Bounded Martin's Maximum with an asterisk

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Abstract

We isolate natural strengthenings of Bounded Martin's Maximum which we call BMM^* and A–$\mathsf{BMM}^{*,++}$ (where A is a universally Baire set of reals), and we investigate their consequences. We also show that if A–$\mathsf{BMM}^{*,++}$ holds true for every set of reals A in $L(\mathbb{R})$, then Woodin's axiom $(*)$ holds true. We conjecture that MM^{++} implies A–$\mathsf{BMM}^{*,++}$ for every A which is universally Baire.

W.H. Woodin, P. Larson, I. Farah, and M. Magidor asked the second author whether the method developed in [1] and [3] can be applied to show other Π_2–statements which are discussed in [13]. In particular, they asked if the statements from Definition 1.2 below can be shown from Bounded Martin's Maximum, BMM, together with the precipitousness of NS_{ω_1}. This led the second author to the formulation of the "maximality" principle BMM^* (cf. Definition 1.9) which says that if a Σ_1 statement φ (with parameters from H_{ω_2}) is "honestly consistent," then φ holds true in V.

A scenario for proving BMM^* from BMM plus NS_{ω_1} is precipitous appears naturally: one would have to show that if a Σ_1 statement is "honestly consistent," then it can be forced by a stationary set preserving forcing. It has been conjectured (cf. e.g. [10, Conjecture 6.8]) that Martin's Maximum^{++} implies Woodin's axiom $(*)$. Showing that if a Σ_1 statement is "honestly consistent," then it can be forced by a stationary set preserving forcing would verify this conjecture, but the present paper has to leave this conjecture unanswered.

We are able to show, though, that a strengthening of BMM^* implies $(*)$. This strengthening allows NS_{ω_1} as well as universally Baire sets A as parameters and will be written as A–$\mathsf{BMM}^{*,++}$, cf. Definition 2.6. Our Theorem 2.7 says that in the presence of large cardinals, $(*)$ follows from A–$\mathsf{BMM}^{*,++}$ for all sets of reals A in $L(\mathbb{R})$. We conjecture that MM^{++} implies A–$\mathsf{BMM}^{*,++}$ for every universally Baire set A.

We assume the reader to have some familiarity with forcing axioms as well as with Woodin's $\mathbb{P}_{\max}$. Classical texts on forcing axioms are [5] and [6] (cf. also [10]). The forcing $\mathbb{P}_{\max}$ was introduced in [13] (cf. also [9]).

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Throughout this paper, we let $\mathsf{NS} = \mathsf{NS}_{\omega_1}$ denote the nonstationary ideal on ω_1. The Bounded Proper Forcing Axiom, BPFA (cf. [6]), says that for every proper poset $\mathbb{P}$ and every $\mathbb{P}$–generic filter G over V,

$$((H_{\omega_2})^V; \in) \prec_{\Sigma_1} ((H_{\omega_2})^{V[G]}; \in).$$

The formulation of Bounded Martin's Maximum, BMM, results from that of BPFA by replacing "proper" with "stationary set preserving." Given a universally Baire set $A \subset \mathbb{R}$, *A-Bounded Martin's Maximum*$^{++}$ (cf. [13, Definition 10.91]) says that for every stationary set preserving poset $\mathbb{P}$ and every $\mathbb{P}$–generic filter G over V,

$$((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A) \prec_{\Sigma_1} ((H_{\omega_2})^{V[G]}; \in, (\mathsf{NS}_{\omega_1})^{V[G]}, A^*),$$

where A^* is $V[G]$'s version of A, i.e., if the trees T and U witness that A is $|\mathbb{P}|^+$–universally Baire with $A = p[T]$, then $A^* = p[T] \cap V[G]$.

A $\mathbb{P}_{\max}$–condition is a countable transitive structure $p = (M; \in, I, a)$ such that M is a model of a fragment of ZFC plus MA_{ω_1}, $p \models$ "I is a normal uniform ideal on ω_1," $a \in \mathcal{P}(\omega_1^M) \cap M$ is such that $\omega_1^M = \omega_1^{L[a,x]}$ for some $x \in \mathbb{R} \cap M$, and p is generically iterable (cf. [13, Definition 3.5]). If $p = (M; \in, I, a)$ and $q = (N; \in, J, b)$ are in $\mathbb{P}_{\max}$, then $q <_{\mathbb{P}_{\max}} p$ iff there is a generic iteration of p which gives rise to an embedding

$$j \colon p = (M; \in, I, a) \to (M^*; \in, I^*, j(a))$$

such that $j(a) = b$, $\{M^*, j\} \in N$, and $J \cap M^* = I^*$. *Woodin's Axiom* $(*)$ (cf. [13, Definition 5.1]) says that AD, the *Axiom of Determinacy*, holds in $L(\mathbb{R})$ and $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{\max}$–extension of $L(\mathbb{R})$, i.e., there is some G which is $\mathbb{P}_{\max}$–generic over $L(\mathbb{R})$ and

$$L(\mathcal{P}(\omega_1)) = L(\mathbb{R})[G].$$

1 Bounded Martin's Maximum*

Let us start with some examples.

Definition 1.1 *Let $B \subset \omega_1$. We say that B is* amenably closed *iff for all $D \subset \omega_1$, if $D \cap \xi \in L[B]$ for all $\xi < \omega_1$, then $D \in L[B]$.*

By [4], "B is amenably closed" may be formulated in the presence of BPFA in a Σ_1 fashion as follows.

Let $B \subset \omega_1$ be amenably closed. The set of all cofinal branches through the tree $T = {}^{<\omega_1}\omega_1 \cap L[B]$ is then contained in $L[B]$ and has cardinality $\aleph_1$ in V since, under BPFA, ω_2^V is inaccessible (in fact Σ_2–reflecting) in every inner model of the form $L[X]$ for $X \subset \omega_1$ (cf. [6]). If BPFA holds true, then T is *weakly special*, i.e., there is a function $f \colon T \to \omega$ such that for all s, t, $t' \in T$, if $f(s) = f(t) = f(t')$, $s \subset t$ and $s \subset t'$, then $t \subset t'$ or $t' \subset t$ (cf. [4]). For each cofinal branch b through T there is then some $s \in T$ such that

$$b = \{t \in T \ : \ \exists t' \supset s \ (t \subset t' \wedge f(t') = f(s)\}.$$

We then have that under BPFA a given $B \subset \omega_1$ is amenably closed iff there is some $\alpha < \omega_2$ and some $f \colon {}^{<\omega_1}\omega_1 \cap J_{\omega_1}[B] \to \omega$ witnessing that ${}^{<\omega_1}\omega_1 \cap J_{\omega_1}[B]$ is weakly special and such that for all $s \in T$,

$$\{t \in T \ : \ \exists t' \supset s \ (t \subset t' \wedge f(t') = f(s)\} \in J_\alpha[B].$$

Definition 1.2 *We will be concerned with the following two statements.*

(1) (Cf. [13, Theorem 5.74 (5)].) Let $S \subset \omega_1$ be stationary and costationary. There is then some $x \in \mathbb{R}$ and some G which is $\mathrm{Col}(\omega, < \omega_1^V)$–generic over $L[x]$ such that $L[x,S] = L[x,G]$.

(2) (Cf. [13, Theorem 6.108 (5)].) Let $A \subset \omega_1$. There is then some amenably closed $B \subset \omega_1$ with $A \in L[B]$.

It is not hard to see that e.g. if BPFA holds true, then both (1) and (2) may be formulated as $\Pi_2^{H_{\omega_2}}$–sentences. For (2), this uses the remark after Definition 1.1.

The following observation is very easy.

Lemma 1.3 *If (1) holds, then $\mathbb{R}$ is closed under $\sharp$'s, and $\boldsymbol{\delta}^1_2 = \omega_2$.*

Proof. Let $z \subset \omega$. In order to show that $z^\sharp$ exists it suffices to see that every $X \in \mathcal{P}(\omega_1) \cap L[z]$ either contains a club or is disjoint from a club, as then the club filter on ω_1, restricted to $L[z]$, is an $L[z]$–ultrafilter. Suppose that $S' \in \mathcal{P}(\omega_1) \cap L[z]$ is stationary and costationary in V. Then $S = (S' \setminus \omega) \cup z$ is also stationary and costationary. By (1), there is some $x \in \mathbb{R}$ and some G which is $\mathrm{Col}(\omega, < \omega_1)$–generic over $L[x]$ with $L[x,S] = L[x,G]$. But $L[x,S] = L[x,z]$, so that there is some $\bar{G}$ which is $\mathrm{Col}(\omega, < \omega_1)$–generic over $L[x,z]$ with $L[x,G] = L[x,z,\bar{G}]$. But then $L[x,z] = L[x,S] = L[x,G] = L[x,z,\bar{G}]$, which contradicts the fact that every real $z \in L[x,G]$ is in $L[x, G \restriction \alpha]$ for some $\alpha < \omega_1$.

To see that $\boldsymbol{\delta}^1_2 = \omega_2$, let $\beta < \omega_2$, and let $A \subset \omega_1$ be such that $\beta < (\omega_1^V)^{+L[A]}$. Let $S' \subset \omega_1$ be stationary and costationary, and let

$$S = \{\omega \cdot \alpha \colon \alpha \in S'\} \cup \{\omega \cdot \alpha + 1 \colon \alpha \in A\}.$$

Then S is again stationary and costationary, and if $x \in \mathbb{R}$ and G $\mathrm{Col}(\omega, < \omega_1)$–generic over $L[x]$ are such that $L[x,S] = L[x,G]$, then

$$(\omega_1^V)^{+L[x]} = (\omega_1^V)^{+L[x,G]} = (\omega_1^V)^{+L[x,S]} \geq (\omega_1^V)^{+L[A]} > \beta,$$

so that $\beta < \boldsymbol{\delta}^1_2$. $\square$

In particular, (1) by itself implies $\neg$CH, the negation of the Continuum Hypothesis. On the other hand, in L, every subset of ω_1 is trivially amenably closed, so that (2) holds in L and does not by itself imply $\neg$CH. The situation is a bit more tricky under forcing axioms. As we said, under BPFA, ω_2^V is inaccessible in every inner model of the form $L[B]$ for $B \subset \omega_1$. Suppose (2) and that ω_2^V is inaccessible in every inner model of the form $L[B]$ for $B \subset \omega_1$. If $W \subset V$ is an inner model of GCH, then we may pick some $A \in W$, $A \subset \omega_1$, such that $HC \cap W = HC \cap L[A]$. If $A \in L[B]$, where $B \subset \omega_1$ is amenably closed, then $\mathcal{P}(\omega_1) \cap W \subset L[B]$, so that $(\omega_1^V)^{+W} < \omega_2$. In particular:

Lemma 1.4 *If* (2) *holds and* H_{ω_2} *is closed under* $\#$*'s, then* CH *fails.*

Whereas Lemma 1.3 shows that (1) by itself is a fairly strong principle, (2) is only strong in the presence of e.g. a precipitous ideal on ω_1:

Lemma 1.5 *If (2) holds and there is a precipitous ideal on* ω_1*, then there is an inner model with a Woodin cardinal.*

Proof. If there is a precipitous ideal on ω_1, then H_{ω_2} is closed under $\sharp$'s. Suppose Lemma 1.5 to fail, and let K denote the core model below a Woodin cardinal. By the remarks before the statement of Lemma 1.4, $(\omega_1^V)^{+K} < \omega_2$. On the other hand, by [2, Theorem 0.3], if there is a precipitous ideal on ω_1, then $(\omega_1^V)^{+K} = \omega_2$. Contradiction! $\Box$

We are now about to propose our strengething of BMM (Bounded Martin's Maximum). Recall that BMM says that if $A \in H_{\omega_2}$, $\varphi(x)$ is a Σ_1–formula, and $\mathbb{P} \in V$ is a poset which preserves stationary subsets of ω_1, then

$$V^{\mathbb{P}} \models \varphi(A) \Longrightarrow V \models \varphi(A).$$

We might strengthen this statement by saying that if $\varphi(A)$ is "consistent," then $\varphi(A)$ is true, where we might try to spell out "consistent" as in the following version of BMM.

Let us write $\mathsf{BMM}^{\mathrm{o}}$ for the statement that if $A \in H_{\omega_2}$, if $\varphi(x)$ is a Σ_1–formula, and if there is some transitive model $\mathfrak{A}$ such that

(a) $\mathfrak{A} \in V^{\mathrm{Col}(\omega, 2^{\aleph_1})}$,

(b) $(H_{\omega_2})^V \subset \mathfrak{A}$,

(c) if $T \subset \omega_1^V$, $T \in V$, $V \models T$ is stationary, then $\mathfrak{A} \models T$ is stationary, and

(d) $\mathfrak{A} \models \mathsf{ZFC}^- + \varphi(A)$,

then $\varphi(A)$ is true in V.

If in (a) we demand $\mathfrak{A}$ to be in V rather than just $V^{\mathrm{Col}(\omega, 2^{\aleph_1})}$, then the hypothesis would already say that $\varphi(A)$ is true in V. If we dropped (c), then a counterexample would be given by $\varphi(A) \equiv$ "A is disjoint from a club" for some $A \subset \omega_1$ which is stationary in V but not in $\mathfrak{A}$.

Clearly, $\mathsf{BMM}^{\mathrm{o}}$ is a strengthening of BMM. By [11], $\mathsf{BMM}^{\mathrm{o}}$ thus implies that V is closed under $\#$'s. This may be used to show that $\mathsf{BMM}^{\mathrm{o}}$ is in fact inconsistent. Let us consider the statement $\varphi(\omega_1) \equiv$ "there is some $x \in \mathbb{R}$ such that $\omega_1 = \omega_1^{L[x]}$." Let V_α be a model of a sufficiently rich finite fragment of ZFC. We may force over V_α by Jensen coding to add some G which is class generic over V_α such that in $V_\alpha[G]$, there is some real x with $V_\alpha[G] = J_\alpha[x]$. As Jensen coding preserves stationary subsets of ω_1 (cf. [11]), Shoenfield absoluteness yields that there is some $\mathfrak{A}$ with (a), (b), (c), and (d) for $A = \omega_1$ and $\varphi(\omega_1) \equiv$ "there is some $x \in \mathbb{R}$ such that $\omega_1 = \omega_1^{L[x]}$." Then $\mathsf{BMM}^{\mathrm{o}}$ would imply that in V there is a real x such that $\omega_1 = \omega_1^{L[x]}$, which contradicts the existence of $x^{\#}$.

The problem with BMM^o is that it ignores that the model $\mathfrak{A}$ has to be "as closed as" V. For BMM this is automatic, as every set generic extension of V is "as closed as" V. We need to make this requirement explicit if we aim to arrive at a consistent weakening of BMM^o that strengthens BMM. We'll spell out the neccessary closure of $\mathfrak{A}$ in terms of universally Baire sets of reals, basically as in [13].

We call a function $F\colon \mathbb{R} \to \mathbb{R}$ *universally Baire* iff its graph $F = \{(x, F(x)) \colon x \in \mathbb{R}\}$ is a universally Baire subset of $\mathbb{R}^2$. Let $U\colon \mathbb{R} \to \mathbb{R}$ be universally Baire, as being witnessed by the class sized trees T and U with $F = p[T]$ and $V^{\mathbb{P}} \models p[U] = {}^\omega\omega \setminus p[T]$ for all $\mathbb{P} \in V$. Then if $\mathbb{P} \in V$ is any poset and if G is $\mathbb{P}$–generic over V, F^G denotes the (possibly partial) function $p[T]^{V[G]}$. It is easy to see that F^G is indeed a function. Also, this function is independent from the choice of T and U, so the notation F^G is unambiguous.

Definition 1.6 *Let $F\colon \mathbb{R} \to \mathbb{R}$ be universally Baire. Let Ω be an uncountable cardinal, and let G be $\mathrm{Col}(\omega, \Omega)$–generic over V. Let $\mathfrak{A} \in V[G]$ be a transitive model of ZFC^- which is countable in $V[G]$. We say that $\mathfrak{A}$ is* closed under F *(or,* F–closed*) iff for all posets $\mathbb{P} \in \mathfrak{A}$ and for all $g \in V[G]$ which are $\mathbb{P}$–generic over $\mathfrak{A}$, $\mathfrak{A}[g]$ is closed under F^G, i.e., $F^G(x) \in \mathfrak{A}[g]$ for all $x \in \mathbb{R} \cap \mathfrak{A}[g]$ in the domain of F^G.*

The following lemma can be proved easily by an absoluteness argument.

Lemma 1.7 *Let $F\colon \mathbb{R} \to \mathbb{R}$ be universally Baire. Let $\mathbb{P} \in V$ be a poset, and let H be $\mathbb{P}$–generic over V. If $V[H']$ is a set–generic extension of $V[H]$, then $F^{H'} \restriction \mathbb{R}^{V[H]} = F^H$.*

Here is an example, of which the case $n = 1$ will be important later. Let $n < \omega$, and let V be closed under $X \mapsto M_n^\#(X)$. Then $F\colon x \mapsto M_n^\#(x)$, construed as a function from $\mathbb{R}$ to $\mathbb{R}$, is universally Baire, cf. [2, Lemma 2.9]. If $\mathfrak{A}$ is closed under F in the sense of Definition 1.6, then $\mathfrak{A}$ must be closed under $X \mapsto M_n^\#(X)$ in the ordinary sense. The same is of course true for mouse operators other than $M_n^\#$.

Definition 1.8 *Let $X \in H_{\omega_2}$, and let $\varphi(x)$ be a Σ_1 formula in the language of set theory. We say that $\varphi(X)$ is* honestly consistent *iff for every $F\colon \mathbb{R} \to \mathbb{R}$ which is universally Baire there is an F–closed transitive model $\mathfrak{A}$ such that*

(a) $\mathfrak{A} \in V^{\mathrm{Col}(\omega, 2^{\aleph_1})}$,

(b) $(H_{\omega_2})^V \subset \mathfrak{A}$,

(c) if $T \subset \omega_1^V$, $T \in V$, $V \models T$ is stationary, then $\mathfrak{A} \models T$ is stationary, and

(d) $\mathfrak{A} \models \mathsf{ZFC}^- \ + \ \varphi(X)$.

Definition 1.9 *By* Bounded Martin's Maximum*, BMM^*, *we mean the conjunction of the following two statements.*

(a) NS_{ω_1} *is precipitous, and*

(b) if $X \in H_{\omega_2}$ and if $\varphi(x)$ is a Σ_1 formula such that $\varphi(X)$ is honestly consistent, then $\varphi(X)$ holds true in V.

Theorem 1.10 *If* BMM^* *holds true, then so does* (1).

Proof. Let $\theta = 2^{\aleph_1}$ and $\rho = (2^\theta)^+$, and let H be $\mathrm{Col}(\omega, <\rho)$–generic over V. Note that $\rho = \omega_1^{V[H]}$. Let $x \in \mathbb{R} \cap V[H]$ be a real coding the structure $(H_{(2^{\aleph_1})^+}; \in, \mathsf{NS})^V$. There is some G which is $\mathrm{Col}(\omega, <\rho)$–generic over $V[x]$ with the property that $V[x, G] = V[H]$. We have that $\bigcup G \colon \omega \times \rho \to \rho$, and for each $\eta < \rho$, $\bigcup G(\cdot, \eta) \colon \omega \to \eta$ is a surjection. Setting

$$\bar{S}_\xi = \{\eta < \rho : \ \bigcup G(0, \eta) = \xi\}$$

for $\xi < \rho$, $(\bar{S}_\xi \colon \xi < \rho)$ is a family of pairwise disjoint subsets of $\rho = \omega_1^{V[H]}$ such that each $\bar{S}_\xi$ is stationary in $V[H]$.

Let $e \colon \rho \to [\rho]^{<\rho} \cap L[x, G]$, $e \in L[x, G]$ be an enumeration of all the bounded subsets of ρ which exist in $L[x, G]$.

Let $\bar{D} = \{\alpha < \rho : \ J_\alpha[x] \models \mathsf{ZFC}^-\}$, let $D' \subset \rho$ be the club of all limit points of $\bar{D}$, and let $D = \bar{D} \setminus D'$. Then D is an unbounded nonstationary subset of ρ. We let $d \colon \omega \times \rho \times \rho \to D$ be some bijection which exists in $L[x]$. Setting $S_\xi = \bar{S}_\xi \cap D'$ for $\xi < \rho$, we have that $(S_\xi \colon \xi < \rho)$ is a family of pairwise disjoint subsets of ρ each of which is stationary in $V[x, G]$ and such that $S_\xi \cap D = \emptyset$ for all $\xi < \rho$.

We now fix $S \in V$, $S \subset \omega_1^V$, stationary and costationary in V. Working inside $L[x, G]$, we may construct a generic iteration

$$((\mathcal{M}_i, \pi_{ij} \colon i \leq j \leq \rho), (G_i \colon i < \rho))$$

of $\mathcal{M}_0 = (H_{(2^{\aleph_1})^+}; \in, \mathsf{NS})^V$ with the following properties.

(i) If ξ, $i < \rho$ and $e(\xi) \in \mathcal{M}_i \setminus \bigcup_{k<i} \mathrm{ran}(\pi_{ki})$ is stationary in $\mathcal{M}_i$, then $S_\xi \setminus \mathrm{crit}(G_i) \subset \pi_{i\rho}(e(\xi))$.

(ii) For $n < \omega$ and η, $\xi < \rho$, $G(n, \eta) = \xi$ iff $d(n, \eta, \xi) \in \pi_{0\rho}(S)$.

In particular, if $T \subset \rho$, $T \in M_\rho$, $M_\rho \models T$ is stationary, then T is stationary in $V[H]$.

Also, $L[x, \pi_{0\rho}(S)] = L[x, G]$. This is true as D, $d \in L[x]$, so that G may be read off from d and $\pi_{0\rho}(S)$ inside $L[x, \pi_{0\rho}(S)]$, i.e., $L[x, G] \subset L[x, \pi_{0\rho}(S)]$. On the other hand, the generic iteration $((\mathcal{M}_i, \pi_{ij} \colon i \leq j \leq \rho), (G_i \colon i < \rho))$ is inside $L[x, G]$, so that we certainly have that $\pi_{0\rho}(S) \in L[x, G]$, so that $L[x, \pi_{0\rho}(S)] = L[x, G]$.

We may lift the iteration maps to act on V, i.e., there is a unique generic iteration

$$((N_i, \tilde{\pi}_{ij} \colon i \leq j \leq \rho), (G_i \colon i < \rho))$$

of $(V; \in, \mathsf{NS})$ such that $\mathcal{M}_i = (H_{(2^{\aleph_1})^+})^{N_i}$ for $i \leq \rho$ and $\pi_{ij} = \tilde{\pi}_{ij} \upharpoonright \mathcal{M}_i$ for $i \leq j \leq \rho$. Let us write $N = N_\rho$.

Now let $F \colon \mathbb{R} \to \mathbb{R}$ be universally Baire, and let T_0, U_0 be the class sized trees witnessing that F is universally Baire (with $F = p[T_0]$). Set $T_\rho = \tilde{\pi}_{0\rho}(T_0)$ and $U_\rho = \tilde{\pi}_{0\rho}(U_0)$, so that $p[T_\rho] = p[T_0]$ and $p[U_\rho] = p[U_0]$.

By Lemma 1.7, every rank initial segment of $V[H]$ is closed under F. Hence in $V[H]$, there is some transitive F–closed $\mathfrak{A}$ with $(H_{(2^{\aleph_1})^+})^N \subset \mathfrak{A}$, $\mathfrak{A} \models T$ is stationary for all $T \subset \rho$, $T \in \mathcal{M}_\rho$, such that $\mathcal{M}_\rho \models T$ is stationary, and such that $\mathfrak{A}$ is a model of ZFC^- plus "there is some real x and some G which is $\mathrm{Col}(\omega,<\rho)$–generic over $L[x]$ with $L[x,\pi_{0\rho}(S)]=L[x,G]$." (Just take an appropriate rank initial segment of $V[H]$ as $\mathfrak{A}$.)

We may use the tree T_ρ to witness the fact that $\mathfrak{A}$ is F–closed. By absoluteness then, in $N^{\mathrm{Col}(\omega,\tilde{\pi}_{0,\rho}(2^{\aleph_1}))}$ there is some transitive F–closed (as being witnessed by T_ρ) $\mathfrak{A}$ with the above properties. Pulling this back via $\tilde{\pi}_{0\rho}$ we get that in $V^{\mathrm{Col}(\omega,2^{\aleph_1})}$ there is some transitive F–closed (as being witnessed by T_0) $\mathfrak{A}$ with $(H_{\omega_2})^V\subset\mathfrak{A}$, $\mathfrak{A}\models T$ is stationary for all $T\subset\omega_1$, $T\in V$, such that $V\models T$ is stationary, and such that $\mathfrak{A}$ is a model of ZFC^- plus "there is some real x and some G which is $\mathrm{Col}(\omega,<\omega_1)$–generic over $L[x]$ with $L[x,S]=L[x,G]$."

We have shown that (1) is honestly consistent. □ (Theorem 1.10)

Theorem 1.11 *If* BMM^* *holds true, then so does* (2).

Proof. Let us again write $\theta=2^{\aleph_1}$ and $\rho=(2^\theta)^+$, and let G be $\mathrm{Col}(\omega,<\rho)$–generic over V. Let H be $\mathrm{Col}(\rho,\rho)$–generic over $V[G]$. We have that $\rho=\omega_1^{V[G,H]}$ and $\diamondsuit$ holds in $V[G,H]$. Let $e^*\colon\rho\to[\rho]^{<\rho}\cap V[G,H]$, $e^*\in V[G,H]$, be an enumeration of all the bounded subsets of ρ which exist in $V[G,H]$. Let $(\tau_i\colon i<\rho)$ witness that $\diamondsuit$ holds in $V[G,H]$. As in the previous proof, we may set

$$\bar{S}_\xi=\{\eta<\rho\;:\;\bigcup G(0,\eta)=\xi\}$$

for $\xi<\rho$, so that $(\bar{S}_\xi\colon\xi<\rho)$ is a family of pairwise disjoint subsets of ρ, each $\bar{S}_\xi$ being stationary in $V[G,H]$.

Let $x\in\mathbb{R}\cap V[G]$ be such that the structure $(H_{(2^{\aleph_1})^+};\in,\mathsf{NS})^V$ is in $L[x]$ and is countable there. Let $x^\#=(J_\alpha[x];\in,U)$, and let $\kappa=\mathrm{crit}(U)$. Let $g\in V[G]$ be $\mathrm{Col}(\omega,<\kappa)$–generic over $x^\#$ (equivalently, over $L[x]$), and let

$$I=\{X\in\mathcal{P}(\kappa)\cap x^\#[g]\;:\;\exists Y\in U\;Y\cap X=\emptyset\}.$$

Then $(x^\#[g];\in,I)\models I$ is a σ–complete uniform normal ideal on κ, and $(x^\#[g];\in,I)$ is generically iterable via I and its images in a way that every iteration map lifts an iteration map resulting from iterating the ground model $x^\#$.

We may let $(W_\xi\colon\xi<\kappa)\in x^\#[g]$ be a partition of κ into I–positive sets. We may also let $e\colon\kappa\to[\kappa]^{<\kappa}\cap L[x,g]$ be an enumeration of all the bounded subsets of κ which exist in $L[x,g]$.

Working inside $(x^\#[g];\in,I)$, we may construct a generic iteration

$$((\mathcal{M}_i,\pi_{ij}\colon i\leq j\leq\kappa),(G_i\colon i<\kappa))$$

of $\mathcal{M}_0=(H_{(2^{\aleph_1})^+};\in,\mathsf{NS})^V$ with the following property.

(i) If ξ, $i<\kappa$ and $e(\xi)\in\mathcal{M}_i\setminus\bigcup_{k<i}\mathrm{ran}(\pi_{ki})$ is stationary in $\mathcal{M}_i$, then $W_\xi\setminus\mathrm{crit}(G_i)\subset\pi_{i\kappa}(e(\xi))$.

In particular, $\mathcal{M}_\kappa\in x^\#[g]$ and $(\mathsf{NS})^{\mathcal{M}_\kappa}=I\cap\mathcal{M}_\kappa$. Working inside $V[G,H]$, we may then construct a generic iteration

$$((\mathcal{M}^*_i,\pi^*_{ij}\colon i\leq j\leq\rho),(G^*_i\colon i<\rho))$$

of $\mathcal{M}^*_0=(x^\#[g];\in,I)$ with the following properties.

(ii) If ξ, $i < \rho$ and $e^*(\xi) \in \pi^*_{0i}(I^+) \setminus \bigcup_{k<i} \mathrm{ran}(\pi_{ki})$, then $\bar{S}_\xi \setminus \mathrm{crit}(G^*_i) \subset \pi_{i\kappa}(e^*(\xi))$.

(iii) If $i < \rho$, then G^*_i is generic over $L[\mathcal{M}^*_i, (\tau_k : k \leq i)]$ (not just over $\mathcal{M}^*_i$).

In particular, $\pi^*_{0\rho}(I) = (\mathsf{NS}_\rho)^{V[G,H]} \cap \mathcal{M}^*_\rho$. Also, if $k \leq i \leq \rho$, then

$$\tau_k \in \mathcal{M}^*_k \Longleftrightarrow \tau_k \in \mathcal{M}^*_i.$$

Let

$$((\mathcal{M}_i, \pi_{ij} : i \leq j \leq \rho), (G_i : i < \rho)) = \pi^*_{0\rho}((\mathcal{M}_i, \pi_{ij} : i \leq j \leq \kappa), (G_i : i < \kappa)),$$

which is a generic iteration of $\mathcal{M}_0 = (H_{(2^{\aleph_1})^+}; \in, \mathsf{NS})^V$. We have that $\pi_{0,\rho}((\mathsf{NS})^V) = \pi_{0,\rho}(I) \cap \mathcal{M}^*_\rho = (\mathsf{NS}_\rho)^{V[G,H]} \cap \mathcal{M}_\rho$, so that every $T \subset \rho$, $T \in \mathcal{M}_\rho$, which is stationary in $\mathcal{M}_\rho$ is also stationary in $V[G,H]$.

Let $D \subset \rho$, $D \in V[G,H]$. Let $S_D = \{i < \rho : \tau_i = D \cap i\}$ which is stationary in $V[G,H]$. Suppose that $D \cap \xi \in \mathcal{M}^*_\rho$ for every $\xi < \rho$. There is then a club $C \subset \rho$ such that $D \cap i \in \mathcal{M}^*_i$ for all $i < \rho$. This gives some stationary $\bar{S} \subset S_D \cap C$ and some $i_0 < \rho$ and $\bar{D} \in \mathcal{M}^*_{i_0}$ such that $\pi^*_{i_0 j}(\bar{D}) = D \cap j$ for all $j \in \bar{S}$. But then $D = \pi^*_{i_0\rho}(\bar{D}) \in \mathcal{M}^*_\rho$. Writing $\mathcal{M}^*_\rho = (J_{\alpha^*}[x, g^*]; \in, \pi_{0\rho}(I))$ and letting $B \subset \rho$ code $x \oplus g^*$ in a simple way, we have shown that B is amenably closed in $V[G,H]$.

We may again lift $((\mathcal{M}_i, \pi_{ij} : i \leq j \leq \rho), (G_i : i < \rho))$ to a generic iteration $((N_i, \tilde{\pi}_{ij} : i \leq j \leq \rho), (G_i : i < \rho))$ of $(V; \in, (\mathsf{NS})^V)$. Let us write $N = N_\rho$.

Let us fix some $A \subset \omega_1$, $A \in V$. Also let $F : \mathbb{R} \to \mathbb{R}$ be universally Baire, and let T_0, U_0 be the class sized trees witnessing that F is universally Baire (with $F = p[T_0]$). Set $T_\rho = \tilde{\pi}_{0\rho}(T_0)$ and $U_\rho = \tilde{\pi}_{0\rho}(U_0)$, so that $p[T_\rho] = p[T_0]$ and $p[U_\rho] = p[U_0]$.

By Lemma 1.7, every rank initial segment of $V[G,H]$ is closed under F. In $V[G,H]$, there is thus some transitive F–closed (as being witnessed by T_ρ) $\mathfrak{A}$ with $(H_{(2^{\aleph_1})^+})^N \subset \mathfrak{A}$, $\mathfrak{A} \models T$ is stationary for all $T \subset \rho$, $T \in N$, such that $N \models T$ is stationary, and such that $\mathfrak{A}$ is a model of ZFC^- plus "there is some amenably closed $B \subset \rho$ with $\tilde{\pi}_{0\rho}(A) \in L[B]$." (Take an appropriate rank initial segment of $V[G,H]$ as $\mathfrak{A}$.)

Hence by absoluteness, in $N^{\mathrm{Col}(\omega, \tilde{\pi}_{0,\rho}(2^{\aleph_1}))}$ there is some transitive F–closed (as being witnessed by T_ρ) $\mathfrak{A}$ with the above properties. Pulling this back via $\tilde{\pi}_{0,\rho}$ we get that in $V^{\mathrm{Col}(\omega, 2^{\aleph_1})}$ there is thus a transitive F–closed (as being witnessed by T_0) $\mathfrak{A}$ with $(H_{\omega_2})^V \subset \mathfrak{A}$, $\mathfrak{A} \models T$ is stationary for all $T \subset \omega_1$, $T \in V$, such that $V \models T$ is stationary, and such that $\mathfrak{A}$ is a model of ZFC^- plus "there is some amenably closed $B \subset \rho$ with $A \in L[B]$."

We have shown that (2) is honestly consistent. □ (Theorem 1.11)

There is an obvious question which we have to leave unanswered: Does BMM plus NS_{ω_1} is precipitous prove BMM^*? We will explore this question further in the next section.

2 BMM^* and Woodin's axiom $(*)$

We now aim to discuss the relationship between BMM^* and $(*)$. In order to do so, we shall need strengthenings of BMM^* which we call $\mathsf{BMM}^{*,++}$ (in analogy with

MM^{++} and BMM^{++}, cf. Definition 2.2) and A–$\mathsf{BMM}^{*,++}$ (where A is a universally Baire set of reals, cf. Definition 2.6). We apologize for the awkward notation.

Definition 2.1 *Let $X \in H_{\omega_2}$, and let $\varphi(x, \dot{I}_{\mathsf{NS}})$ be a Σ_1 formula in the language of set theory, augmented by a predicate $\dot{I}_{\mathsf{NS}}$ for the non–stationary ideal on ω_1. We say that $\varphi(X, \dot{I}_{\mathsf{NS}})$ is* honestly consistent *iff for every $F\colon \mathbb{R} \to \mathbb{R}$ which is universally Baire there is an F–closed transitive model $\mathfrak{A}$ such that*

(a) $\mathfrak{A} \in V^{\mathrm{Col}(\omega, 2^{\aleph_1})}$,

(b) $(H_{\omega_2})^V \subset \mathfrak{A}$,

(c) if $T \subset \omega_1^V$, $T \in V$, $V \models T$ is stationary, then $\mathfrak{A} \models T$ is stationary, and

(d) $\mathfrak{A} \models \mathsf{ZFC}^- \ + \ \varphi(X, \dot{I}_{\mathsf{NS}})$.

Definition 2.2 *By* Bounded Martin's Maximum$^{*,++}$, $\mathsf{BMM}^{*,++}$, *we mean the conjunction of the following two statements.*

(a) NS_{ω_1} *is precipitous, and*

(b) if $X \in H_{\omega_2}$ and if $\varphi(x, \dot{I}_{\mathsf{NS}_{\omega_1}})$ is a Σ_1 formula in the language of set theory, augmented by a predicate for the non–stationary ideal on ω_1, such that $\varphi(X, \dot{I}_{\mathsf{NS}_{\omega_1}})$ is honestly consistent, then $\varphi(X, \dot{I}_{\mathsf{NS}_{\omega_1}})$ holds true in V.

In Definitions 2.1 and 2.2 we understand that the predicate $\dot{I}_{\mathsf{NS}_{\omega_1}}$ is interpreted by $(\mathsf{NS}_{\omega_1})^{\mathfrak{A}}$ and $(\mathsf{NS}_{\omega_1})^V$ inside $\mathfrak{A}$ and V, respectively. Of course, $\mathsf{BMM}^{*,++}$ strengthens both BMM^* as well as BMM^{++}.

After the first version of this paper had been written, J. Zapletal mentioned the following principle to us.

Definition 2.3 *(3) (Cf. [14].) Let $A \subset \omega_1$. There is then some $B \subset \omega_1$ with $A \in L[B]$ such that for every $D \in \mathcal{P}(\omega_1) \cap L[B]$, if $L[B] \models$ "D is stationary," then $V \models$ "D is stationary."*

Our proof of Theorem 1.11 presented above also produces the following result.

Theorem 2.4 *If* $\mathsf{BMM}^{*,++}$ *holds true, then so does* (3).

Definition 2.5 *Let $X \in H_{\omega_2}$, let $A \subset \mathbb{R}$ be universally Baire, and let $\varphi(x, \dot{A}, \dot{I}_{\mathsf{NS}_{\omega_1}})$ be a Σ_1 formula in the language of set theory, augmented by predicates $\dot{A}$ and $\dot{I}_{\mathsf{NS}_{\omega_1}}$ for A and for the non–stationary ideal on ω_1, respectively. We say that $\varphi(X, \dot{A}, \dot{I}_{\mathsf{NS}_{\omega_1}})$ is* honestly consistent *iff for every $F\colon \mathbb{R} \to \mathbb{R}$ which is universally Baire there is an F–closed transitive model $\mathfrak{A}$ such that*

(a) $\mathfrak{A} \in V^{\mathrm{Col}(\omega, 2^{\aleph_1})}$,

(b) $(H_{\omega_2})^V \subset \mathfrak{A}$,

(c) if $T \subset \omega_1^V$, $T \in V$, $V \models T$ is stationary, then $\mathfrak{A} \models T$ is stationary, and

(d) $\mathfrak{A} \models \mathsf{ZFC}^- + \varphi(X, \dot{A}, \dot{I}_{\mathsf{NS}_{\omega_1}})$.

Definition 2.6 *Let $A \subset \mathbb{R}$ be universally Baire. By A–*Bounded Martin's Maximum$^{*,++}$*, A–$\mathsf{BMM}^{*,++}$, we mean the conjunction of the following two statements.*

(a) NS_{ω_1} *is precipitous, and*

(b) if $X \in H_{\omega_2}$ and if $\varphi(x, \dot{A}, \dot{I}_{\mathsf{NS}_{\omega_1}})$ is a Σ_1 formula in the language of set theory, augmented by predicates for $\dot{A}$ and for the non–stationary ideal on ω_1, such that $\varphi(X, \dot{A}, \dot{I}_{\mathsf{NS}_{\omega_1}})$ is honestly consistent, then $\varphi(X, \dot{A}, \dot{I}_{\mathsf{NS}_{\omega_1}})$ holds true in V.

In Definitions 2.5 and 2.6 we again understand that the predicate $\dot{I}_{\mathsf{NS}_{\omega_1}}$ is interpreted by $(\mathsf{NS}_{\omega_1})^{\mathfrak{A}}$ and $(\mathsf{NS}_{\omega_1})^V$ inside $\mathfrak{A}$ and V, respectively; moreover, if the trees T and U witness that A is universally Baire with $A = p[T]$, then $\dot{A}$ is supposed to be interpreted by A inside V and by $p[T] \cap \mathfrak{A} = A^* \cap \mathfrak{A}$ inside $\mathfrak{A}$, where $A^* = p[T] \cap V^{\mathrm{Col}(\omega, 2^{\aleph_1})}$ is the version of A inside $V^{\mathrm{Col}(\omega, 2^{\aleph_1})}$.

We now prove the following result which is in the spirit of [13, Theorems 10.127, 128, 129, and 137]. This result also shows that BMM^* is consistent, in case the reader may have wondered. This is true because if we let V be the least inner model of ZFC which has ω Woodin cardinals $\delta_0 < \delta_1 < \dots$ and is closed under $X \mapsto M_\omega^{\#\#}(X)$, if G is $\mathrm{Col}(\omega, < \sup_{n<\omega} \delta_n)$–generic over V, and if

$$\mathbb{R}^* = \bigcup \{\mathbb{R} \cap V[G \restriction \delta_n] \colon n < \omega\},$$

then we may construct inside $V[G]$ an inner model

$$L^{M_\omega^{\#\#}}(\mathbb{R}^*)$$

of ZF plus AD which is the least inner model whose set of reals is $\mathbb{R}^*$ and is closed under $X \mapsto M_\omega^{\#\#}(X)$, and

$$L^{M_\omega^{\#\#}}(\mathbb{R}^*)^{\mathbb{P}_{\max}}$$

satisfies the hypotheses of Theorem 2.7 as well as $(*)$.

Theorem 2.7 *Suppose that $M_\omega^{\#\#}$ exists*[3] *and is fully iterable.*[4] *Suppose* NS_{ω_1} *is precipitous. Then the following statements are equivalent.*

(A) $(*)$

(B) For every set A of reals with $A \in L(\mathbb{R})$, A–Bounded Martin's Maximum$^{*,++}$ holds true.

Proof. We first show $(B) \Longrightarrow (A)$. Let $sat(\mathsf{NS})$ denote the saturation of NS, i.e., the least cardinal μ such that every antichain in $\mathcal{P}(\omega_1)/\mathsf{NS}$ has cardinality less than μ. In what follows, we shall write κ for $2^{<sat(\mathsf{NS})} = \mathrm{Card}(H_{sat(\mathsf{NS})})$. If $2^{\aleph_1} = \aleph_2$, then $\kappa = 2^{\aleph_1} = \aleph_2$ if NS is saturated, and $\kappa = 2^{\aleph_2}$ otherwise.

$(B) \Longrightarrow (A)$ is now an immediate consequence of the following result.

[3] $M_\omega^{\#\#}$ is a mouse with ω Woodin cardinals and a top measure which is closed under $\#$'s.
[4] E.g., suppose that there is a proper class of Woodin cardinals.

Theorem 2.8 *Let M be an inner model of ZF such that $\mathbb{R} \subseteq M$, and let $\Gamma = \mathcal{P}(\mathbb{R}) \cap M$. Let $\kappa = 2^{<sat(\mathsf{NS})}$. Assume the following hypotheses.*

(a) NS *is precipitous.*

(b) AD *holds true in M.*

(c) Every set of reals in Γ is κ^+–universally Baire.

(d) If A is a set of reals in Γ, φ is a Π^1_2–formula, and g is $\mathrm{Col}(\omega, \kappa)$–generic over V, then
$$\varphi(A) \Longleftrightarrow \varphi(A^g),$$
where A^g is $V[g]$'s version of A, i.e., if the trees T and U witness that A is κ^+–universally Baire with $A = (p[T])^V$, then $A^g = (p[T])^{V[g]}$.

(e) For every set A of reals in Γ, A–Bounded Martin's Maximum$^{*,++}$ holds true.

Let $A_0 \subset \omega_1$ be such that $\omega_1^{L[A_0]} = \omega_1$. Then there is some $G \in V$ such that G is $\mathbb{P}_{\max}$–generic over M and

$$L(\mathbb{R})[G] = L(\mathbb{R})[A_0] = L(\mathcal{P}(\omega_1)). \tag{1}$$

Proof of Theorem 2.8. Let us fix M as in the statement of the theorem. Let us also fix, until the end of this proof, some $A_0 \subset \omega_1$ such that $\omega_1^{L[A_0]} = \omega_1$. Let G be the set of all $p = (M_0; \in, J_0, a_0) \in \mathbb{P}_{\max}$ such that there is some generic iteration

$$((\mathcal{M}_i, \pi_{i,j} \colon i \leq j \leq \omega_1), (G_i \colon i < \omega_1))$$

of $\mathcal{M}_0 = p$ such that $\pi_{0,\omega_1}(a_0) = A_0$ and, writing $\mathcal{M}_{\omega_1} = (M_{\omega_1}; \in, J_{\omega_1}, A_0)$, every set in

$$J^+_{\omega_1} = (\mathcal{P}(\omega_1) \cap M_{\omega_1}) \setminus J_{\omega_1}$$

is stationary in V.

We claim that G is $\mathbb{P}_{\max}$–generic over M and that (1) holds true for G. In order to verify this, we shall need to prove the following three Claims which will be shown from the hypotheses of Theorem 2.8.

Claim 2.9 *G is a filter.*

Claim 2.10 *If $D \in M$ is a dense subset of $\mathbb{P}_{\max}$, then $D \cap G \neq \emptyset$.*

By a standard $\mathbb{P}_{\max}$–argument, if $p \in G$, then there is a *unique* generic iteration

$$((\mathcal{M}_i, \pi_{i,j} \colon i \leq j \leq \omega_1), (G_i \colon i < \omega_1))$$

of $\mathcal{M}_0 = p$ such that $\pi_{0,\omega_1}(a_0) = A_0$. Assuming Claims 2.9 and 2.10 and following [13], we shall then write $\mathcal{P}(\omega_1)_G$ for the set of all $X \subset \omega_1$ for which there is some $p \in G$ such that if

$$((\mathcal{M}_i, \pi_{i,j} \colon i \leq j \leq \omega_1), (G_i \colon i < \omega_1))$$

is the generic iteration of $\mathcal{M}_0 = p$ with $\pi_{0\omega_1}(a_0) = A_0$, then $X \in \mathrm{ran}(\pi_{i,\omega_1})$ for some $i < \omega_1$.

Claim 2.11 $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$.

If NS were assumed saturated, then Claim 2.9 would be given by [13, Theorem 4.74] and Claim 2.11 would follow from [13, Lemma 3.12 and Corollary 3.13]. Under the hypotheses (a) and (e) instead, one can prove Claims 2.9 and 2.11 by an easy application of the forcing developed in [1]: Using hypothesis (a), [1] designs a stationary set preserving forcing which (for a given regular cardinal $\theta \geq \aleph_2$) adds a generic iteration

$$(\mathcal{M}_i, \pi_{i,j} \colon i \leq j \leq \omega_1)$$

of a countable model $\mathcal{M}_0 = (M_0; \in, I_0)$ such that $\mathcal{M}_{\omega_1} = (H_\theta; \in, \mathsf{NS})$. This immediately gives Claim 2.11 by Bounded Martin's Maximum^{++}. Also, if p, $q \in G$, then we may assume without loss of generality that p, q, $A_0 \cap \omega_1^{M_0} \in M_0$, so that Bounded Martin's Maximum^{++} also yields Claim 2.9.

It remains to verify Claim 2.10.

Let us fix $D \subset \mathbb{P}_{\max}$, $D \in M$, a dense set in $\mathbb{P}_{\max}$, and let $D^* \in \Gamma$ be a set of reals coding D according to some natural coding device. As D^* is κ^+–universally Baire, we may pick trees T and U on $\omega \times 2^\kappa$ such that

$$D^* = p[T] \text{ and } \Vdash_{\mathrm{Col}(\omega,\,\kappa)} p[U] = {}^\omega\omega \setminus p[T].$$

The following is a variant of the argument for Theorems 1.10 and 1.11.

Let us pick some g which is $\mathrm{Col}(\omega, \kappa)$–generic over V, so that $(\kappa^+)^V = \omega_1^{V[g]}$ and $H_{sat(\mathsf{NS})}$ is countable in $V[g]$. By our hypothesis (a) and the proof of [13, Lemma 3.10], $p_0 = ((H_{sat(\mathsf{NS})})^V; \in, (\mathsf{NS})^V, A_0)$ is then a $\mathbb{P}_{\max}$ condition in $V[g]$. The statement

$$\forall p \in \mathbb{P}_{\max} \exists q \in \mathbb{P}_{\max} (q \leq_{\mathbb{P}_{\max}} p \wedge q \in D) \tag{2}$$

which expresses that D is dense in $\mathbb{P}_{\max}$ is Π^1_2 in $\mathbb{P}_{\max} \oplus D$ in the codes, so that by hypothesis (d) there is some $q = (N_0; \in, J_0, A_0') \in V[g]$ belonging to the set of $\mathbb{P}_{\max}$–conditions coded by $(D^*)^g$ and such that $q <_{\mathbb{P}_{\max}} p_0$. Let

$$j_0 \colon ((H_{sat(\mathsf{NS})})^V; \in, (\mathsf{NS})^V, A_0) \to (N_0; \in, J_0, A_0')$$

such that p_0, $j_0 \in N_0$ witness that $q < p_0$.

Let

$$(S_\xi \colon \xi < (\kappa^+)^V) \in V[g]$$

be a partition of $(\kappa^+)^V$ into stationary sets. Working inside $V[g]$, we may then choose a generic iteration

$$(\mathcal{N}_i, \sigma_{i,j} \colon i \leq j \leq \kappa^+),$$

of $\mathcal{N}_0 = (N_0; \in, J_0, A_0') = q$ such that, writing $\mathcal{N}_{(\kappa^+)^V} = (N; \in, J, A')$,

$$\forall S \in (\mathcal{P}((\kappa^+)^V) \cap N) \setminus J \ \exists \xi < (\kappa^+)^V \ \exists \beta < (\kappa^+)^V \ S_\xi \setminus \beta \subset S.$$

(Cf. e.g. [1, proof of Lemma 5] and also the proofs of Theorems 1.10 and 1.11.) In particular,

$$J = (\mathsf{NS})^{V[g]} \cap N. \tag{3}$$

Writing

$$j = \sigma_{0,(\kappa^+)^V}(j_0) \colon ((H_{sat(\mathsf{NS})})^V; \in, (\mathsf{NS})^V, A_0) \to \sigma_{0,(\kappa^+)^V}(p_0) = (M_{(\kappa^+)^V}; \in, I, A'),$$

we thus also have that

$$I = J \cap M_{(\kappa^+)^V} = (\mathsf{NS})^{V[g]} \cap M_{(\kappa^+)^V}.$$

As V is (κ^+)–iterable in $V[g]$ by our hypothesis (a) and the proof of [13, Lemma 3.10], we may lift the generic iteration of $((H_{sat(\mathsf{NS})})^V; \in, (\mathsf{NS})^V, A_0)$ which gave rise to j_0 to a generic iteration of $(V; \in, (\mathsf{NS})^V, A_0)$. Let us write

$$\hat{\jmath} \colon V \to M$$

for the induced iteration map, so that $\hat{\jmath} \supset j$.

Now let $x \in p[T] \cap V[g]$ code $\mathcal{N}_0$, and let $(x, y) \in [T] \cap V[g]$. This gives

$$(x, \hat{\jmath}\text{"}y) \in [\hat{\jmath}(T)]. \tag{4}$$

By D^*–Bounded Martin's Maximum$^{*,++}$, the proof of Claim 2.10 will be finished if we show that the natural Σ_1 statement $\varphi(A_0, \dot{D}^*, \dot{I}_{\mathrm{NS}_{\omega_1}})$ expressing the existence of a $\mathbb{P}_{\max}$–condition in G coded by a real in D^* is an honestly consistent statement, in the sense of Definition 2.5. The proof that $\varphi(A_0, \dot{D}^*, \dot{I}_{\mathrm{NS}_{\omega_1}})$ is honestly consistent in the sense of Definition 2.5 is essentially as in the proofs of Theorems 1.10 and 1.11:

Let $F : \mathbb{R} \to \mathbb{R}$ be a universally Baire function in V, $\eta > \kappa$ a cardinal, $\overline{T}$ and $\overline{U}$ a pair of trees on $\omega \times 2^\eta$ witnessing the η^+–universal Baireness of F (with $F = p[\overline{T}]$), and set $T^* = \hat{\jmath}(\overline{T})$ and $U^* = \hat{\jmath}(\overline{U})$, so that $p[\overline{T}] = p[T^*]$ and $p[\overline{U}] = p[U^*]$.

In $V^{\mathrm{Col}(\omega, 2^\eta)}$ there is a $p[T^*]$–closed model $\mathfrak{A}$ such that $H^M_{\omega_2} \subseteq \mathfrak{A}$, every set in $(\mathcal{P}(\omega_1) \setminus \mathrm{N}S_{\omega_1})^M$ is stationary in $\mathfrak{A}$, and such that $\mathfrak{A}$ satisfies ZFC^- together with $\varphi(A_0, [\hat{\jmath}(T)], \mathrm{NS}_{\omega_1})$ (the existence of $\mathfrak{A}$ in $\mathrm{Col}(\omega, 2^\eta)$ is witnessed by some rank–initial segment of $V[g]$). By absoluteness, $\mathrm{Col}(\omega, \hat{\jmath}(2^\eta))$ forces over M that there is a $p[\hat{\jmath}(\overline{T})]$–closed model $\mathfrak{A}$ such that $H^M_{\omega_2} \subseteq \mathfrak{A}$, every set in $(\mathcal{P}(\omega_1) \setminus \mathrm{N}S_{\omega_1})^M$ is stationary in $\mathfrak{A}$, and such that $\mathfrak{A}$ satisfies ZFC^- together with $\varphi(A_0, [\hat{\jmath}(T)], \mathrm{NS}_{\omega_1})$. Finally, by elementarity of $\hat{\jmath}(T)$ we get that $V^{\mathrm{Col}(\omega, 2^\eta)}$ forces over V that there is a $p[\overline{T}]$–closed model $\mathfrak{A}$ such that $H^V_{\omega_2} \subseteq \mathfrak{A}$, every set in $(\mathcal{P}(\omega_1) \setminus \mathrm{N}S_{\omega_1})^V$ is stationary in $\mathfrak{A}$, and such that $\mathfrak{A}$ satisfies ZFC^- together with $\varphi(A_0, D^*, \mathrm{NS}_{\omega_1})$. □ (Theorem 2.8)

We are now going to prove $(A) \Longrightarrow (B)$ of Theorem 2.7. This will be arranged by varying the argument for [13, Theorem 10.99], cf. also the proof of [13, Theorem 10.127].

We shall use the following lemma to produce A–iterable $\mathbb{P}_{\max}$–conditions, where A is a set of reals. (Cf. [13, Definition 4.3] on the definition of the concept of "A–iterability.") The proof of [13, Lemma 4.40] presents a different method for producing A–iterable structures, but we thought that writing up the method for proving Lemma 2.12 would be of independent interest.

Lemma 2.12 *Suppose that $M_\omega^{\#\#}$ exists and is fully iterable. Let $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$. There is then some $x \in \mathbb{R}$ and some $\mathbb{Q} \in M_\omega^{\#\#}(x)$ which has the δ–c.c. and is of size δ in $M_\omega^{\#\#}(x)$, where δ is the least Woodin cardinal of $M_\omega^{\#\#}(x)$, such that if $g \in V$ is $\mathbb{Q}$–generic over $M_\omega^{\#\#}(x)$, then*

$$M_\omega^{\#\#}(x)[g]$$

is an A–iterable $\mathbb{P}_{\max}$–condition.

Proof. Let A be definable from $x \in \mathbb{R}$ and (finitely many) $\mathbb{R}$–indiscernibles inside $L(\mathbb{R})$. Let $\mathbb{Q} \in M_\omega^{\#\#}(x)$ be a standard forcing iteration of length δ to force both NS to be saturated as well as MA_{ω_1}, where δ is the least Woodin cardinal of $M_\omega^{\#\#}(x)$. We claim that if $g \in V$ is $\mathbb{Q}$–generic over $M_\omega^{\#\#}(x)$, then $M_\omega^{\#\#}(x)[g]$ is an A–iterable $\mathbb{P}_{\max}$–condition.

Let us write $M = M_\omega^{\#\#}(x)[g]$. We know from [13, Lemma 3.10] that M is generically iterable and is hence a $\mathbb{P}_{\max}$–condition. It thus remains to be seen that M is A–iterable.

The set $A \cap \mathcal{N}$ is uniformly definable over *any* z–mouse $\mathcal{N}$ with infinitely many Woodin cardinals and a top measure, where x is coded into $z \in H_{\omega_1}$, in the following way. Let $y \in A$ iff $L(\mathbb{R}) \models \varphi(y, x, \eta_0, \dots, \eta_{k-1})$, where $\eta_0 < \dots < \eta_k$ are $\mathbb{R}$–indiscernibles. Let $\mathcal{N}'$ result from $\mathcal{N}$ by iterating the top measure of $\mathcal{N}$ and its images $k+1$ times, and let $\kappa_0 < \dots < \kappa_k$ be the sequence of the critical points. Then

$$(5) \qquad y \in A \cap \mathcal{N} \iff \Vdash^{\mathcal{N}'}_{\mathrm{Col}(\omega, <\sup_n(\delta_n))} \models L_{\kappa_k}(\mathbb{R}^*) \models \varphi(y, x, \kappa_0, \dots, \kappa_{k-1}),$$

where $\delta_0 < \delta_1 < \dots$ are the Woodin cardinals of $\mathcal{N}$ (and thus also of $\mathcal{N}'$) and $\mathbb{R}^*$ denotes the collection of all reals which are added by proper initial segments of the forcing $\mathrm{Col}(\omega, < \sup_n(\delta_n))$ (cf. [12, p. 1663]). In particular, $A \cap \mathcal{N} \in \mathcal{N}$, and thus $A \cap M \in M$.

It remains to be seen that if

$$j \colon M \to N$$

is a generic iteration of M, then $j(A \cap M) = A \cap N$. Suppose not. Let $\zeta_1 < \zeta_2 < \dots$ be the Woodin cardinals of M (i.e., the Woodin cardinals of $M_\omega^{\#\#}(x)$ above $\delta + 1$). Let

$$j \colon M|(\delta^{+M}) \to N$$

be a generic iteration of $M|(\delta^{+M})$ with $j(A \cap M) \neq A \cap N$, and let M^* be an iterate of M via extenders with critical points and lengths between δ and ζ_1 such that j is generic over M^* for the extender algebra at ζ_1. Using (5), $M^*[j]$ can see that $j \colon M|(\delta^{+M}) \to N$ is a generic iteration with $j(A \cap M) \neq A \cap N$, and by pulling back the statement that there is such a generic iteration we thus get that in $M^{\mathrm{Col}(\omega, \zeta_1)}$ there is some generic iteration $j \colon M|(\delta^{+M}) \to N$ with $j(A \cap M) \neq A \cap N$.

However, inside M, $A \cap M$ is ζ_1^+–universally Baire, again using (5). Namely, we may let $T \in M$ be a tree of height ω searching for y, $\bar{M}$, k, h such that $k \colon \bar{M} \to (M|\sup_n(\zeta_n))^\# \in M$ is elementary, h is $\mathrm{Col}(\omega, k^{-1}(\zeta_1))$–generic over $\bar{M}$

with $y \in \bar{M}[h]$, and y is in $A \cap \bar{M}[h]$ as computed using the recipe (5) for $\mathcal{N} = \bar{M}[h]$. If $(y, \bar{M}, k, h) \in [T]$, then we write $y \in p[T]$. We also let $U \in M$ be defined in exactly the same way, except for that "y is in $A \cap \bar{M}[h]$" gets replaced by "y is not in $A \cap \bar{M}[h]$." If $(y, \bar{M}, k, h) \in [U]$, then we write $y \in p[U]$. The trees T and U are easily seen to witness that $A \cap M$ is ζ_1^+–universally Baire inside M.

Now let $j \colon M|(\delta^{+M}) \to N$ be a generic iteration inside $M^{\mathrm{Col}(\omega,\zeta_1)}$ with $j(A \cap M) \neq A \cap N$. We have that $A \cap M = p[T] \cap M$, and thus $j(A \cap M) = p[j(T)] \cap N = (p[j(T)] \cap M^{\mathrm{Col}(\omega,\zeta_1)}) \cap N$. However, $(p[j(T)] \cap M^{\mathrm{Col}(\omega,\zeta_1)}) = (p[T] \cap M^{\mathrm{Col}(\omega,\zeta_1)})$ by the fact that T, U witness that $A \cap M$ is ζ_1^+–universally Baire in M. Therefore $j(A \cap M) = p[T] \cap N = A \cap N$. Contradiction! $\square$ (Lemma 2.12)

We have to prove $(A) \Longrightarrow (B)$ of Theorem 2.7.

We assume that $M_\omega^{\#\#}$ exists and is fully iterable and also that $(*)$ holds true. Let us fix a set B of reals in $L(\mathbb{R})$ and let also $A \in H_{\omega_2}$. Let $\varphi(x, \dot{B}, \dot{I}_{\mathsf{NS}_{\omega_1}})$ be a Σ_1 formula in the language of set theory, augmented by predicates for B and for the non–stationary ideal on ω_1. Suppose that $\varphi(A, \dot{B}, \dot{I}_{\mathsf{NS}_{\omega_1}})$ is honestly consistent in the sense of Definition 2.5. We aim to show that $\varphi(A, \dot{B}, \dot{I}_{\mathsf{NS}_{\omega_1}})$ holds true in V.

Suppose not. We may assume without loss of generality that $A \subset \omega_1$ and in fact that A is $\mathbb{P}_{\max}$–generic over $L(\mathbb{R})$ (cf. [13, Theorem 4.60]). Let $\dot{A}$ be the canonical name for A. Now say that

$$p = (M, \in, I, a) \ \Vdash \neg\varphi(\dot{A}, \check{B}, \dot{I}_{\mathsf{NS}_{\omega_1}}), \tag{6}$$

where $p \in G_A = \{q = (N, \in, I', a') \in \mathbb{P}_{\max} \colon a' = A \cap \omega_1^N\}$. We shall derive a contradiction by finding some $q <_{\mathbb{P}_{\max}} p$ with $q \Vdash \varphi(\dot{A}, \check{B}, \dot{I}_{\mathsf{NS}_{\omega_1}})$.

By our hypothesis, the function $F \colon \mathbb{R} \to \mathbb{R}$ with $F(x) =$ (the canonical real code for) $M_\omega^{\#\#}(x)$, $x \in \mathbb{R}$, is universally Baire. Let $\mathfrak{A}$ be an F–closed witness to the fact that $\varphi(A, \dot{B}, \dot{I}_{\mathsf{NS}_{\omega_1}})$ is honestly consistent.

Let $M_\omega^{\#\#}(X) \in \mathfrak{A}$ be such that X is transitive and $(\mathcal{P}(\omega_1) \cap \mathfrak{A}) \cup \{(\mathsf{NS}_{\omega_1})^{\mathfrak{A}}\} \in X$. Let δ be the least Woodin cardinal of $M_\omega^{\#\#}(X)$, and let g be $\mathbb{Q}$–generic over $M_\omega^{\#\#}(X)$, where $\mathbb{Q}$ is, in $M_\omega^{\#\#}(X)$, a standard forcing iteration of size δ with the δ–c.c. forcing both that NS is precipitous and that MA_{ω_1} holds. By Lemma 2.12, inside $V^{\mathrm{Col}(\omega,2^{\aleph_1})}$ we have that

$$q = (M_\omega^{\#\#}(X)[g]; \in, \mathsf{NS}^{M_\omega^{\#\#}(X)[g]}, A)$$

is a B^*–iterable $\mathbb{P}_{\max}$ condition with $q <_{\mathbb{P}_{\max}} p$, and

$$q \models \varphi(A, B^*, \mathsf{NS}_{\omega_1}),$$

so that $q \Vdash \varphi(\dot{A}, \check{B}, \dot{I}_{\mathsf{NS}_{\omega_1}})$.

The assertion that there is such a q is now absolute between V and $V^{\mathrm{Col}(\omega,2^{\aleph_1})}$. We obtained a contradiction! $\square$ (Theorem 2.7)

It remains open whether $(*)$ can be forced over models of choice containing large cardinals or whether $(*)$ indeed follows from a forcing axiom. In [13, Theorem 10.70], Woodin proves that $(*)$ does not follow from $\mathrm{MM}^{++}(2^{\aleph_0})$. In [7] and [8], Paul Larson shows that $(*)$ does not follow from $\mathsf{MM}^{+\omega}$, and he asks whether $(*)$ follows from MM^{++} (cf. [8, Question 7.2]). Woodin asks whether $(*)$ can be forced from large cardinals as [13, Question (18) a), p. 924], cf. also [9, p. 2158].

Theorem 2.7 yields an obvious scenario for showing that MM^{++} implies $(*)$. Basically, one would have to show that if a Σ_1 statement φ with parameters as in A–$\mathsf{BMM}^{*,++}$ is honestly consistent in the sense of Definition 2.5, then φ can be forced by a stationary set preserving forcing. We don't know how to do that, though.

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