YET ANOTHER CHARACTERIZATION OF REMARKABLE CARDINALS

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1. Remarkable games

We present a characterization of remarkable cardinals in terms of the existence of winning strategies in a class of Ehrenfeucht-Fraïssé-like games. This is in the spirit of [1, section 4].

Let κ be an infinite cardinal. We consider two two-player games.

The first one will be denoted by G^1_κ:

I	α		x_0		x_1		$\dots$	
II		λ, β		y_0		y_1		$\dots$

Rules: $\alpha > \kappa$, $\lambda < \beta < \kappa$, $\{x_0, x_1, \dots\} \subset V_\beta$, $\{y_0, y_1, \dots\} \subset V_\alpha$, $x_k \in V_\lambda \Longrightarrow y_k = x_k$, and for every formula φ in the language of set theory and for all $k < \omega$,

$$V_\beta \models \varphi(\lambda, x_0, \dots, x_{k-1}) \Longleftrightarrow V_\alpha \models \varphi(\kappa, y_0, \dots, y_{k-1}).$$

II wins a run of G^1_κ if she is not the first one to break any rule.

The second one will be denoted by G^{crit}_κ:

I	α		X_0		X_1		$\dots$	
II		λ, β		j_0		j_1		$\dots$

Rules: $\alpha > \kappa$, $\lambda < \beta < \kappa$, $X_0 \subset X_1 \subset \dots \subset V_\beta$, $j_0 \subset j_1 \subset \dots$, and for all $k < \omega$: $\mathrm{Card}(X_k) \leq \lambda$, $j_k \colon X_k \to V_\alpha$, $j_k \restriction X_k \cap V_\lambda = \mathrm{id}$, and for every formula φ in the language of set theory, for all $n < \omega$, and for all $x_0, \dots x_n \in X_k$,

$$V_\beta \models \varphi(\lambda, x_0, \dots, x_n) \Longleftrightarrow V_\alpha \models \varphi(\kappa, j_k(x_0), \dots, j_k(x_n)).$$

II wins a run of G^{crit}_κ if she is not the first one to break any rule.

Both games G^1_κ and G^{crit}_κ are open, hence determined by the Gale-Stewart Theorem.

Proposition 1.1. *Let κ be a cardinal. The following are equivalent.*

(1) *κ is remarkable.*
(2) *II has a winning strategy in G^1_κ.*
(3) *II has a winning strategy in G^{crit}_κ.*

Proof. (3) implies (2) is trivial.

Let's show (2) implies (1). Assume (2), and let σ be a winning strategy for player II, and suppose that G is $\mathrm{Coll}(\omega, < \kappa)$-generic over V. Let $\alpha > \kappa$, and let λ and β be σ's 1st reply in a play where I's 1st move is α. In $V[G]$, we fix an enumeration $\{b_i \mid i < \omega\}$ of V_β. Notice that, in $V[G]$, σ is still a winning strategy for player II, because the game is closed and there are no new finite sets in $V[G]$. So, by playing according to σ against the moves b_n of player I given by this fixed enumeration, player II is easily seen to produce an elementary embedding $j : V_\beta \to V_\alpha$, $j \in V[G]$, $\mathrm{crit}(j) = \lambda$, and $j(\lambda) = \kappa$. We verified (1).

Next, we show (1) implies (3). Let $\alpha > \kappa$. We may without loss of generality assume that $\mathrm{cf}(\alpha) > \kappa$, so that ${}^\kappa V_\alpha \subset V_\alpha$. Again suppose that G is $\mathrm{Coll}(\omega, < \kappa)$-generic over V. Then there are $\lambda < \beta < \kappa$ such that in $V[G]$ there is an elementary embedding $j^+ : V_{\beta+1} \to V_{\alpha+1}$ such that $\mathrm{crit}(j) = \lambda$ and $j(\lambda) = \kappa$. By elementarity we will have that $V_{\beta+1}$ knows that $\mathrm{cf}(\beta) > \lambda$, so that setting $j = j^+ \restriction V_\beta$, $j : V_\beta \to V_\alpha$ is an elementary embedding such that $\mathrm{crit}(j) = \lambda$, $j(\lambda) = \kappa$, and ${}^\lambda V_\beta \cap V \subset V_\beta$.

Let τ be a $\mathrm{Coll}(\omega, < \kappa)$-name for j. Let $p \in G$, $p \Vdash$ "τ is an elementary embedding from$\check{V}_\beta$ to $\check{V}_\alpha$ such that $\mathrm{crit}(\tau) = \check{\lambda}$ and $\tau(\check{\lambda}) = \check{\kappa}$. Let us look at the following auxiliary game G_κ^{aux}, played in V:

I	X_0		X_1		$\ldots$	
II		p_0, j_0		p_1, j_1		$\ldots$

Rules: $X_0 \subset X_1 \subset \ldots \subset V_\beta$, $p \geq p_0 \geq p_1 \ldots$, and for all $k < \omega$: $p_k \Vdash \tau \restriction \check{X}_k = \check{j}_k$.

It is easy to see that II has winning strategy in G_κ^{aux}. Suppose II plays X_k in a play of G_κ^{aux} in her k^{th} move where all the rules are obeyed so far. Write $q = p_{k-1}$ if $k > 0$ and $q = p$ if $k = 0$. Let G' be $\mathrm{Coll}(\omega, < \kappa)$-generic over V such that $q \in G'$. Write $j' = \tau^{G'}$. We have that $j' \restriction X_k \in V$ by the Kunen argument: if $f \colon \lambda \to X_k$ is surjective, $f \in V$, then $f \in V_\beta$ and for all $\xi < \lambda$, $j'(f(\xi)) = j'(f)(\xi)$. There is then some $q' \leq q$ such that $q' \Vdash \tau \restriction \check{X}_k = (j' \restriction X_k)\check{}$, and we may let II reply to X_k by playing a pair p_k, j_k in a way that she keeps obeying the rules.

But any strategy σ^* for II in G_κ^{aux} easily yields a strategy for II in the original game G_κ^{crit} for a play where I's 1st move is α and II's 1st reply is λ, β, and II then follows σ^* but hides her side moves $p_0, p_1, \ldots$ □

References

[1] Bagaria, J., Gitman, V., and Schindler, R., *Generic Vopěnka's Principle, Remarkable cardinals, and the weak Proper Forcing Axiom.* Archive for Mathematical Logic, Volume 56, Issue 1-2, 2017, pp. 1-20.

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