Ulrich Blau. ***Die Logik der Unbestimmtheiten und Paradoxien.*** Philosophische Impulse, vol. 8. Synchron Wissenschaftsverlag der Autoren, Heidelberg, 2008, 960 pp.

Soon after I had come to Munich in 1986 to study philosophy I heard a rumor that one of the professors there was working on a huge manuscript attacking all the paradoxes of logic which unfortunately was supposed to be too technical for a philosopher's digestion. In 1990, I finally got hold of a draft of this manuscript, and I started taking courses with its author, Ulrich Blau. These courses comprised the most impressive experience of my undergraduate career; Ulrich is an amazingly thorough, unorthodox, and deep thinker, an exhilarant platonist, and a hilarious controverter. At all times he has appeared to me as the incarnation of Husserl's slogan "Zu den Sachen selbst!" ("To the things themselves!") by being one of those rare examples of German philosophers who concentrate on discussing substantial issues rather than discussing other people's discussions. In his case the issues are: the logical analysis of natural language (with a stress on vagueness, intention, and quotation), the semantic and epistemic paradoxes (the liar sentence, the hangman paradox), and a set and class theoretical logic of reflection (with an "ultimate" extension of the field of real numbers into the transfinite and the infinitesimal). His approaches are always original and his solutions well-motivated, strong, and powerful. In his world view, the liar paradox, mathematical platonism, the formally unexpressible, and Nāgārjuna are tightly connected with each other. The Blau manuscript kept growing over the years, and it now finally appeared in print as ***Die Logik der Unbestimmtheiten und Paradoxien***. Fluent understanding of German is a prerequisite for being able to read this book, but a knowledge of logic is not, as it has a self-contained chapter on first order logic, called **L** here. There are people with more aggressive marketing strategies for their products than Ulrich is willing to use, so I can only hope that this will be another instance where "it is the stillest words that bring on the storm."

The current book includes the results of Ulrich Blau's previous one, ***Die dreiwertige Logik der Sprache***, de Gruyter, Berlin and New York, 1978, X + 275 pp., in which it is shown that the analysis of the phenomena of vagueness and non-referentiality of natural language produce a system **LN** of first order logic expanding **L** with three truth values *true*, *false*, and *indefinite* ("unbestimmt") and two forms of negation (corresponding to the two interpretations of "Pizarro did *not* find Eldorado" depending on whether we assume Eldorado to exist or not to exist). The new book also develops a logic **LQ** of quotation which expands **L** and which has quotation marks and two different kinds of variables, *de re* and *de dicto*, to range over objects and expressions, respectively. The system **LQ** is shown to contain arithmetic and is subject to the incompleteness phenomena. It is the most convincing and powerful logic of quotation I know to have appeared in the literature.

But I would like to have my review of ***Die Logik der Unbestimmtheiten und Paradoxien*** focus on his proposed solution of the semantic paradoxes and the set and class theories which get involved. A paradox appears if intuitive reasoning keeps entailing conclusions which contradict what is proven by a formal system. Blau's key idea is to allow the presence of a truth predicate in a formal language with arithmetic at the cost of making the process of reflection along an absolutely indefinite well-ordering Ω^* part of the formal system. His approach, first announced in his *Die Logik der Unbestimmtheiten und Paradoxien (Kurzfassung)*, ***Erkenntnis***, vol. 22, (1985), pp. 369–459, is somewhat reminiscent of "revisionist theories" of truth, but it is also much more advanced and superior to them, as none of the systems of revisionist theories which are in stock tries to catch up with the expressive power of our intuitive semantic reasoning about that system (like the one producing the revision sequences from attempts to find the extension of the truth predicate); but this is exactly what Blau works out, and he has answers where all of the revision theories rest in silence.

The liar paradox is given by the sentence φ = "This sentence is not true." We understand this sentence, and we reason that if it is false or just "meaningless," then it is true, and that if it is true, then it is false. This very process of verification, Blau says, assigns the truth value *open* ("offen" $\neq$ indefinite) to φ at the lowest level of reflection. Therefore, at the next level of reflection we verified φ to be *true*, and at the second next level of reflection we verified φ to be *false*, etc. If φ is true/false at the α^{th} level of reflection, then φ is false/true at the $\alpha + 1^{\mathrm{st}}$ level of reflection. Once we passed through all the finite levels of reflection or more generally through all the levels below some limit ordinal λ, we verified φ to be open at the ω^{th} or at the λ^{th} level of reflection, respectively. Blau argues convincingly (cf. pp. 451) that other proposed solutions to the semantic paradoxes lead into dead ends. His solution is a *tour de force* with a big amount of persuasive power. If this idea is to be integrated into a formal system, though, then the expressive power of the meta-language should be made available to the object language as well so that the latter may express statements like "At stage α, the sentence ψ is true." By the necessary well-foundedness of the verification process, at a given level α of reflection we only have access to the truth values of statements at earlier levels $\beta < \alpha$ and not to truth values that are only to be decided at higher levels $\beta > \alpha$. But doesn't the liar paradox then resurrect in new disguise? What about the sentence φ^* = "This sentence is open at *all* levels of reflection"? It is indeed open at all levels of reflection, but therefore, as we reason intuitively, it is true. The subtety here, though, is given by the word "all" in φ^*. We need to take a closer look at Blau's logic of reflection and the "ordinals" α which index levels of reflection and which are provided by set and class theories.

Three logics of reflection are presented in ***Die Logik der Unbestimmtheiten und Paradoxien***, namely **LR**, **MR**, and **KR**. The system **LR** expands both **LN** and **LQ**. The role of having quotation available will be to have a device for coding integers at hand, and the overall idea behind generalizing both **LN** and **LQ** is to arrive at a system which analyzes the entire natural language. In addition to (,), sentential connectives, quantifiers, variables *de re* and *de dicto*, function and relation symbols, **LR** has an indicator $\star$ (standing for the current level of reflection) and a symbol T^τ (for "is true at level τ") for each term τ or for τ being equal to $\star$. As arithmetic can be modelled already in **LQ**, a term τ may end up as being interpreted by an integer $n \in \mathbb{N}$ in which case T^τ will get its obvious meaning; $T^\star$ will mean "is true at the current level." (Formally, T^τ will be allowed as a unary sentential connective as well as a predicate.) As self-referentiality is present already in **LQ**, the system **LR** can formulate a sentence φ which is equivalent to $\neg T^\star\varphi$, i.e., **LR** can formulate the liar sentence. The logic **LR** has six truth values: *true*, *false*, *indefinite*, *open*, $\bar{\mathrm{T}}$ (not true, but open if false or indefinite), and $\bar{\mathrm{F}}$ (not false, but open if true or indefinite), and the set of levels of reflection is identical with $\omega = \mathbb{N}$. The semantics is defined in a natural way through verification rules. $T^\star\psi$ is true at level $n+1$ iff ψ is true at level n. An attempt to verify, falsify, or identify ψ as indefinite at a given level of reflection need not produce an answer in which case ψ will end up as being open, $\bar{\mathrm{T}}$, or $\bar{\mathrm{F}}$ at that level. The **LR**-version of the sentence φ^* discussed above can be written as a sentence $\varphi^*_{\mathbf{LR}}$ which is equivalent to $\forall n \in \mathbb{N} \neg O^n \varphi^*_{\mathbf{LR}}$. (Here, O, standing for "open," is definable in terms of T.) Whereas the liar paradox might be considered as "solved," this sentence $\varphi^*_{\mathbf{LR}}$ produces a new paradox: $\varphi^*_{\mathbf{LR}}$ is open at all levels n, and therefore it is intuitively true. Another new paradox is given by the simpler sentence $\forall n \in \mathbb{N}\ T^n 2+2=4$, which is open at all levels of **LR**, but $2+2=4$ is true at all levels, whence $\forall n \in \mathbb{N}\ T^n 2+2=4$

is also intuitively true. These paradoxes are to be resolved by introducing more levels of reflection: ω and beyond.

This is where the line of thought is forced to bring set and class theory into play. Blau develops set theory and a hierarchy of class theories $\mathcal{M}'^{\alpha}_{\mathbb{V}}$. $\mathcal{M}'^{0}_{\mathbb{V}}$ is set theory. For any α, $\mathcal{M}'^{\alpha}_{\mathbb{V}}$ results from $\mathcal{M}'^{\beta}_{\mathbb{V}}$, $\beta < \alpha$, by adding variables ranging over all classes which are definable in one the languages of $\mathcal{M}'^{\beta}_{\mathbb{V}}$, $\beta < \alpha$. Blau shows that $\mathcal{M}'^{\alpha}_{\mathbb{V}}$ is intertranslatable with $\mathcal{M}^{\alpha}_{\mathbb{V}}$, where for any α, $\mathcal{M}^{\alpha}_{\mathbb{V}}$ results from $\mathcal{M}^{\beta}_{\mathbb{V}}$, $\beta < \alpha$, by adding the truth predicate for the languages $\mathcal{M}'^{\beta}_{\mathbb{V}}$, $\beta < \alpha$ (and $\mathcal{M}^{0}_{\mathbb{V}} = \mathcal{M}'^{0}_{\mathbb{V}}$). He also discusses impredicative classes and the relevant philosophical issues of platonism with respect to sets and (predicative or impredicative) classes and of relativism vs. absolutism. Almost *en passant* he also presents a platonistic argument in favor of CH, the continuum hypothesis. (Cf. also his *Ein platonistisches Argument für Cantors Kontinuumshypothese*, ***Dialectica***, vol. 52 (1998), pp. 175–202.)

Now following Blau's notation, let Ω denote the class of all ordinals, and let $\Omega^\bullet$ denote the length of the least class-sized well-ordering which has Ω as a strict initial segment and which is closed under "ordinal" addition and multiplication. The system **MR** comes from **LR** (basically) by adding a constant $\overline{\alpha}$ for every $\alpha \in \Omega^\bullet$, and the system **KR** comes from **MR** by adding set and class theoretical truth predicates. The class of levels of reflection now becomes identical with $\Omega^\bullet$. The semantics is again defined in a natural way through verification rules. If λ is a limit, then $T^\star\psi$ is true at level λ if ψ is true at all but boundedly many levels $\alpha < \lambda$. The paradox given by $\varphi^*_{\mathbf{LR}}$ is then "solved" (the sentence will become true from level ω onward), but the new variant of φ^* is a sentence $\varphi^*_{\mathbf{LM/KR}}$ which is equivalent to $\forall\alpha \in \Omega^\bullet\ \neg O^\alpha \varphi^*_{\mathbf{LM/KR}}$ and which produces a new paradox: $\varphi^*_{\mathbf{MR/KR}}$ is indeed open at all levels α, and therefore it is intuitively true.

Blau goes on to argue that in order to resolve the paradox given by $\varphi^*_{\mathbf{LM/KR}}$, one needs an ultimate logic of reflection which expands say **MR** in that it does not come with a *fixed* class of levels of relection (like ω in the case of **LR** and $\Omega^\bullet$ in the case of **MR**) but with an *open totality* Ω^* of all levels of reflection which is "extendable indefinitely" ("unbegrenzt fortsetzbar"): the apparent paradox provided by φ^* will disappear if and only if we realize that the quantifier "for *all* levels of reflection ..." is as absolutely indefinite and formally incomprehensible as *the* longest (class sized) well ordering (cf. Thesis 15 on p. 132 and the discussion on p. 756). This move of ontological speculation in the light of the liar paradox constitutes the platonist's counterpart to the constructivist's conception of the reals as an "open totality" in the light of Cantor's diagonal argument. The progression Ω^*, for Blau, yields a shortcut from mathematics to a mysticism that is "as clear as day." Even though Ω^* is formally unexpressible, Blau develops a "transreal" number theory based on it.

Die Logik der Unbestimmtheiten und Paradoxien could be one of the most important contributions to the theory of semantic paradoxes of all times. It is a quarry of philosophical insights, of convincing results concerning the logical analysis of natural language, a compendium of interesting logical systems, a forceful attack to solve the semantic paradoxes, and, finally, it is also a beautiful piece of literature (I keep enjoying reading its wonderful dialogues!). It is to be hoped that Ulrich Blau will eventually receive the amount of attention that he truly deserves, and his proposals are overripe for an in-depth discussion in the logico-philosophical community.

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