $\mathrm{cof}(lh(\mathcal{T})) = \omega$, we can show as in Lemma 3.38 that there is a $\mathcal{T}$-stable $\mathcal{X}$ and so we can define a wellfounded branch as before.

Now if $\mathcal{T}$ is maximal and $\mathrm{cof}(lh(\mathcal{T})) > \omega$, we can show that each nice $\mathcal{X} \prec V_\Omega$ is $\mathcal{T}$-stable. The proof is as in [Ste05, Lemma 1.25].

Let $\mathcal{X}, \mathcal{Y}$ be nice substructures with $\mathcal{X} \prec \mathcal{Y}$. Set $\eta := \sup(c_{\mathcal{X},\mathcal{Y}})$. We are done if we can show $\eta = lh(\mathcal{T}_\mathcal{Y})$.[33]

Suppose the converse holds and $\eta < lh(\mathcal{T}_\mathcal{Y})$. Since $\mathcal{T}$ is maximal we get by Lemma 3.34:

$$e^{\mathcal{T}_\mathcal{X}}_{b_\mathcal{X}}(\delta^{\mathcal{N}_\mathcal{X}}) = \delta(\mathcal{T}_\mathcal{X})$$

Claim 1. Then we have $e^{\mathcal{T}_\mathcal{Y}}_{0,\eta}(\delta^{\mathcal{N}_\mathcal{Y}}) = \delta(\mathcal{T}_\mathcal{Y}\restriction\eta)$.

Proof. The proof follows [Ste05, Lemma 1.25]. W.l.o.g. we can assume that $\pi_{\mathcal{X},\mathcal{Y}}{}''\delta^{\mathcal{N}_\mathcal{X}}$ is cofinal in $\delta^{\mathcal{N}_\mathcal{Y}}$.[34] Let $\rho < \delta^{\mathcal{N}_\mathcal{Y}}$ and $\rho' < \delta^{\mathcal{N}_\mathcal{X}}$ be such that $\rho < \pi_{\mathcal{X},\mathcal{Y}}(\rho')$. Since $e^{\mathcal{T}_\mathcal{X}}_{b_\mathcal{X}}(\rho') < e^{\mathcal{T}_\mathcal{X}}_{b_\mathcal{X}}(\delta^{\mathcal{N}_\mathcal{X}}) = \delta(\mathcal{T}_\mathcal{X})$ there is a $\beta \in b_\mathcal{X}$ with $cr(e^{\mathcal{T}_\mathcal{X}}_{\beta,b_\mathcal{X}}) > e^{\mathcal{T}_\mathcal{X}}_{0,\beta}(\rho')$. But then for $\gamma := \pi_{\mathcal{X},\mathcal{Y}}(\beta)$ we can conclude that $cr(e^{\mathcal{T}_\mathcal{Y}}_{\gamma,\eta}) > e^{\mathcal{T}_\mathcal{Y}}_{0,\gamma}(\pi_{\mathcal{X},\mathcal{Y}}(\rho'))$. So

$$e^{\mathcal{T}_\mathcal{Y}}_{0,\eta}(\rho) < e^{\mathcal{T}_\mathcal{Y}}_{0,\eta}(\pi_{\mathcal{X},\mathcal{Y}}(\rho')) = e^{\mathcal{T}_\mathcal{Y}}_{0,\gamma}(\pi_{\mathcal{X},\mathcal{Y}}(\rho')) < cr(e^{\mathcal{T}_\mathcal{Y}}_{\gamma,\eta}) < \delta(\mathcal{T}_\mathcal{Y}\restriction\eta)$$

Now since $e^{\mathcal{T}_\mathcal{Y}}_{0,\eta}$ is continuous at $\delta^{\mathcal{N}_\mathcal{Y}}$, we have the desired result. $\square$(*Claim* 1)

[33] Since $\mathrm{cof}(lh(\mathcal{T}_\mathcal{Y})) > \omega$ there is at most one cofinal branch, so since both $b_\mathcal{Y}$ and $c_{\mathcal{X},\mathcal{Y}}$ are cofinal they are equal.

[34] Otherwise we slightly change our definition of "nice". Then we demand that nice substructures are cofinal in $\delta^\mathcal{N}$. Since μ is singular and $\mathcal{N}$ has size μ it follows $\mathrm{cof}(\delta^\mathcal{N}) < \mu$. We can arrange that there are cofinally many $\tilde{\mu} < \mu$ such that $\tilde{\mu}^\omega = \tilde{\mu}$, so we let nice substructures have size $\geq \mathrm{cof}(\delta^\mathcal{N})$.

But then, since the tree $\mathcal{T}_\mathcal{Y}$ is a normal continuation of $\mathcal{T}_\mathcal{Y}\restriction\eta$, we have $lh(E_\eta^{\mathcal{T}_\mathcal{Y}}) > \delta(\mathcal{T}_\mathcal{Y}\restriction\eta) = e_{0,\eta}^{\mathcal{T}_\mathcal{Y}}(\delta^{\mathcal{N}_\mathcal{Y}})$, which contradicts the fact that $\mathcal{T}_\mathcal{Y}$ is a tree on $\mathcal{N}_\mathcal{Y}|\delta^{\mathcal{N}_\mathcal{Y}}$. So indeed $\eta = lh(\mathcal{T}_\mathcal{Y})$, and hence $\mathcal{X}$ is $\mathcal{T}$-stable.

Now we can again define

$$\Sigma_Y(\mathcal{T}) := \bigcup\{c_\mathcal{X} : \mathcal{X} \text{ is } \mathcal{T}\text{-stable}\},$$

which defines a cofinal wellfounded branch in $\mathcal{T}$. $\square$

Hence there is an iteration strategy Σ in HOD_X which condenses well for arbitrary trees on $\mathcal{N}$ in HOD_X.

Definition 3.42. For the rest of this section, let Σ be the iteration strategy in HOD_X which we get from Lemma 3.41.

Now we can build premice over sets of ordinals A coding $\mathcal{N}$ which know how to iterate $\mathcal{N}$. The "condenses well" property enables us to do this in a way such that the resulting models have fine structure. Such mice are called Σ-hybrid. W. Hugh Woodin found a way to build premice which are closed under Σ and all of whose levels have fine structure. (Cf. [SSa].)

Definition 3.43. Suppose $A \subseteq \mathsf{On}$ codes $\mathcal{N}$ in a simple way. The J-structure $\mathcal{M} := J_\beta^{\vec{E},\vec{S}}(A)$ is called a Σ-*hybrid premouse* iff the following conditions are fulfilled:

1. $\vec{E}$ codes a sequence of extenders $(E_\iota : \iota \in X_0)$, where $X_0 \subseteq \beta + 1$. $(E_\iota : \iota \in X_0)$ has to satisfy [MS94b, Definition 1.0.4] (cf. also [SSZ02, Definition 2.4]) with the understanding that concerning $E_\iota \neq \emptyset$, the relevant initial segment of $\mathcal{M}$ is $J_\iota^{\vec{E},\vec{S}}(A)$ rather than $J_\iota^{\vec{E}}(A)$.
2. $\vec{S}$ codes fragments of Σ in the following sense: $\vec{S}$ codes a sequence $((\mathcal{T}_\iota, \gamma_\iota) : \iota \in X_1)$ such that
   (a) $X_0 \cap X_1 = \emptyset$,
   (b) if $\iota \in X_1$ and if $\bar{\iota} \leq \iota, \bar{\iota} \in X_1$ is least with $\mathcal{T}_{\bar{\iota}} = \mathcal{T}_\iota$, then $\mathcal{T}_{\bar{\iota}}$ is the $\mathcal{M}|\bar{\iota}$-least iteration tree which is built according to Σ such that $\Sigma(\mathcal{T}_{\bar{\iota}})$ does not exist in $\mathcal{M}|\bar{\iota}$ (as being provided by $\vec{S}\restriction\bar{\iota}$),
   (c) $\mathcal{T}_\delta = \mathcal{T}_\iota$ for all $\delta \in [\bar{\iota}, \iota] \cap X_1$, and

(d) $\{\gamma_\delta : \delta \in [\bar{\iota}, \iota] \cap X_1\} = \Sigma(\mathcal{T}_\iota) \cap \iota \mathrm{otp}([\bar{\iota}, \iota])$.

We also denote a Σ-hybrid premouse by $J_\beta^{\vec{E},\Sigma}(A)$.

The "condenses well" property ensures that Σ-hybrid premice satisfy condensation in the sense that elementary substructures of any Σ-hybrid premouse collapse to Σ-hybrid premice.

Lemma 3.44. *Let $J_\beta^{\vec{E},\Sigma}(A)$ be a Σ-hybrid premouse and let $\Omega \leq \beta$. Suppose $\mathcal{X} \prec J_\Omega^{\vec{E},\Sigma}(A)$, $\mathcal{X} \in \mathsf{HOD}_X$ is an elementary substructure with $A \cup \{A\} \subseteq \mathcal{X}$ and $J_\gamma^{\vec{F},\Gamma}(A) \cong \mathcal{X}$ is the transitive collapse of $\mathcal{X}$.*

Then $\Gamma = \Sigma \restriction J_\gamma^{\vec{F},\Gamma}(A)$.

We want to show that certain Σ-hybrid premice satisfy condensation not only in HOD_X, but also in V. First we prove, that if there were a substructure $\mathcal{X}$ of an initial segment of some hybrid premouse $J_\beta^{\vec{E},\Sigma}(A)$ with $\mathcal{X} \in \mathsf{V}$ which does not collapse to a Σ-hybrid premouse, then there would be such a witness of size $\overline{\overline{A}}$.

Lemma 3.45. *Let $A \in \mathrm{Pow}(< \mu_+)$ code $\mathcal{N}$ and $J_\beta^{\vec{E},\Sigma}(A)$ Σ-hybrid. Suppose condensation for $J_\beta^{\vec{E},\Sigma}(A)$ fails in V, i. e. there is an elementary embedding $\pi\colon J_\gamma^{\vec{F},\Gamma}(A) \to J_\beta^{\vec{E},\Sigma}(A), \pi \in \mathsf{V}$ such that $\Gamma \neq \Sigma \restriction J_\gamma^{\vec{F},\Gamma}(A)$.*

Then there is such a witness of size $\overline{\overline{A}}$.

Proof. Let $J_\gamma^{\vec{F},\Gamma}(A)$ witness that condensation for $J_\beta^{\vec{E},\Sigma}(A)$ is false in V and let π be the associated embedding. Since all these sets are wellorderable, we can assume that there is a set Y of ordinals such that the same situation exists in $\mathsf{HOD}_{X,Y}$. So we work in $\mathsf{HOD}_{X,Y}$.

Now we can take an elementary substructure $\mathcal{X}$ of a large initial segment of the universe with $A \cup \{A, \pi, \Sigma\} \subseteq \mathcal{X}$ such that $\overline{\overline{\mathcal{X}}} = \overline{\overline{A}}$. Let σ be the uncollapsing map. Then we have the following commutative diagram:

$$
\begin{tikzcd}
J_\gamma^{\vec{F},\Gamma}(A) \arrow[r, "\pi"] & J_\beta^{\vec{E},\Sigma}(A) \\
J_{\gamma'}^{\vec{F}',\Gamma'}(A) \arrow[r, "\pi'"] \arrow[u, "\sigma"] & J_{\beta'}^{\vec{E}',\Sigma'}(A) \arrow[u, "\sigma"]
\end{tikzcd}
$$

Now either $J^{\vec{F}',\Gamma'}_{\gamma'}(A)$ or $J^{\vec{E}',\Sigma'}_{\beta'}(A)$ witnesses the lemma, since otherwise

$$\Gamma' = \Sigma\restriction J^{\vec{F}',\Gamma'}_{\gamma'}(A) \quad \text{and} \quad \Sigma' = \Sigma\restriction J^{\vec{E}',\Sigma'}_{\beta'}(A)$$

and therefore

$$J^{\vec{F}',\Gamma'}_{\gamma'}(A) = J^{\vec{F}',\Sigma}_{\gamma'}(A) \quad \text{and} \quad J^{\vec{E}',\Sigma'}_{\beta'}(A) = J^{\vec{E}',\Sigma}_{\beta'}(A).$$

But then $J^{\vec{F}',\Gamma'}_{\gamma'}(A)$ and $J^{\vec{E}',\Sigma'}_{\beta'}(A)$ have the same "Σ-predicate" and by elementarity of σ $J^{\vec{F},\Gamma}_{\gamma}(A)$ and $J^{\vec{E},\Sigma}_{\beta}(A)$ also have the same "Σ-predicate". This is a contradiction. $\square$

The first step of our "projective like" induction is to get a sharp for the least model which knows how to iterate $\mathcal{N}$. This is corresponding to the situation in the inadmissible case where we produce a sharp for the minimal $\mathcal{M}$-closed model.

Definition 3.46. Let $A \in \mathsf{HOD}_X$ be a set of ordinals coding $\mathcal{N}$. Then $L^\Sigma(A)$ denotes the *least Σ-hybrid mouse over A containing all ordinals.* We have $L^\Sigma(A) \subseteq \mathsf{HOD}_X$.

This model has fine structure due to the "condenses well" property. Moreover, it satisfies a stronger form of condensation in HOD_X than usual Σ-hybrid premice: Let Ω be a limit ordinal and $J^\Sigma_\Omega(A) \trianglelefteq L^\Sigma(A)$. If $\mathcal{X} \prec J^\Sigma_\Omega(A)$ is an elementary substructure such that $A \cup \{A\} \subseteq \mathcal{X}$, $\mathcal{X} \in \mathsf{HOD}_X$ and $\mathcal{M} \cong \mathcal{X}$ is the transitive collapse, then $\mathcal{M} \trianglelefteq L^\Sigma(A)$.

We want to prove that $L^\Sigma(A)$ satisfies condensation in V. For this it suffices to show that condensation holds not only in HOD_X, but also in all $\mathrm{Col}(\omega, <\mu_+)$-generic extensions of HOD_X.

Lemma 3.47. *Let $A \in \mathrm{Pow}(<\mu_+)$ code $\mathcal{N}$ and $J^{\vec{E},\Sigma}_\beta(A)$ be Σ-hybrid, satisfying condensation in $\mathsf{HOD}_X^{\mathrm{Col}(\omega,<\mu_+)}$.*

Then condensation for $J^{\vec{E},\Sigma}_\beta(A)$ also holds in V.

Proof. Suppose there is an $\mathcal{X} \in \mathsf{V}$ containing A as a subset such that

$$J^{\vec{F},\Gamma}_\gamma(A) \cong \mathcal{X} \prec J^{\vec{E},\Sigma}_\beta(A) \quad \text{and} \quad \Gamma \neq \Sigma\restriction J^{\vec{F},\Gamma}_\gamma(A).$$

First by Lemma 3.45 we can suppose $\overline{\overline{\gamma}} = \overline{\overline{A}} < \mu_+$. But then an absoluteness argument between $\mathsf{V}^{\mathrm{Col}(\omega, <\mu_+)}$ and $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu_+)}$ implies that in $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu_+)}$ there is also such an $\mathcal{X}$.

But this is absurd since condensation for $J_\beta^{\vec{E},\Sigma}(A)$ holds in $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu_+)}$. $\square$

Thus we show the following:

Lemma 3.48. *Let $A \subseteq \mathrm{Pow}(<\mu_+), A \in \mathsf{HOD}_X$ code $\mathcal{N}$.*

Then $L^\Sigma(A)$ satisfies condensation in $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu_+)}$.

Proof. First note that the iteration strategy Σ restricted to iteration trees of size $< \mu_+ = \tilde{\omega}_1 := \omega_1^{\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu_+)}}$ which are in HOD_X is just the restriction of the $\tilde{\omega}_1$-iteration strategy which is given by Lemma 3.35. This iteration strategy condenses well in $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu_+)}$. So for the same reason for which $L^\Sigma(A)$ satisfies condensation in HOD_X, $L^\Sigma(A)$ satisfies condensation in $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu_+)}$ up to $\tilde{\omega}_1$. Now if $J_\gamma^\Gamma(A)$ is the transitive collapse of some elementary substructure of a large initial segment of $L^\Sigma(A)$ in $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu_+)}$, then the goal is to get in $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu_+)}$ an elementary embedding from $J_\gamma^\Gamma(A)$ into some $J_\beta^\Sigma(A)$ where $\beta < \tilde{\omega}_1$. Then we can use condensation of $L^\Sigma(A)$ up to $\tilde{\omega}_1$.

As before suppose $\pi\colon J_\gamma^\Gamma(A) \to L^\Sigma(A)$ is elementary with $J_\gamma^\Gamma(A) \ntrianglelefteq L^\Sigma(A)$, where $J_\gamma^\Gamma(A)$ has size $\overline{\overline{A}} < \mu_+$. Let $\mathcal{X} := \mathrm{ran}(\pi)$. The μ_+-c. c. of $\mathrm{Col}(\omega, <\mu_+)$ enables us to cover $\mathcal{X}$ by a set Z of size $< \mu_+$ in HOD_X. So there is in HOD_X an elementary substructure $\mathcal{Z} \supseteq Z$ of $L^\Sigma(A)$ which has size $< \mu_+$ and is such that $\mathcal{X} \prec \mathcal{Z} \prec L^\Sigma(A)$. But $\mathcal{Z} \cong J_\beta^\Sigma(A) \trianglelefteq L^\Sigma(A)$ for some $\beta < \mu_+$ since $L^\Sigma(A)$ satisfies condensation in HOD_X and $\mathcal{Z} \in \mathsf{HOD}_X$. So in $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu_+)}$ there is an elementary $\pi'\colon J_\gamma^\Gamma(A) \to J_\beta^\Sigma(A)$. Since $L^\Sigma(A)$ satisfies condensation up to $\tilde{\omega}_1 > \beta$, we can conclude $J_\gamma^\Gamma(A) \trianglelefteq L^\Sigma(A)$. $\square$

The last two lemmata guarantee that $L^\Sigma(A)$ satisfies condensation in V for each $A \in \mathrm{Pow}(<\mu_+) \cap \mathsf{HOD}_X$ which simply codes $\mathcal{N}$. This suffices to begin with the induction through our "projective like" hierarchy.

Lemma 3.49. *Suppose $A \in \mathrm{Pow}(<\mu_+) \cap \mathsf{HOD}_X$ simply codes $\mathcal{N}$. Let $L^\Sigma(A)$ be the least Σ-hybrid premouse in HOD_X.*

Then there exists a sharp for $L^\Sigma(A)$.

Proof. Let η be a singular V-cardinal large enough and closed under the function Θ. Since η^+ is weakly compact it is inaccessible in $L^\Sigma(A)$, so we have $\eta^{+L^\Sigma(A)} < \eta^+$ and therefore $\mathrm{cof}(\eta^{+L^\Sigma(A)}) < \eta$. Let $Y \subseteq \eta^{+L^\Sigma(A)}$ be cofinal and of order type $\mathrm{cof}(\eta^{+L^\Sigma(A)})$.

Working in $\mathsf{HOD}_X[Y]$, we build a substructure $\mathcal{X} \prec V_\Omega$ which is closed under ω-sequences. This substructure shall be cofinal in $\eta^{+L^\Sigma(A)}$ of size $< \eta$ and contain $A \cup \{A, \Sigma, \eta, \eta^{+L^\Sigma(A)}\}$ as a subset. Let $H \cong \mathcal{X}$ be the transitive collapse. Notice that $L^\Sigma(A)|(\mathsf{On} \cap H) \subseteq H$, since $L^\Sigma(A)$ satisfies condensation in V and therefore in $\mathsf{HOD}_X[Y]$.

Let η', λ be the preimages of $\eta, \eta^{+L^\Sigma(A)}$ under the uncollapsing map π.

Claim 1. $\lambda = \eta'^{+L^\Sigma(A)}$.

Proof. Suppose $\lambda < \eta'^{+L^\Sigma(A)}$. Then let $\beta \geq \lambda$ be the least ordinal, such that $\rho_\omega(J^\Sigma_\beta(A)) \leq \eta'$. Since the substructure is cofinal in $\eta^{+L^\Sigma(A)}$, we can now lift $J^\Sigma_\beta(A)$ via π to a mouse $\mathcal{M}$ which extends $L^\Sigma(A)|\eta^{+L^\Sigma(A)}$ and projects to η. This works as in Lemma 3.16. So $\mathcal{M}$ is the ultrapower of $J^\Sigma_\beta(A)$ by the extender derived from $\pi\restriction J^\Sigma_\lambda(A)$. Let i be the canonical ultrapower embedding.

Let $\mathcal{X}$ be an elementary substructure of a large initial segment of the universe such that $i'' J^\Sigma_\beta(A) \cup \{\mathcal{M}\} \subseteq \mathcal{X}$. Suppose $\pi\colon H' \to \mathcal{X}$ is the uncollapsing map and $\bar{\mathcal{M}} = \pi^{-1}(\mathcal{M})$. Then $\bar{\mathcal{M}}$ can be reembedded into $J^\Sigma_\beta(A)$ via $i^{-1} \circ \pi$ and is therefore an initial segment of $L^\Sigma(A)$, which also implies $\mathcal{M} \trianglelefteq L^\Sigma(A)$ by elementarity. This is of course a contradiction since no initial segment of $L^\Sigma(A)$ which extends $L^\Sigma(A)|\eta^{+L^\Sigma(A)}$ can project to η. $\square$(*Claim* 1)

So let $\mathcal{U}$ be the ultrafilter derived from $\pi\restriction\lambda$. The claim ensures that this is an ultrafilter on $J^\Sigma_\lambda(A)$. Then $(J^\Sigma_\lambda(A); \in, \mathcal{U})$ witnesses the existence of a sharp for $L^\Sigma(A)$.

Moreover if $(\mathcal{M}; \in, \bar{\mathcal{U}})$ is an element in the linear iteration of $(J^\Sigma_\lambda(A); \in, \mathcal{U})$ with $\mathcal{U}$, then also $\mathcal{M} \trianglelefteq L^\Sigma(A)$. This follows by an easy condensation argument. For this let $(\mathcal{M}; \in, \bar{\mathcal{U}})$ be a linear iterate of $(J^\Sigma_\lambda(A); \in, \mathcal{U})$ and let i be the associated embedding. We can assume that $(\mathcal{M}; \in, \bar{\mathcal{U}})$ is the first element in the iteration, i.e. $\mathcal{M} = \mathrm{ult}(J^\Sigma_\lambda(A), \mathcal{U})$. Then we take an elementary substructure $\mathcal{X}$ of a large initial segment of the universe such that $A \subseteq \mathcal{X}$. If H' is the transitive collapse of $\mathcal{X}$ and $\sigma\colon H' \cong \mathcal{X}$ is the uncollapsing map then we have the following situation:

$$\begin{tikzcd}
(J^{\Sigma}_{\lambda}(A); \in, \mathcal{U}) \arrow[r, "i"] & (\mathcal{M}; \in, \bar{\mathcal{U}}) \\
(J^{\Sigma}_{\lambda'}(A); \in, \mathcal{U}') \arrow[u, "\sigma"] \arrow[r, "i'"] & (\mathcal{M}'; \in, \bar{\mathcal{U}}') \arrow[u, "\sigma"]
\end{tikzcd}$$

Now we can use [SZ, Lemma 8.12] to show that $\mathcal{M}'$ is elementarily embeddable into $J^{\Sigma}_{\lambda}(A)$ and therefore by condensation an initial segment of $L^{\Sigma}(A)$. The elementarity of σ then yields $\mathcal{M} \trianglelefteq L^{\Sigma}(A)$. $\square$

We can now define certain mice likewise as in the inadmissible case which entirely ensure that $(W^{\star}_{\alpha+1})$ holds.

Definition 3.50. Let A be a set of ordinals coding $\mathcal{N}$. Then let $P^{\Sigma\sharp}_{n}(A)$ be the least iterable Σ-hybrid premouse over A which is active and satisfies "there are n Woodin cardinals", and let $P^{\Sigma\sharp\sharp}_{n}(A)$ be the least iterable Σ-hybrid active premouse over A which is closed under $P^{\Sigma\sharp}_{n}$. If such a premouse does not exist, this is undefined.

Lemma 3.51. *$P^{\Sigma\sharp}_{n}(A)$ exists for each set of ordinals $A \in \mathsf{HOD}_X$ coding $\mathcal{N}$.*

Proof. The proof for $n = 0$ is given above: $P^{\Sigma\sharp}_{0}(A)$ is just the sharp for $L^{\Sigma}(A)$. The case $n > 0$ is essentially the same as in the inadmissible case, but there are small differences which are due to the fact that the iteration strategy Σ is defined only for trees in HOD_X. So we will explain the case $n = 1$ more precisely.

First one can show that for each $A \in \mathrm{Pow}(< \mu_+) \cap \mathsf{HOD}_X$ which codes $\mathcal{N}$, $P^{\Sigma\sharp\sharp}_{0}(A)$ exists. Let $R^{\Sigma}(A \oplus A_0)$ be the minimal Σ-hybrid, $P^{\Sigma\sharp}_{0}$-closed model over $A \oplus A_0$ of height κ^+, and let Ω be the first indiscernible for $R^{\Sigma}(A \oplus A_0)$. Note that $R^{\Sigma}(A \oplus A_0) \subseteq \mathsf{HOD}_X$. Now we build in $R^{\Sigma}(A \oplus A_0)$ the Σ-hybrid K^c over A below Ω, with result $K^{c\Sigma}(A)$. We are done if we find some γ such that for $\mathcal{Q} := K^{c\Sigma}(A) \| \gamma$ there is a δ with $\mathcal{Q} \models$ "δ is Woodin" and $P^{\Sigma\sharp}_{0}(\mathcal{Q}|\delta) = \mathcal{Q}$, because then $\mathcal{Q} = P^{\Sigma\sharp}_{1}(A)$.

Again we use the following claim

Claim 1. In $R^\Sigma(A \oplus A_0)$ either $P_1^{\Sigma\sharp}(A)$ exists or $K^{c\Sigma}(A)$ is $\Omega+1$-iterable.

We show that $K^{c\Sigma}(A)$ cannot be $\Omega + 1$-iterable. Suppose the converse holds and $K^{c\Sigma}(A)$ is $\Omega + 1$-iterable. Again if $K^{c\Sigma}(A)$ is $\Omega + 1$-iterable, then we can isolate the Σ-hybrid core model over A built in $R^\Sigma(A \oplus A_0)$, $\mathsf{K}^\Sigma(A)^{R^\Sigma(A\oplus A_0)}$. We abbreviate $\mathsf{K}^\Sigma := \mathsf{K}^\Sigma(A)^{R^\Sigma(A\oplus A_0)}$.

We derive a contradiction by running the same arguments as in the inadmissible case: Let δ, δ^+ be weakly compact cardinals larger than $\sup(A)$ but less than κ from Definition 2.49. Then let $f\colon \delta \to \delta^{+\mathsf{K}^\Sigma}$ be a bijection. f is Vopěnka-generic over H_κ^{HOD}, and also over $R^\Sigma(A \oplus A_0)$ as A_0 codes H_κ^{HOD}. By Lemma 2.46 we get a countably complete ultrafilter $\tilde{U}$ on $\mathrm{Pow}(\delta) \cap R^\Sigma(A \oplus A_0)[f]$, and again $R^\Sigma(A \oplus A_0)[f][\tilde{U}]$ is a generic extension of $R^\Sigma(A \oplus A_0)[f]$. So we can build the ultrapower of $R^\Sigma(A \oplus A_0)[f]$ by $\tilde{U}$:

$$\pi\colon R^\Sigma(A \oplus A_0)[f] \to \mathrm{ult}(R^\Sigma(A \oplus A_0)[f], \tilde{U}), \quad cr(\pi) = \delta$$

Moreover, $\pi \restriction V_{\Omega+1}^{R^\Sigma(A\oplus A_0)[f]} \in R^\Sigma(A \oplus A_0)[f][\tilde{U}]$.

By the countable completeness of $\tilde{U}$, the ultrapower is wellfounded and we can identify it with the transitive collapse. Now consider the restriction of π to the core model.

$$\pi\restriction\mathsf{K}^\Sigma\colon \mathsf{K}^\Sigma \to \pi(\mathsf{K}^\Sigma).$$

We will reach a contradiction by showing that K^Σ satisfies weak covering at δ in $R^\Sigma(A \oplus A_0)[f]$ and that the ultrapower map π is continuous at $\delta^{+\mathsf{K}^\Sigma}$, i. e. $\pi(\delta^{+\mathsf{K}^\Sigma}) = \sup(\pi''\delta^{+\mathsf{K}^\Sigma})$.

Claim 2.
1. The hybrid core model K^Σ satisfies weak covering at δ in each inner ZFC model $W \supseteq R^\Sigma(A \oplus A_0)[f]$, i. e. $W \models \mathrm{cof}(\delta^{+\mathsf{K}^\Sigma}) \geq \overline{\overline{\delta}}$.
2. Let $\mathbb{Q} \in V_\Omega^{R^\Sigma(A\oplus A_0)}$ be a forcing notion and $G \in \mathsf{V}$ be $\mathbb{Q}$-generic over $R^\Sigma(A \oplus A_0)$. Suppose there is a $j \in R^\Sigma(A \oplus A_0)[G]$ and a W which is $\Omega + 1$-iterable in $R^\Sigma(A \oplus A_0)[G]$ such that $j\colon \mathsf{K}^\Sigma \to W$ is elementary. Then $j(\gamma) = \sup(j''\gamma)$ for each γ which is regular, but not measurable in K^Σ.

Proof. First note that K^Σ satisfies condensation in $R^\Sigma(A \oplus A_0)$, i. e. whenever we build an elementary substructure of K^Σ and collapse it, we get a hybrid premouse which has the same "Σ-predicate" as K^Σ. This can be used to show that K^Σ satisfies condensation not just in $R^\Sigma(A \oplus A_0)$ but also in V (with a similar proof as for L^Σ).

Now we can show that whenever $\bar{K}$ is elementarily embeddable into K^Σ, then each iterate $\mathcal{M}$ of $\bar{K}$ is also a Σ-mouse. This is because if $\mathcal{M}$ were not a Σ-mouse we could reflect this fact, i. e. there would be π, $\bar{K}'$, and $\mathcal{M}'$ with:

$$\begin{tikzcd}
\bar{K} \arrow[r] & \mathcal{M} \\
\bar{K}' \arrow[u, "\pi"] \arrow[r] & \mathcal{M}' \arrow[u, "\pi"]
\end{tikzcd}$$

But now $\mathcal{M}'$ can be reembedded into $\bar{K}$ and therefore into K^Σ which implies that $\mathcal{M}'$ is a Σ-mouse. The elementarity of π then yields that also $\mathcal{M}$ is a Σ-mouse.

The same argument can be used to show that each "lift-up" of $\bar{K}$ is a Σ-mouse.

So each elementary substructure of K^Σ, each iterate of such a substructure, and each "lift-up" is again a Σ-hybrid mouse. These facts enable us to prove the "weak covering" property of K^Σ in the generic extension $R^\Sigma(A \oplus A_0)[f]$ as in [MSS97]. So we have $R^\Sigma(A \oplus A_0)[f] \models \mathrm{cof}(\delta^{+\mathsf{K}^\Sigma}) \geq \bar{\bar{\delta}}$. But now, since $f\colon \delta \to \delta^{+\mathsf{K}^\Sigma}$ is a bijection there is no $R^\Sigma(A \oplus A_0)[f]$-cardinal in $(\delta, \delta^{+\mathsf{K}^\Sigma}]$; so $R^\Sigma(A \oplus A_0)[f] \models \mathrm{cof}(\delta^{+\mathsf{K}^\Sigma}) = \bar{\bar{\delta}}$. δ is regular in V and therefore in inner any model W, so if W is as in 1 then we have $W \models \mathrm{cof}(\delta^{+\mathsf{K}^\Sigma}) \geq \bar{\bar{\delta}}$.

Moreover, if $G \in \mathsf{V}$ is $\mathbb{Q}$-generic over $R^\Sigma(A \oplus A_0)$ then one can see that K^Σ is still a universal weasel in the generic extension $R^\Sigma(A \oplus A_0)[G]$. We also need that classes which are thick in K^Σ are still thick in K^Σ if we consider "thickness" in the sense of $R^\Sigma(A \oplus A_0)[G]$. Furthermore we need that for each Γ which is thick in K^Σ in the sense of $R^\Sigma(A \oplus A_0)[G]$ there is a $\Gamma' \subseteq \Gamma$, $\Gamma' \in R^\Sigma(A \oplus A_0)$ which is thick in K^Σ in the sense of $R^\Sigma(A \oplus A_0)$.

The second statement of the claim can now be proved as in [Ste96, Theorem 8.14(3)]. $\square$(*Claim* 2)

Now we can argue exactly as in the inadmissible case, since the "weak covering at δ" property is carried to the ultrapower by π. $\square$

Lemma 3.52. *$(W^\star_{\alpha+1})$ holds. Moreover, if $\mathcal{P}$ is the mouse witnessing $(W^\star_{\alpha+1})$ with respect to a $\Sigma_n(J_\alpha(\mathbb{R}^g))$-set of reals, then $\mathcal{P}$ is closed under $C_{\Sigma_n(J_\alpha(\mathbb{R}^g))}$.*

Proof. Let $U \subseteq \mathbb{R}^g$ be in $J_{\alpha+1}(\mathbb{R}^g)$ and $k < \omega$. We are searching for a coarse (k, U)-Woodin mouse. Suppose U is Σ_n-definable in the real parameter z. We can assume that $z = \rho^{g\restriction\bar{\iota}}$ for some $\bar{\iota} < \mu_+$. Set $\bar{g} := g\restriction\bar{\iota}$ and

$$\mathcal{P} = P^{\Sigma}_{k+n}{}^{\sharp}(\langle \mathcal{N}, \rho\rangle).$$

First note that as in the inadmissible case we have the $\mathcal{Q}$-structures for determining the canonical ω_1-iteration strategy Γ of $\mathcal{P}[\bar{g}]$ in $\mathsf{HOD}_X[g]$. Let $\mathcal{A} = (A_i : i < \omega)$ be the self-justifying system of sets which are ordinal definable over $J_\gamma(\mathbb{R}^g)$ for some $\gamma < \alpha$ from the parameter $(\tau, \bar{g})$; cf. page 68.

Suppose j is least such that $\rho_j(J_\alpha(\mathbb{R}^g)) = \mathbb{R}^g$. Then by construction of $\mathcal{A}$ it codes a universal $\underset{\sim}{\Sigma}_j(J_\alpha(\mathbb{R}^g))$-set. Since there is a $\underset{\sim}{\Sigma}_j(J_\alpha(\mathbb{R}^g))$-surjection $f \colon \mathbb{R}^g \to J_\alpha(\mathbb{R}^g)$ and $\underset{\sim}{\Sigma}_{j+l+1}(J_\beta(\mathbb{R})) \cap \mathrm{Pow}(\mathbb{R}) = \exists^{\mathbb{R}}(\underset{\sim}{\Pi}_{j+l}(J_\beta(\mathbb{R}))) \cap \mathrm{Pow}(\mathbb{R})$ for each $k < \omega$ (cf. proof of Lemma 2.40), Σ_j-truth at level α is therefore coded into

$$W := \bigcup \{\langle i\rangle^\frown x : x \in A_i\}.$$

Claim 1. For any $\nu < ht(\mathcal{P})$ there is a term $\dot{W}_\nu \in \mathcal{P}[\bar{g}]^{\mathrm{Col}(\omega,\nu)}$ capturing W, i. e. whenever $i \colon \mathcal{P}[\bar{g}] \to \mathcal{R}[\bar{g}]$ is a simple iteration map by Γ and h is $\mathrm{Col}(\omega, i(\nu))$-generic over $\mathcal{R}[\bar{g}]$, then

$$i(\dot{W}_\nu)^h = W \cap \mathcal{R}[\bar{g}][h].$$

Proof. Basically, $\dot{W}_\nu$ asks what the $\tau^{\mathcal{N}}_{A_i}$ [35] are moved to in the iteration of $\mathcal{N}$ which makes $\mathcal{P}|\nu^+$ generic over the extender algebra of the iterate. This iteration is done inside of $\mathcal{P}$, using what it knows of Σ.

Let $(\rho, p) \in \dot{W}_\nu$ iff $\rho \in \mathcal{P}[\bar{g}]^{\mathrm{Col}(\omega,\nu)}$ is a name for a real $y = \langle j\rangle^\frown x$ and $p \in \mathrm{Col}(\omega, \nu)$ is such that for all $\mathrm{Col}(\omega, \nu)$-generic h:

> $\mathcal{P}[\bar{g}][h] \models$ "There is a countable simple iterate $\mathcal{N}'$ of $\mathcal{N}$ according to Σ with iteration map e such that x is $\mathbb{E}^{\mathcal{N}'}_{e(\delta^{\mathcal{N}})}$-generic[36] over $\mathcal{N}'$,

[35] $\tau^{\mathcal{N}}_{A_i} \in \mathcal{N}^{\mathrm{Col}(\omega,\delta^{\mathcal{N}})}$ is the unique standard term which weakly captures A_i over $\mathcal{N}$; see [Sted, Definition 2.4].

[36] Recall that $\mathbb{E}^{\mathcal{N}'}_{e(\delta^{\mathcal{N}})}$ is the extender algebra of $\mathcal{N}'$ at $e(\delta^{\mathcal{N}})$.

and there is a $\mathrm{Col}(\omega, e(\delta^{\mathcal{N}}))$-generic h' over $\mathcal{N}'$ with $x \in \mathcal{N}'[h']$ and $x \in e(\tau^{\mathcal{N}}_{A_j})^{h'}$."

So let $i\colon \mathcal{P}[\bar{g}] \to \mathcal{R}[\bar{g}]$ be a simple iteration map according to Γ and let h be $\mathrm{Col}(\omega, i(\nu))$-generic over $\mathcal{R}[\bar{g}]$. We show $i(\dot{W}_\nu)^h = W \cap \mathcal{R}[\bar{g}][h]$.

If $y \in W \cap \mathcal{R}[\bar{g}][h]$, then $y = \langle j \rangle^\frown x$ for some $j < \omega$, $x \in A_j$. Since x is a real we can use Woodin's genericity theorem to make x generic for the extender algebra at $e(\delta^{\mathcal{N}})$ over a simple countable iterate $\mathcal{N}'$ of $\mathcal{N}$, where e is the corresponding iteration map. Now let h' be $\mathrm{Col}(\omega, e(\delta^{\mathcal{N}}))$-generic over $\mathcal{N}'$ such that $x \in \mathcal{N}'[h']$. Such an h' exists since $\mathcal{N}'[x]$ is an $e(\delta^{\mathcal{N}})$-c. c. generic extension of $\mathcal{N}'$. But now $e(\tau^{\mathcal{N}}_{A_j})$ weakly captures A_j over $\mathcal{N}'$[37], so $x \in e(\tau^{\mathcal{N}}_{A_j})^{h'}$. Hence by definition of $\dot{W}_\nu$ we have $\rho^h \in i(\dot{W}_\nu)^h$.

For the other direction let $y \in i(\dot{W}_\nu)^h$. Then $y = \langle j \rangle^\frown x$ and there is in $\mathcal{R}[\bar{g}][h]$ a countable simple iterate $\mathcal{N}'$ of $\mathcal{N}$ with iteration map e, and a $\mathrm{Col}(\omega, e(\delta^{\mathcal{N}'}))$-generic filter h' over $\mathcal{N}'$ with $x \in \mathcal{N}'[h']$ and $x \in e(\tau^{\mathcal{N}}_{A_j})^{h'}$. But this implies $x \in A_j$, so $y \in W$. $\square$(*Claim* 1)

Claim 2. Let δ be the k^{th} Woodin cardinal of $\mathcal{P}$. Then for any $\Sigma_n(J_\alpha(\mathbb{R}^g), z)$-set Y, there is a term $\dot{Y} \in \mathcal{P}[\bar{g}]$ such that for each simple iteration map $i\colon \mathcal{P}[\bar{g}] \to \mathcal{R}[\bar{g}]$ according to Γ and for each $\mathrm{Col}(\omega, i(\delta))$-generic filter h over $\mathcal{R}[\bar{g}]$ we have

$$i(\dot{Y})^h = Y \cap \mathcal{R}[\bar{g}][h].$$

Proof. $\dot{Y}$ is constructed from the term $\dot{W}_\nu$ given by the previous claim, where ν is the $k+n^{\text{th}}$ Woodin cardinal of $\mathcal{P}$. The term $\dot{W}_\nu$ captures the set W which codes a universal $\undertilde{\Sigma}_j(J_\alpha(\mathbb{R}^g))$-set. Now we can use Lemma 2.40 to construct the desired term $\dot{Y}$. $\square$(*Claim* 2)

Now one can see that $\mathcal{P}[\bar{g}]$ is the desired coarse witness. The trees in $\mathcal{P}[\bar{g}]$, which are moved appropriately by Γ, are obtained just as in the inadmissible case. $\square$

[37] Here we use the weak $\mathcal{A}$-iterability of $\mathcal{N}$.

4. EVERY UNCOUNTABLE CARDINAL IS SINGULAR

This chapter is about the application of the core model induction to the hypothesis "ZF + each uncountable cardinal is singular".

We briefly recall Definition 2.49.

1. ε is such that for each set of ordinals X the following holds: Each subset of ω_1 is Vopěnka-generic over $H_\varepsilon^{\mathsf{HOD}_X}$, and each $\omega_1 + 1$-iteration strategy in HOD_X for any countable premouse is already in $H_\varepsilon^{\mathsf{HOD}_X}$.
2. ζ is such that for each set of ordinals X the following holds: If G is a $\mathrm{Col}(\omega, \varepsilon)$-generic object over HOD_X, then G is already generic over $H_\zeta^{\mathsf{HOD}_X}$. Moreover, $H_\zeta^{\mathsf{HOD}_X}[G]$ contains every $\omega_1^{\mathsf{HOD}_X[G]} + 1$-iteration strategy for any countable mouse which is in $\mathsf{HOD}_X[G]$.
3. κ is a Θ-closed cardinal such that for each X and each $G \subseteq \mathsf{On}, G \in \mathsf{V}$, there are cofinally many Θ-closed $\mu < \kappa$ with $\mathsf{HOD}_X[G] \models \mu^\varepsilon = \mu$.

So we can define:

4. $A_0 \subseteq \kappa, A_0 \in \mathsf{HOD}$ codes H_κ^{HOD} in some simple way,
5. $\lambda = \kappa^{+\mathsf{Lp}(A_0)}$,
6. $X \subseteq \lambda$ is cofinal of order type ω, and
7. $\mu < \kappa$ is a Θ-closed cardinal such that $\mathsf{HOD}_X \models \mu^\varepsilon = \mu$.

Theorem 1.5. *Let* V *be a model of* ZF *in which each uncountable cardinal is singular.*

Then $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ *holds in* $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu^{+\mathsf{HOD}_X})}$.

For the rest of this chapter we fix a $\mathrm{Col}(\omega, < \mu^{+\mathsf{HOD}_X})$-generic object g over V and abbreviate $\mathbb{R}^g := \mathbb{R}^{\mathsf{HOD}_X[g]}$. Note that $\mathbb{R}^g = \bigcup_{\iota < \mu^{+\mathsf{HOD}_X}} \mathbb{R}^{\mathsf{HOD}_X[g \restriction \iota]}$. Moreover we set $\mu_+ := \mu^{+\mathsf{HOD}_X}$.

4.1 The projective case

Again we start with the projective case, i. e. we show that $M_n^\sharp(A)$ exists for each $n < \omega$ and each $A \subseteq \mathsf{On}$. For this we use

Lemma 4.1. *Let $n < \omega$ and suppose B is a set of ordinals such that HOD_B is closed under $M_{n-1}^\sharp$, but $M_n^\sharp(A)$ does not exist for some $A \in \mathsf{HOD}_B$. Let $Z \subseteq \mathsf{On}$ be* Vop*-generic over HOD_B.*

Then $\mathsf{Lp}^{\mathsf{HOD}_B}(A)$ is fully iterable in $\mathsf{HOD}_B[Z]$.

Proof. We can suppose $B = \emptyset$.

The proof is essentially the same as the proof of $(1)_n$ in Lemma 3.7. We show that $\mathsf{Lp}^{\mathsf{HOD}}(A)$ is fully iterable via the $\mathcal{Q}$-structure iteration strategy. So suppose not and let $\mathcal{T}$ be a counterexample of minimal length. Then $\mathcal{T}$ is a tree on $\mathcal{M} := \mathsf{Lp}^{\mathsf{HOD}}(A)|\alpha$ for some α. Such an α exists because $\mathsf{Lp}^{\mathsf{HOD}}(A)$ is a lower part model. Then there is a $p \in \mathrm{Vop}$ which forces this property over HOD:

$$p \Vdash^{\mathrm{Vop}}_{\mathsf{HOD}} \check{\mathcal{M}} \text{ is not iterable, witnessed by } \dot{\mathcal{T}}$$

Working in HOD, let Ω be large and $\pi \colon H \rightarrow V_\Omega$ be elementarily such that H is transitive and countable. Let $\mathcal{N}, \dot{\mathcal{S}}, \mathrm{Vop}', p'$ be the images of $\mathcal{M}, \dot{\mathcal{T}}, \mathrm{Vop}, p$ under π^{-1}.

Then $H \models$ "$p' \Vdash \check{\mathcal{N}}$ is not iterable, witnessed by $\dot{\mathcal{S}}$". Let $G \in \mathsf{HOD}$ be Vop'-generic over H and containing p', so that

$$H[G] \models \mathcal{N} \text{ is not iterable, witnessed by } \dot{\mathcal{S}}^G$$

But $H[G]$ contains the necessary $\mathcal{Q}$-structures[1] to argue as in the proof of Lemma 3.6 that $\dot{\mathcal{S}}^G$ has a cofinal branch b such that an initial segment of $\mathcal{N}_b^{\dot{\mathcal{S}}^G}$ is a $\mathcal{Q}$-structure. So we get a contradiction. □

[1] The $\mathcal{Q}$-structure for an iteration tree $\mathcal{T}$ is given by an initial segment of $M_{n-1}^\sharp(\mathcal{M}(\mathcal{T}))$.

Now we can prove PD in $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu_+)}$.

Lemma 4.2. *Suppose that each uncountable cardinal is singular.*

Then $M_n^\sharp(A)$ exists for each set of ordinals A.

Proof. First let $n = 0$. We will show that $A^\sharp$ exists for every set of ordinals $A \in \mathsf{HOD}$. If we have shown that $A^\sharp$ exists in HOD then we can use Lemma 3.8 to get that $A^\sharp$ exists for all $A \subseteq \mathsf{On}, A \in \mathsf{V}$.

So let $\theta > \sup(A)$ be a cardinal, $\nu := (\theta^+)^{\mathsf{L}[A]}$, and $A \subseteq \nu, A \in \mathsf{V}$ cofinal of order type ω. A is generic over HOD, so as ν has cofinality ω in $\mathsf{HOD}[A]$

$$\mathsf{HOD}[A] \models \theta^+ > \nu = (\theta^+)^{\mathsf{L}[A]}$$

holds true, and therefore $\mathsf{HOD}[A] \models$ "$A^\sharp$ exists". But this is the real $A^\sharp \in \mathsf{V}$. Moreover, $A^\sharp$ exists in HOD, because it is hereditarily ordinal definable from $A \in \mathsf{HOD}$.

Now let $n \geq 1$. Suppose that $M_{n-1}^\sharp(A)$ exists for all sets of ordinals A in V. HOD is closed under $M_{n-1}^\sharp$ by Lemma 3.8. We will show that $M_n^\sharp(A)$ exists for all sets of ordinals $A \subseteq \mathsf{On}$, by proving that HOD is closed under $M_n^\sharp$. So let $A \in \mathsf{HOD}$ and suppose $M_n^\sharp(A)$ does not exist in HOD.

Let $\eta > \sup(A)$ be a Θ-closed cardinal and set $H := H_\eta^{\mathsf{HOD}}$. Suppose that $\theta > \eta$ is countably closed in any inner model of ZFC[2] and $A \subseteq \nu := \theta^{+\mathsf{Lp}^{\mathsf{HOD}}(H)}$, $A \in \mathsf{V}$ is cofinal in ν of order type ω. $\mathsf{Lp}^{\mathsf{HOD}}(H)$ is iterable in $\mathsf{HOD}[A]$ by the previous lemma. Now we can use [SW01, Theorem 2.4] in $\mathsf{HOD}[A]$ which produces an $\Omega \geq \eta$ and a countably closed ultrafilter U on $\mathrm{Pow}(\Omega) \cap \mathsf{Lp}^{\mathsf{HOD}}(H)$ such that U is weakly amenable to $\mathsf{Lp}^{\mathsf{HOD}}(H)$. Consider the structure

$$\mathcal{U} := (\mathsf{Lp}^{\mathsf{HOD}}(H) | \Omega^{+\mathsf{Lp}^{\mathsf{HOD}}(H)}; \in, U)$$

Under the assumption that $M_n^\sharp(A)$ does not exist, we can now use [FMS01, Lemma 2.3] inside $\mathcal{U}$. So $K^c(A)^{\mathcal{U}}$ up to Ω exists and is $\Omega + 1$-iterable in $\mathcal{U}$.

Now we can isolate the true core model $\mathsf{K}(A)^{\mathcal{U}}$ below Ω. But the core model is invariant under forcing so that $\mathsf{K}(A)^{\mathcal{U}} = \mathsf{K}(A)^{\mathcal{U}[G]}$ for all G which are $\mathbb{P}$-generic over $\mathcal{U}$ for some forcing $\mathbb{P} \in V_\Omega^{\mathcal{U}}$.

[2] I. e. if $\gamma < \theta$ then also $\gamma^\omega < \theta$ holds. For example, every Θ-closed cardinal is countably closed in each inner model of ZFC.

Let $\gamma < \eta$ be a cardinal and $G \in \mathsf{V}$ a witness for the singularity of both γ and $\gamma^{+\mathsf{K}^{\mathcal{U}}}$. Then G is Vopěnka-generic over H and therefore over $\mathcal{U}$.

$$\operatorname{cof}(\gamma^{+\mathsf{K}(A)^{\mathcal{U}[G]}}) = \operatorname{cof}(\gamma^{+\mathsf{K}(A)^{\mathcal{U}}}) = \omega < \overline{\overline{\gamma}}$$

holds in $\mathcal{U}[G]$, which contradicts the "weak covering" property for $\mathsf{K}(A)$ in $\mathcal{U}[G]$.

So $M_n^\sharp(A)^{\mathcal{U}}$ exists. But $M_n^\sharp(A)^{\mathcal{U}} = M_n^\sharp(A)^{\mathsf{HOD}}$, since every countable substructure of $M_n^\sharp(A)^{\mathcal{U}}$ and any putative iteration tree we have to consider is an element of $H_\eta^{\mathsf{HOD}} = H \subseteq \mathcal{U}$. So $M_n^\sharp(A)^{\mathcal{U}}$ is a mouse also in HOD and therefore equal to $M_n^\sharp(A)^{\mathsf{HOD}}$.

Thus $\mathsf{HOD} \models$ "$M_n^\sharp(A)$ exists for each A" and hence V is closed under $M_n^\sharp$ for all sets of ordinals A by Lemma 3.8. □

But now we can use Lemmata 3.7 and 3.8, to show that the closure under $M_n^\sharp$ of V carries over to $\mathsf{HOD}_X^{\mathrm{Col}(\omega, <\mu_+)}$.

4.2 The inadmissible cases

Now suppose α is a critical, $\mathbb{R}$-inadmissible ordinals which begins a Σ_1-gap. As in the weakly compact case we distinguish between three subcases:

1. α is the successor of a critical ordinal, or
2. α is a limit of countable cofinality, or
3. α has uncountable cofinality.

The uncountable-cofinality case

Suppose α is a limit ordinal which is $\mathbb{R}$-inadmissible, begins a Σ_1-gap, and has uncountable cofinality. The main framework can be taken from Section 3.2.

Let again $\varphi(v_0, v_1) \in \Sigma_1$ and $x \in \mathbb{R}^g$ determine a failure of admissibility. Since $\mathbb{R}^g = \bigcup_{\iota<\mu_+} \mathbb{R}^{\mathsf{HOD}_X[g\restriction\iota]}$, there is a τ and a ι such that $x = \tau^{g\restriction\iota}$, and some $p_0 \in g\restriction\iota$ which forces the properties listed so far. So we have

$$\forall y \in \mathbb{R}^g \; \exists \gamma < \alpha \; J_\gamma(\mathbb{R}^g) \models \varphi(x, y)$$

and φ is true cofinally often. Moreover, for each h which is $\mathrm{Col}(\omega, < \iota)$-generic over HOD_X and contains p_0 we have

$$\mathsf{HOD}_X[h] \models \exists q \in \mathrm{Col}(\omega, < \mu_+)\ q \Vdash \forall y \in \mathbb{R}\ J_{\check{\alpha}}(\mathbb{R}) \models \varphi(\check{\tau^h}, y).$$

In HOD_X, let $A \in \mathrm{Pow}(< \mu^{+\mathsf{HOD}_X})$ code

$$\mathfrak{c} := \tau \oplus H_\zeta^{\mathsf{HOD}_X}$$

in a simple fashion. Again σ_A is a forcing term such that whenever $G \times H$ is $\mathrm{Col}(\omega, < \iota) \times \mathrm{Col}(\omega, A)$-generic over HOD_X, then

1. $\sigma_A^{G\times H} \in \mathbb{R}^G$
2. $\left(\sigma_A^{G\times H}\right)_0 = \tau^G$
3. $\left\{\left(\sigma_A^{G\times H}\right)_i : i \in \omega\right\} = \{\rho^{G\times H} : \rho \text{ is simply coded into } A, \rho^{G\times H} \in \mathbb{R}^G\}$

For $n < \omega$ let φ_n be the Σ_1-formula

$$\varphi_n(v) \equiv \exists\gamma\big(\gamma + \omega n \text{ exists} \wedge J_\gamma(\mathbb{R}) \models \forall i > 0\ \varphi((v)_0, (v)_i)\big),$$

and let ψ be the natural sentence, such that for any A-premice $\mathcal{M}$

> $\mathcal{M} \models \psi$ iff whenever $G \times H$ is $\mathrm{Col}(\omega, < \iota) \times \mathrm{Col}(\omega, A)$-generic over $\mathcal{M}$ with $p_0 \in G$, then for all n there is a strictly increasing sequence $(\gamma_i : i \leq n)$ such that for all $i \in [1, n)$
>
> 1. $\mathcal{M}[G \times H] \| \gamma_i$ is a $\langle \varphi_{i+1}, \sigma_A^{G\times H}\rangle$-pre-witness and
> 2. there is a $\delta \in (\gamma_i, \gamma_{i+1}]$ such that $\rho_\omega(\mathcal{M}[G\times H]\|\delta) = \sup(A)$.

Definition 4.3. For any set of ordinals A coding $\mathfrak{c}$, let $\mathcal{M}(A)$ be the shortest initial segment of $\mathsf{Lp}(A)$ which satisfies ψ, if it exists, and let $\mathcal{M}(A)$ be undefined otherwise.

If $\mathcal{M}(A)$ exists it is countably iterable in V and therefore each pre-witness is inherently a $\langle \varphi_n, \sigma_A^{G\times H}\rangle$-witness.

As in the weakly compact case we will define $\mathcal{M}(A)$ for more than only those $A \in \mathrm{Pow}(< \mu_+)$ which code $\mathfrak{c}$. We will show that $\mathcal{M}(A)$ exists for all sets of ordinals A coding $\mathfrak{c}$. This again works in four steps.

1. First we show that $\mathcal{M}(A)$ exists for all bounded subsets of μ_+ which are in HOD_X and code $\mathfrak{c}$.
2. By a lift-up argument and the fact that A_0 codes H_κ^{HOD} we can show that $\mathcal{M}(A \oplus \mathfrak{c})$ is defined for all $A \in \mathrm{Pow}(< \kappa) \cap \mathsf{HOD}$.
3. Then one can see that $\mathcal{M}(A)$ exists for all bounded subsets of κ in V which code $\mathfrak{c}$.
4. Finally we can use the lift-up argument once again to show that $\mathcal{M}(A)$ is defined for all sets of ordinals A coding $\mathfrak{c}$.

Lemma 4.4. *$\mathcal{M}(A)$ exists for any $A \in \mathrm{Pow}(< \mu_+) \cap \mathsf{HOD}_X$ which simply codes $\mathfrak{c}$.*

The method for proving this lemma is exactly that of Lemma 3.10. First we can show

Lemma 4.5. *$\mathcal{M}(A)^*$ exists for any $A \in \mathrm{Pow}(< \mu_+) \cap \mathsf{HOD}_X$ which simply codes $\mathfrak{c}$ and is countably iterable in HOD_X.*

Remark. Here $\mathcal{M}(A)^*$ will be the desired $\mathcal{M}(A)$. The reason why we use * is that we don't know until now that $\mathcal{M}(A)^*$ is countably iterable in whole V.

For the countable iterability of $\mathcal{M}(A)^*$ in V we use the following lemmata which are the equivalents to Lemmata 3.12 and 3.13:

Lemma 4.6. *For all $A \in \mathrm{Pow}(< \mu_+) \cap \mathsf{HOD}_X$ which simply code $\mathfrak{c}$ and all γ such that $\rho_\omega(\mathcal{M}(A)^* \| \gamma) \leq \sup(A)$, we get*

$$\Vdash_{\mathcal{M}(A)^* \| \delta_\gamma}^{\mathrm{Col}(\omega, \varepsilon)} (\mathcal{M}(A)^* \| \gamma)\check{} \text{ is countably iterable}$$

Lemma 4.7. *Let A simply code $\mathfrak{c}$ such that $\mathcal{M}(A)^*$ exists and suppose that for all γ with $\rho_\omega(\mathcal{M}(A)^* \| \gamma) \leq \sup(A)$, we have "$\Vdash_{\mathcal{M}(A)^* \| \delta_\gamma}^{\mathrm{Col}(\omega, \varepsilon)} (\mathcal{M}(A)^* \| \gamma)\check{}$ is countably iterable".*

Then $\mathcal{M}(A)^*$ is countably iterable in V as well as in any inner model $W \supseteq \mathsf{HOD}_X$.

So $\mathcal{M}(A)^*$ is countably iterable not only in HOD_X but also in V and therefore $\mathcal{M}(A) = \mathcal{M}(A)^*$ exists, proving Lemma 4.4.

Again we want to use a reflection argument, so we need that the $\mathcal{M}$-operator behaves correctly.

Definition 4.8. An operator $\mathcal{O}$ *relativizes well at* μ iff there is a formula $\Phi(v_0, v_1, v_2)$ such that whenever $\overline{\overline{A}} = \mu$, A is coded into some B with $\overline{\overline{B}} = \mu$, $\mathcal{O}(A)$ exists, and W is a transitive model of ZFC^- such that $\mathcal{O}(B) \in W$, then $\mathcal{O}(A)$ is the unique $x \in W$ such that $W \models \Phi(x, A, \mathcal{O}(B))$.

Lemma 4.9. *The $\mathcal{M}$-operator relativizes well at μ in HOD_X.*

The next lemmata are as before.

Lemma 4.10. *$\mathcal{M}(A \oplus \mathfrak{c})$ exists for all $A \in \mathrm{Pow}(< \kappa) \cap \mathsf{HOD}$.*

Lemma 4.11. *$\mathcal{M}(A)$ exists for all $A \in \mathrm{Pow}(< \kappa) \cap \mathsf{V}$ which simply code $\mathfrak{c}$.*

Lemma 4.12. *$\mathcal{M}(A)$ exists for all sets of ordinals A which simply code $\mathfrak{c}$.*

Remark. This lemma works for every set of ordinals, not only for those which are bounded in κ^+.

Proof sketch. Consider $\mathsf{Lp}(A)$. Set $\lambda_A := \sup(A)^{+\mathsf{Lp}(A)}$ and let $Y \subseteq \lambda_A$ be cofinal of order type ω.[3] Now we can build a substructure of a large initial segment of $\mathsf{HOD}_{A,Y}$ which is cofinal in λ_A and continue as in Lemma 3.18. $\square$

Now we are ready to define the mice which we need to prove $(W^{\star}_{\alpha+1})$.

Definition 4.13. For any n and any set of ordinals A which simply codes $\mathfrak{c}$, let $P^{\sharp}_n(A)$ be the least active countably iterable $\mathcal{M}$-closed A-premouse which has n Woodin cardinals, and let $P^{\sharp\sharp}_n(A)$ be the least active countably iterable A-premouse which is $P^{\sharp}_n$-closed. If such a premouse does not exist, this is undefined.

Lemma 4.14. *$P^{\sharp}_n(A)$ exists for each n and all sets of ordinals A which simply code $\mathfrak{c}$.*

[3] This is the difference to the case that each uncountable successor cardinal is weakly compact. In that case we restricted ourself to $\lambda_A < \kappa^+$, since we needed that the substructure we built has size $< \kappa$ and is cofinal in λ_A. So we had to know that $\mathrm{cof}(\lambda_A) < \kappa$, which is true if $\lambda_A < \kappa^+$.

Proof. Let $n = 0$. As before we build the minimal $\mathcal{M}$-closed model. For this let

$$\begin{aligned} \mathcal{N}_0 &:= J_1(A), \\ \mathcal{N}_{\gamma+1} &:= \mathcal{M}(\mathcal{N}_\gamma), \\ \mathcal{N}_\lambda &:= \bigvee_{\alpha<\lambda} \mathcal{N}_\alpha \text{ for } \lambda \text{ limit.} \end{aligned}$$

Now set $L^{\mathcal{M}}(A) := \bigvee_{\alpha\in\mathsf{On}} \mathcal{N}_\alpha$. Since the $\mathcal{M}$-operator condenses to itself, one can adapt the proof for L to see that $L^{\mathcal{M}}(A)$ is a fine structural model such that each substructure containing $A \cup \{A\}$ as a subset condenses to an initial segment of $L^{\mathcal{M}}(A)$. Of course we have $L^{\mathcal{M}}(A) \subseteq \mathsf{HOD}_A$.

Now let ν be a Θ-closed cardinal $> \sup(A)$. Suppose $Z \in \mathsf{V}$ is a set of ordinals of size ω which is cofinal in both ν and $\nu^{+L^{\mathcal{M}}(A)}$. Then we have that ν is a countably closed cardinal in $\mathsf{HOD}_A[Z]$ with $\nu^{+L^{\mathcal{M}}(A)} < \nu^{+\mathsf{HOD}_A[Z]}$.

But then the existence of $P_0^\sharp(A)$ follows from the usual covering argument as for example in Claim 1 in the proof of Lemma 3.21.

Let $n = 1$. A similar argument as above shows that for all sets of ordinals A which code $\mathfrak{c}$, $P_0^{\sharp\sharp}(A)$ exists. Let

$R(A \oplus A_0)$ be the minimal $P_0^\sharp$-closed model over $A \oplus A_0$

and

Ω the first indiscernible for $R(A \oplus A_0)$.

Now in $R(A\oplus A_0)$ we build K^c over A below Ω via the construction in [Ste96], with result $K^c(A)$. We use the following claim, which one can prove as in [Ste05, Lemma 1.33].

Claim 1. In $R(A \oplus A_0)$ either $P_1^\sharp(A)$ exists or $K^c(A)$ is $\Omega + 1$-iterable.

If $P_1^\sharp(A)$ does not exist, we can isolate $\tilde{\mathsf{K}} := \mathsf{K}(A)^{R(A\oplus A_0)}$, the true core model over A built in $R(A \oplus A_0)$. But the core model is invariant under forcing which yields $\mathsf{K}(A)^{R(A\oplus A_0)[G]} = \tilde{\mathsf{K}}$ whenever G is a $\mathbb{P}$-generic filter over $R(A \oplus A_0)$ for some $\mathbb{P} \in V_\Omega^{R(A\oplus A_0)}$.

Let $\gamma < \zeta$ be a cardinal and Z be a set of ordinals which is a witness for the singularity of γ as well as the singularity of $\gamma^{+\tilde{\mathsf{K}}}$. Then Z is Vopěnka-generic over $H_\zeta^{\mathsf{HOD}_x}$ and therefore also over $R(A \oplus A_0)$.[4]

But

$$R(A \oplus A_0)[Z] \models \operatorname{cof}(\gamma^{+\mathsf{K}(A)}) = \operatorname{cof}(\gamma^{+\tilde{\mathsf{K}}}) = \omega < \overline{\overline{\gamma}}$$

contradicts weak covering.

So $P_1^\sharp(A)$ exists.

For $n > 1$ the proof is essentially the same. The additional ingredient is that the $P_n^\sharp$-operator relativizes well if $P_n^\sharp(A)$ exists for all sets of ordinals A. $\square$

Exactly as before we get

Lemma 4.15. $(W_{\alpha+1}^\star)$ *holds. Moreover, if* $\mathcal{P}$ *is the mouse witnessing* $(W_{\alpha+1}^\star)$ *with respect to a* $\Sigma_n(J_\alpha(\mathbb{R}^g))$*-set of reals, then* $\mathcal{P}$ *is closed under* $C_{\Sigma_n(J_\alpha(\mathbb{R}^g))}$.

The successor-of-a-critical and countable-cofinality case

Both these cases are exactly the same as in the case that every uncountable cardinal successor is weakly compact and every uncountable limit cardinal is singular.

4.3 The end-of-gap cases

For technical reasons we specify our choice of the generic object g. As usual one has a dense embedding e from a dense subset $D \subseteq \operatorname{Col}(\omega, <\mu_+)$ into the partial order $\operatorname{Col}(\omega, \omega_1) \times \operatorname{Col}(\omega, <\mu_+)$. This embedding defines a one-to-one correspondence between the $\operatorname{Col}(\omega, \omega_1) \times \operatorname{Col}(\omega, <\mu_+)$-generic filters over V and the $\operatorname{Col}(\omega, <\mu_+)$-generic filters over V such that the appropriate generic extensions are the same.

[4] Note that A_0 codes $H_\zeta^{\mathsf{HOD}_x}$ (and more).

Definition 4.16. Fix a $\mathrm{Col}(\omega, \omega_1)$-generic h over V and a $\mathrm{Col}(\omega, < \mu_+)$-generic k over $\mathsf{V}[h]$ such that $\mathsf{V}[h][k] = \mathsf{V}[g]$. Since e exists in HOD_X also $\mathsf{HOD}_X[h][k] = \mathsf{HOD}_X[g]$ holds true.

For this section we can use most parts of Section 3.3.

Lemma 4.17. *There is in HOD_X a suitable premouse $\mathcal{N}$ and a fullness-preserving μ_+-iteration strategy Σ which condenses well. Moreover:*

$$\Vdash^{\mathrm{Col}(\omega, < \mu_+)}_{\mathsf{HOD}_X} \begin{array}{l} \check{\mathcal{N}} \textit{ is } \omega_1\textit{-iterable, witnessed by the unique fullness-} \\ \textit{preserving iteration strategy } \dot{\Sigma}_0\textit{, which condenses well.} \end{array}$$

Remark. In contrast to the case where we work in $\mathsf{HOD}_X^{\mathrm{Col}(\omega, < \mu^{+\mathsf{V}})}$ we can now determine the size of $\mathcal{N}$. Since $\mathcal{N}$ has size ω_1 in the forcing extension, we could in the former case only say that $\mathcal{N}$ has HOD_X-size less than $\mu^{+\mathsf{V}}$. But since there are many HOD-cardinals in $[\mu, \mu^{+\mathsf{V}})$ we could not determine its exact size. In this case if we force with $\mathrm{Col}(\omega, < \mu^{+\mathsf{HOD}_X})$ then $\mathcal{N}$ has exactly size μ in HOD_X.

First we want to prove an analogous lemma to Lemma 3.37. What we prove is in fact that not only small premice in V which are elementarily embeddable into $\mathcal{N}$ are iterable, but also small premice in $\mathsf{V}^{\mathrm{Col}(\omega, \omega_1)}$.

For this we need the following lemma.

Lemma 4.18. *Let h' be $\mathrm{Col}(\omega, \omega_1)$-generic over V. Then for each set of ordinals $A \in \mathsf{V}[h']$ there is a set of ordinals $Z \in \mathsf{V}$ with $A \in \mathsf{HOD}_X[Z][h']$.*

Proof. Let $\dot{A}$ be a nice name for A. Since $\mathrm{Col}(\omega, \omega_1)$ is easily wellorderable in V we can find a set of ordinals $Z \in \mathsf{V}$ which codes $\dot{A}$. So $\dot{A} \in \mathsf{HOD}_X[Z]$ and therefore $A = \dot{A}^{h'} \in \mathsf{HOD}_X[Z][h']$. $\square$

Definition 4.19. From now on HOD_X stands for HOD_X built in V, even if we work in some generic extension of V.

Lemma 4.20. *Let h be the $\mathrm{Col}(\omega, \omega_1)$-generic filter over V defined in 4.16. Suppose $\mathcal{M} \in \mathsf{V}[h]$ is a premouse such that*

$$\mathsf{V}[h] \models \exists \pi\ \pi\colon \mathcal{M} \to \mathcal{N} \textit{ elementary, } \overline{\overline{\mathcal{M}}} \leq \bar{\mu} < \mu$$

Then there is a $(\bar{\mu}^+)^{\mathsf{V}[h]}$*-iteration strategy* $\Sigma_{\mathcal{M}}$ *for short trees in* $\mathsf{V}[h]$ *which condenses well. If* $A \subseteq \mathsf{On}, A \in \mathsf{V}$ *is such that* $\mathcal{M} \in \mathsf{HOD}_X[A][h]$*, then* $\Sigma_{\mathcal{M}} \restriction \mathsf{HOD}_X[A][h] \in \mathsf{HOD}_X[A][h]$.

Moreover, in each $\mathsf{HOD}_X[A][h]$ *which contains* $\mathcal{M}$*, there is an extension of* $\Sigma_{\mathcal{M}}$ *which also works for maximal trees and condenses well. Denote this extension by* $\Sigma_{\mathcal{M},A}$.

Proof. The proof is almost the same as that of Lemma 3.37. So let $\mathcal{M}$ be given and $A \subseteq \mathsf{On}, A \in \mathsf{V}$ be such that $\mathcal{M} \in \mathsf{HOD}_X[A][h]$.

Let $\eta := (\bar{\mu}^+)^{\mathsf{V}[h]}$ and $H := H_\eta^{\mathsf{HOD}_X[A][h]}$. Then all iteration trees in $\mathsf{HOD}_X[A][h]$ of length less that η which we consider are already in H.

H has size $2^{\bar{\mu}}$ in $\mathsf{HOD}_X[A][h]$, which is $\leq \omega_1^{\bar{\mu}\cdot\omega_1}$ computed in $\mathsf{HOD}_X[A]$ and therefore equal to $2^{\bar{\mu}}$ computed in $\mathsf{HOD}_X[A]$.[5] Since this is less than μ we can consider H as a bounded subset of μ. Thus, if $\dot{H} \in \mathsf{HOD}_X[A]$ is a name for H and Z is a bounded subset of μ coding $\dot{H}$, then

$$H \in \mathsf{HOD}_X[Z][h]$$

Now let k be the $\mathrm{Col}(\omega, <\mu_+)$-generic filter over $\mathsf{V}[h]$ from Definition 4.16 such that $\mathsf{V}[h][k] = \mathsf{V}[g]$. Then k is also generic over $\mathsf{HOD}_X[Z][h]$, and since Z is small we can absorb $Z \times h \times k$ by a $\mathrm{Col}(\omega, <\mu_+)$-generic l over HOD_X:

$$\mathsf{HOD}_X[Z][h][k] = \mathsf{HOD}_X[l]$$

But this implies that there is an elementary embedding from the premouse $\mathcal{M}$ into $\mathcal{N}$ in the $\mathrm{Col}(\omega, <\mu_+)$-generic extension of HOD_X. So we are in the same situation as in Lemma 3.37 and can complete the proof using the same argument. □

This enables us to prove the equivalents of Lemmata 3.38 and 3.41.

Lemma 4.21. *Let h be as in Definition 4.16. There is in* $\mathsf{V}[h]$ *a partial iteration strategy* Σ *for* $\mathcal{N}$ *which works for short trees of arbitrary length and condenses well.*

Moreover, for each set of ordinals $A \in \mathsf{V}$ *we have an extension* Σ_A *of* Σ *in* $\mathsf{HOD}_X[A][h]$ *which also works for maximal trees.*

[5] We assume $\omega_1^{\mathsf{HOD}_X[A]} < \bar{\mu}$ w. l. o. g.

Proof. The proof is as in Lemma 3.38. If $\mathcal{T}$ is a maximal tree, then the proof of 3.41 is as that of 3.38.

This proof is easier since there is only one case: If $\mathcal{T}$ is an iteration tree built according to the iteration strategy defined so far, then $\mathrm{cof}(lh(\mathcal{T})) = \omega$ in V. $\square$

Let $\tilde{\Sigma}$ be the extended iteration strategy in $\mathsf{HOD}_X[h]$, which is given by Lemma 4.21. Then $\Sigma := \tilde{\Sigma}\restriction\mathsf{HOD}_X \in \mathsf{HOD}_X$ by the homogeneity of $\mathrm{Col}(\omega, \omega_1)$. Moreover, Σ condenses well in HOD_X.

Definition 4.22. For any set of ordinals $A \in \mathsf{HOD}_X$ coding $\mathcal{N}$ let $L^\Sigma(A)$ be the *least Σ-hybrid J-structure over A containing all ordinals.*

The "condenses well" property of Σ ensures that this model is fine structural. Moreover, we have that $L^\Sigma(A) \subseteq \mathsf{HOD}_X$. This enables us to show the equivalent of Lemma 3.47.

Lemma 4.23. *Let $A \in \mathsf{HOD}_X$ be a set of ordinals coding $\mathcal{N}$. Then $L^\Sigma(A)$ satisfies condensation in V.*

Proof. Suppose not. Then there is in V, and therefore in some $\mathsf{HOD}_X[Y] \subseteq \mathsf{V}$, an elementary embedding

$$\pi \colon J^\Gamma_\gamma(A) \to L^\Sigma(A)$$

such that $J^\Gamma_\gamma(A) \ntrianglelefteq L^\Sigma(A)$. Working in $\mathsf{HOD}_X[Y]$ we can take a countable Skolem hull $\mathcal{X}$ of a large initial segment of the universe containing π and Σ. If we collapse $\mathcal{X}$, we get

$$\begin{tikzcd}
J^\Gamma_\gamma(A) \arrow[r, "\pi"] & L^\Sigma(A) \\
J^{\Gamma'}_{\gamma'}(A') \arrow[u, "\sigma"] \arrow[r, "\pi'"] & J^{\Sigma'}_{\beta'}(A') \arrow[u, "\sigma"]
\end{tikzcd}$$

where A' codes a countable premouse $\mathcal{N}' = \sigma^{-1}(\mathcal{N})$, and $\gamma' < \beta'$ are countable in $\mathsf{HOD}_X[Y]$.

But now by absoluteness of wellfoundedness[6] there exist in $\mathsf{HOD}_X[h]$ the following J-structures and maps:

$$\begin{tikzcd} & L^{\Sigma}(A) \\ J_{\gamma^*}^{\Gamma^*}(A^*) \arrow[r, "\pi^*"] & J_{\beta^*}^{\Sigma^*}(A^*) \arrow[u, "\sigma^*"'] \end{tikzcd}$$

such that $J_{\gamma^*}^{\Gamma^*}(A^*) \ntrianglelefteq J_{\beta^*}^{\Sigma^*}(A^*)$.

But $\Sigma^* = \Sigma^{\sigma^*} \restriction J_{\beta^*}^{\Sigma^*}(A^*)$ also condenses well by the argument in Claim 2 of Lemma 3.37. So $J_{\beta^*}^{\Sigma^*}(A^*)$ satisfies condensation, which is a contradiction. $\square$

Now we are in the same situation as in the end-of-gap case of "each uncountable successor cardinal is weakly compact".

Lemma 4.24. *Suppose $A \in \mathsf{HOD}_X$ simply codes $\mathcal{N}$. Let $L^{\Sigma}(A)$ be the least Σ-hybrid premouse in HOD_X.*

Then there exists a sharp for $L^{\Sigma}(A)$.

Proof. The proof is essentially the same as the proof from the "weakly compact" hypothesis. Again we let η be a large enough Θ-closed V-cardinal. By our hypothesis η is automatically singular. Again we have $\eta^{+L^{\Sigma}(A)} < \eta^+$ since η^+ is a limit cardinal in $L^{\Sigma}(A)$ (see Lemma 2.48).

But then we can argue as in the "weakly compact" case. $\square$

Now we can again define mice which entirely ensure that $(W^{\star}_{\alpha+1})$ holds.

Definition 4.25. Let A be a set of ordinals coding $\mathcal{N}$. Then let $P_n^{\Sigma^\sharp}(A)$ be the least iterable Σ-hybrid premouse over A which is active and satisfies "there are n Woodin cardinals", if such a premouse exists, and undefined otherwise.

But now we get as before

Lemma 4.26. *$P_n^{\Sigma^\sharp}(A)$ exists for each set of ordinals $A \in \mathsf{HOD}_X$ coding $\mathcal{N}$.*

[6] See [Sch01, Lemma 0.2].

and finally

Lemma 4.27. *$(W^\star_{\alpha+1})$ holds. Moreover, if $\mathcal{P}$ is the mouse witnessing $(W^\star_{\alpha+1})$ with respect to a $\Sigma_n(J_\alpha(\mathbb{R}^g))$-set of reals, then $\mathcal{P}$ is closed under $C_{\Sigma_n(J_\alpha(\mathbb{R}^g))}$.*

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