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# Woodin's $\mathbb{P}_{max}$ Axiom (*), Martin's Maximum, and the Inner Model Program

Hilbert's First Problem revisited

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MM
Mathematics

- Georg Cantor (1873): While there are as many algebraic real numbers as there are natural numbers, there are in total more real numbers than natural numbers.

**Definition**

Cantor's Continuum Hypothesis, CH: If $A \subset \mathbb{R}$ is uncountable, then $A$ has as many elements as there are real numbers, or: $2^{\aleph_0} = \aleph_1$.

Cantor (1878): [...] so fragt es sich, in *wie viel* [...] Klassen die linearen Mannigfaltigkeiten zerfallen, wenn Mannigfaltigkeiten von gleicher Mächtigkeit in eine und dieselbe Klasse, Mannigfaltigkeiten von verschiedener Mächtigkeit in verschiedene Klassen gebracht werden. Durch ein Induktionsverfahren, auf dessen Darstellung wir hier nicht näher eingehen, wird der Satz nahe gebracht, daß die Anzahl der [...] sich ergebenden Klassen [...] eine endliche und zwar, daß sie gleich *Zwei* ist.

- David Hilbert's First Problem (1900): Show that CH is true!

In his talk at the IMC in Paris, Hilbert says: Die Untersuchungen von Cantor über solche Punktmengen machen einen Satz sehr wahrscheinlich, dessen Beweis jedoch trotz eifrigster Bemühungen bisher noch Niemanden gelungen ist; dieser Satz lautet: Jedes System von unendlich vielen reellen Zahlen d. h. jede unendliche Zahlen- (oder Punkt)menge ist entweder der Menge der ganzen natürlichen Zahlen 1, 2, 3, ... oder der Menge sämmtlicher reellen Zahlen und mithin dem Continuum, d. h. etwa den Punkten einer Strecke aequivalent [...]

# New Axioms

- Kurt Gödel (1938): CH cannot be refuted in ZFC.

In 1947, Gödel wrote: [...] one may on good reason suspect that the role of the continuum problem in set theory will be this, that it will finally lead to the discovery of new axioms which will make it possible to disprove Cantor's conjecture.

- Paul Cohen (1963): CH cannot be proven in ZFC.

In 1966, Cohen wrote: A point of view which the author feels may eventually come to be accepted is that CH is obviously false. [...] $\aleph_1$ is the set of countable ordinals and this is merely a special and the simplest way of generating a higher cardinal. The set C [the continuum] is, in contrast, generated by a totally new and more powerful principle, namely the Power Set Axiom. It is unreasonable to expect that any description of a larger cardinal which attempts to build up that cardinal from ideas deriving from the Replacement Axiom can ever reach C.

# New axioms, cont'd

Over the years, various sets of natural axioms emerged which decide questions which were left open by ZFC.

- **Large cardinal axioms**?: Lévy-Solovay (1967) showed that CH is still independent from ZFC plus large cardinal hypotheses.
- **Determinacy hypotheses**?: Mycielski-Steinhaus (1962) showed that AD (the full Axiom of Determinacy) contradicts the axiom of choice. AD can thus only hold in inner models which don't contain *all* sets of reals. Moreover, Davis (1964) shows that AD implies the perfect set theorem.

- **Constructibility** (from Gödel's $L$ via core models to Woodin's "ultimate-$L$"): Generalizing Gödel's construction of $L$, various people (Solovay, Jensen, Dodd, Mitchell, Steel, etc.) have built *inner models* with substantial large cardinals (up to, roughly, a Woodin limit of Woodin cardinals). In all these models, Jensen's $\diamondsuit$ (and hence CH) holds true for the same reason.
  Steel, Woodin, Sargsyan: The HODs of determinacy models all seem to fall into the category of such inner models. Woodin (2023): They all satisfy UA, the ultrapower axiom. Woodin formulated "$V$ = ultimate-$L$" as an axiom to say that $V$ locally looks like the HOD of a determinacy model.
  **Open questions.** Can canonical inner models have suppercompact cardinals? Can HODs of determinacy models have rank-into-rank cardinals?

M. Magidor: If a mathematical object can be imagined in a reasonable way, then it exists!

Forcing axioms, or more generally: Maximality principles, try to formalize this approach.

A delicate point: do you want to maximize the $\Pi_2^{H_{\omega_2}}$ or the $\Sigma_2^{H_{\omega_2}}$ theory? Can't have both. CH is a $\Sigma_2^{H_{\omega_2}}$ statement, GCH may be construed as an instantiation of maximality (M. Foreman). The situation with respect to $\Pi_2^{H_{\omega_2}}$ maximality, though, is better understood than the one with respect to $\Sigma_2^{H_{\omega_2}}$ maximality.

## Definition

Foreman-Magidor-Shelah (1988): Formulated Martin's Maximum, MM, a strengthening of Martin's Axiom, MA: If the forcing $\mathbb{P}$ preserves stationary subsets of $\omega_1$ and if $\mathcal{D}$ is a family of $\aleph_1$ many sets which are all dense in $\mathbb{P}$, then there is a $\mathcal{D}$-generic filter.

MM gives many natural answers to questions which are undecidable on the basis of ZFC, e.g.:

- There is a non-free Whitehead group (Shelah 1974).
- Kaplansky's Conjecture holds true (Solovay-Woodin 1976).
- Every automorphism of the Calkin algebra of a separable Hilbert space is inner (Farah 2011).
- Friedman's Problem (Foreman-Magidor-Shelah 1988).

The classical way to force MM is start with a model of ZFC plus a supercompact cardinal, $\delta$, and perform a semi-proper iteration of length $\delta$. A reflection principle will hold in the generic extension which verifies full MM. In fact, a strengthening of MM may be arranged to hold in the extension:

**Definition**

Foreman-Magidor-Shelah (1988): Martin's Maximum$^{++}$, MM$^{++}$: If the forcing $\mathbb{P}$ preserves stationary subsets of $\omega_1$, if $\mathcal{D}$ is a family of $\aleph_1$ many sets which are all dense in $\mathbb{P}$, and if $\{\tau_i : i < \omega_1\}$ is a collection of names for stationary subsets of $\omega_1$, then there is a $\mathcal{D}$-generic filter such that $\tau_i^g = \{\xi : \exists p \in g\ p \Vdash \xi \in \tau_i\}$ is stationary for each $i < \omega_1$.

## Definition

Woodin (1990's): Formulates $(*)$, a maximality principle for sets of size $\aleph_1$: The Axiom of Determinacy holds in $L(\mathbb{R})$, and there is a filter $g \subset \mathbb{P}_{\text{max}}$ which is generic over $L(\mathbb{R})$ such that $H_{\omega_2} \subset L(\mathbb{R})[g]$.

$(*)$ is complete (in $\Omega$-logic) with respect to questions about $H_{\omega_2}$:

- $\boldsymbol{\delta}^1_2 = \omega_2$.
- $\psi_{\text{AC}}$ and $\phi_{\text{AC}}$, variants of Friedman's Problem.
- Admissible club guessing and the club bounding principle.

- W.H. Woodin showed that $(*)$ is $\Pi_2^{H_{\omega_2}}$ maximal: in the presence of large cardinals, if a given $\Pi_2^{H_{\omega_2}}$ statement is $\Omega$ consistent, then that statement is $\Omega$ provable from ZFC plus $(*)$.

The classical way to force $(*)$ is start with a model of ZF plus $V = L(\mathbb{R})$ plus AD (the Axiom of Determinacy) and force with $\mathbb{P}_{\text{max}}$.

What about the size of the continuum?

**Theorem**

*Foreman-Magidor-Shelah (1988) and Woodin (1990's): Both* MM *and* $(*)$ *imply that* $2^{\aleph_0} = \aleph_2$.

MM and $(*)$ were known to have many consequences in common, but they were also known to not follow from each other.

Open questions since the mid 1990's: What is the relation between MM and $(*)$? Does Martin's Maximum$^{++}$ imply $(*)$? Can $(*)$ be forced over a model of ZFC with a large cardinal? Is (*), like MM, consistent with all large cardinals?

These questions may also construed as a test question about Woodin's $\Omega$ conjecture.

**Definition**

The $\Omega$ conjecture states that if a local (i.e, $\Sigma_2$) statement is $\Omega$ consistent, then it may be forced over $V$ from large cardinals.

$(*)$ has been known to be $\Omega$ consistent. The $\Omega$ conjecture was formulated by W.H. Woodin in the context of studying $\mathbb{P}_{\mathsf{max}}$ and $(*)$.

# A theorem joint with David Asperó

**Theorem**

*D. Asperó and R. Schindler (2019): Martin's Maximum$^{++}$ implies the $\mathbb{P}_{\max}$ axiom $(*)$.*

This result appeared in the May 2021 issue (Volume 193, no. 3, pp. 793-835) of the Annals of Mathematics.

**Proof.**

Fix $A \subset \omega_1$. Let $D \subset \mathbb{P}_{\max}$ be dense, $D \in L(\mathbb{R})$. By $\mathrm{MM}^{++}$, it suffices to show that there is a forcing $\mathbb{P}$ such that

- $\mathbb{P}$ preserves stationary subsets of $\omega_1$, and
- $\mathbb{P}$ forces that there is a $\mathbb{P}_{\max}$ condition $p \in D^{V^{\mathbb{P}}}$ together with a generic iterate $p^*$ of $p$ such that $a^{p^*} = A$ and $I^{p^*} = \mathrm{NS}_{\omega_1}^{V^{\mathbb{P}}} \cap p^*$.

Such a $\mathbb{P}$ may be construced as an "$\mathcal{L}$-forcing" (partially building upon methods developed by Ronald Jensen). The conditions in $\mathbb{P}$ give finitely much information about the objects to be added plus information about "virtual side conditions." □

The entirely new feature is that the side conditions of $\mathbb{P}$ aren't in the ground model.

$(*)$ is *equivalent* to a schema of $\Omega$ consistent $\Pi_2^{H_{\omega_2}}$ statements in the language of set theory augmented by predicates for $\mathsf{NS}_{\omega_1}$ as well as sets of reals in $L(\mathbb{R})$.

Moreover, by our proof, $(*)$ is in fact - in the presence of large cardinals - also *equivalent* to a bounded form of $\mathsf{MM}^{++}$.

Our proof shows that every *honestly consistent* statement which is $\Sigma_1$ in predicates from $H_{\omega_2} \cup \{\mathsf{NS}_{\omega_1}\} \cup (\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$ may be forced by a stationary set preserving forcing.

Our proof shows that if an $M_1^{\#}$-closed inner model of BMM which is correct about stationary subsets of $\omega_1$ believes a given $\Sigma_2^{H_{\omega_2}}$ statement $\varphi$, then $\varphi$ is true in $V$.

My proof with Asperó confirms the $\Omega$ conjecture in an important case.

Its proof might open up a scenario for actually proving Woodin's $\Omega$ conjecture.

However, $\Pi_2^{H_{\omega_2}}$ is like $\Sigma_1^{H_{\omega_2}}$ in an arbitrary parameter from $H_{\omega_2}$, whereas dealing with $\Sigma_2$ needs one to take care of an additional parameter.

The $\Omega$ conjecture can be showed from a version of UBH, the Unique Branch Hypothesis. Earlier proofs of "$(*)$ may be forced" also made use of versions of UBH (plus $\mathrm{cf}(\theta(\text{u.B. sets})) > \omega_1$).

## Definition

Woodin: $(*)^{++}$: There is a model $L(\mathbb{R}, \Gamma)$ of AD and a filter $g \subset \mathbb{P}_{\max}$ which is generic over $L(\mathbb{R}, \Gamma)$ such that $\mathcal{P}(\mathbb{R}) \subset L(\mathbb{R}, \Gamma)[g]$.

Woodin: All the known models of MM violate $(*)^{++}$, as $(*)^{++}$ implies that $\theta^{L(\mathbb{R},\Gamma)} = \aleph_3$.

Open question: Is $(*)^{++}$ compatible with MM?

This question basically asks if $\theta(\text{u.B. sets})) = \omega_3$ may be forced from large cardinals. It is thus related to the question about definable counterexamples to the continuum hypothesis.

# Extensions of $(*)$, cont'd

By work of S. Todorcevic, large cardinals imply that a selective ultrafilter on $\omega$ is $\mathcal{P}(\omega)/\text{fin}$-generic over $L(\mathbb{R})$. This has been generalized by others, e.g. Chodounský-Zapletal.

$(*)^{++}$ is equivalent to the statement that there is a model $L(\mathbb{R}, \Gamma)$ of AD and a filter $g \subset \mathbb{P}_{\max}$ which is generic over $L(\mathbb{R}, \Gamma)$ such that *every* ultrafilter on $\omega$ is in $L(\mathbb{R}, \Gamma)[g]$.

# Duality

My result with Asperó connects the two approaches:

- produce an interesting ZFC model by forcing over a ZFC model with large cardinals, and
- produce an interesting ZFC model by forcing over a ZF model of determinacy

with one another.
By work of Larson, Sargsyan, Woodin, and others, bounded fragments/implications of $\mathrm{MM}^{++}$ may be forced over determinacy models.
By work of A. Lietz and myself, $(*)$ may be forced over a model of ZFC plus an inaccessible limit of $\kappa^{++}$ supercompacts.
Open questions: Can $\mathrm{MM}^{++}$ be forced over a determinacy model? Can $(*)$ be forced over a ZFC model with an inaccessible limit of Woodin cardinals?

In subsequent work, my student Andreas Lietz proved the $\Omega$ conjecture in another important case.

**Definition**

(A. Lietz) Q-Maximum (QM) is the statement: There if some $f : \omega_1 \to \mathsf{HC}$ witnessing $\diamondsuit(\omega_1^{<\omega})$ such that Martin's Maximum holds true with respect to all forcings which preserve $f$.

Lietz shows that QM may be forced from a supercompact limit of supercompact cardinals and that "$\mathrm{NS}_{\omega_1}$ is $\omega_1$ dense" follows from QM. In particular, "$\mathrm{NS}_{\omega_1}$ is $\omega_1$ dense" may be forced over a ZFC model with large cardinals (in fact, by a semi-proper forcing), answering a long-standing open question of W.H. Woodin.

It is an empirical fact that the most sophisticated extensions of ZFC which decide the value of the continuum prove $2^{\aleph_0} \in \{\aleph_1, \aleph_2\}$.

Challenges for future research:

- Explore "$V$ = ultimate-$L$." Explore GCH as (a consequence of) a maximality principle.
- Embed MM$^{++}$ into a "complete" theory of $V$. Further develop the theory of "higher forcing axioms."
- In the light of the results of Goldstern-Kellner-Shelah, develop well-justifiable theories which prove that $2^{\aleph_0} > \aleph_2$ and which possibly even produce effective counterexamples to $2^{\aleph_0} \leq \aleph_2$ or which give pairwise different values to as many entries in Cichoń's diagram as possible.

144 years after Cantor formulated it, the Continuum Problem remains one of *the* driving forces of set theoretical research.