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Varsovian models II, cont'd Dec 2017

We aim to verify that $\mathcal{V}_1$ is the mantle of $M = M_{swsw}$. It suffices to prove that $\mathcal{V}_1$ is contained in any ground.

Let us fix a ground W for M. There is then some θ s.t. $W[g] = M[g']$ for some g, g' where g is $Col(\omega, \theta)$-generic over W and g' is $Col(\omega, \theta)$-generic over M. It suffices to prove that $\mathcal{V}_0 \subset HOD^{M[g']}$.

We first aim to show $\mathcal{M}^1_\infty \subset HOD^{M[g']}$.

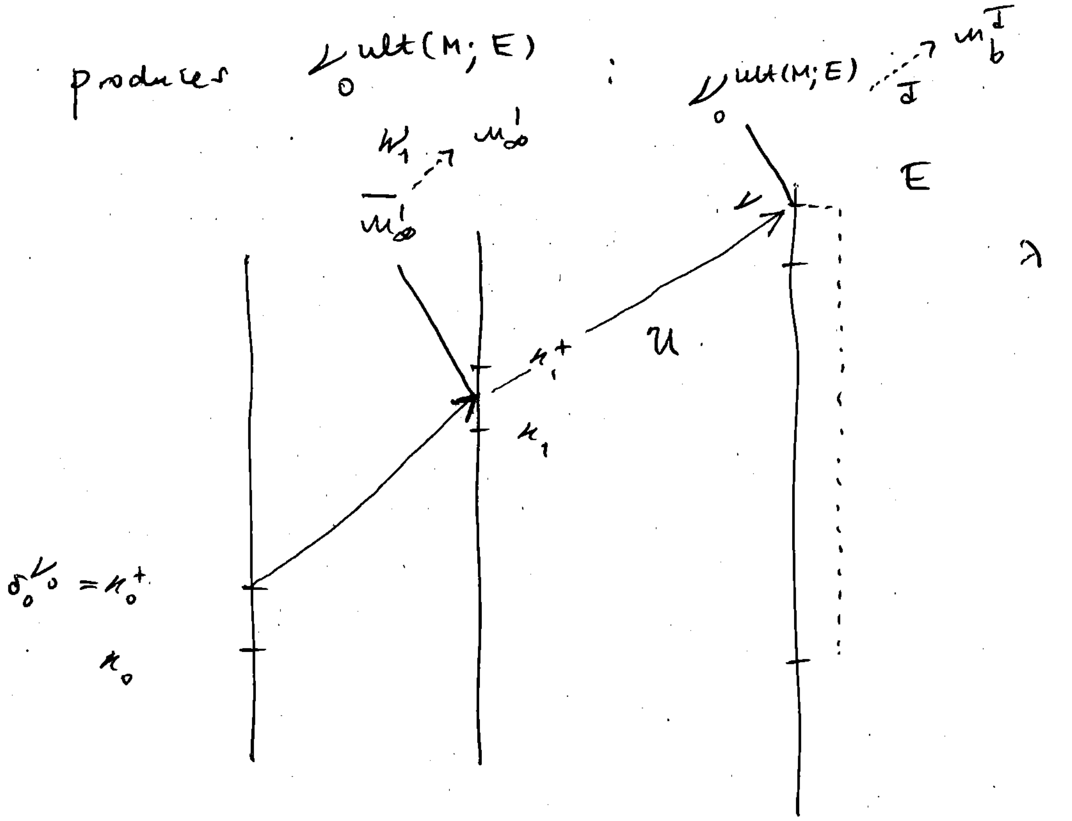
Let $\nu \gg \max\{\theta, \kappa_1^+\}$ such that $E = E^M_\nu$ is a (total) extender with critical point κ_0.

In V, let $\mathcal{W}_0 {}^\frown \mathcal{W}_1$ be the normal tree on $\mathcal{V}_0$ which produces $\mathcal{M}^1_\infty$, where $\mathcal{W}_0$ lives below $\delta_0^{\mathcal{V}_0}$ and $\mathcal{W}_1$ only uses extenders with critical point above the image of $\delta_0^{\mathcal{V}_0}$.

Let us write $\overline{\mathcal{M}^1_\infty}$ for $\mathcal{M}_\infty^{\mathcal{W}_0}$.

$\overline{\mathcal{M}}^1_\infty$ of course exists in M, it is the direct limit of all iterates of $\mathcal{L}_0$ via trees $\mathcal{T}$ on $\mathcal{L}_0$ s.t. $\mathcal{T} \in M|\kappa_1$, $\mathcal{T}$ lives on $\mathcal{L}_0|\delta_0^{\mathcal{L}_0}$, and $[0,\infty]_{\mathcal{T}}$ does not drop.

In M, there is a tree $\mathcal{U}$ on $\mathcal{M}^1_\infty$ which produces $\mathcal{L}_0^{ult(M;E)}$:



We may look at $ult(\mathcal{M}^1_\infty ; \pi^{\mathcal{U}}_{0\infty} \restriction \delta_0^{\overline{\mathcal{M}}^1_\infty})$, which is an iterate of $\mathcal{M}^1_\infty$ via $\mathcal{U}$ (i.e., by having $\mathcal{U}$ act on $\mathcal{M}^1_\infty$ rather than $\overline{\mathcal{M}}^1_\infty$). We may reorganize $\mathcal{W}_1 {}^\frown \mathcal{U}$-on-$\mathcal{M}^1_\infty$ as $\mathcal{U}$-on-$\overline{\mathcal{M}}^1_\infty {}^\frown \mathcal{I}$ which gives a tree

$\mathcal{I}$ on $\mathcal{L}_0^{\mathrm{ult}(M;E)}$ producing $\mathrm{ult}(\mathcal{M}_\infty^1; \pi_{0\infty}^{\kappa} \restriction \delta_0^{\overline{\mathcal{M}_\infty^1}})$.

Let us actually write $\mathcal{I}$ for this tree except for its final branch, and let b be its final branch, so that

$$(*) \qquad \mathrm{ult}(\mathcal{M}_\infty^1; \pi_{0\infty}^{\kappa} \restriction \delta_0^{\overline{\mathcal{M}_\infty^1}}) = \mathcal{M}_b^{\mathcal{I}} \quad ;$$

$\mathcal{I} \in M$, but $b \notin M$. However, $\mathcal{M}_b^{\mathcal{I}} \in M$ by $(*)$.

$\mathcal{W}_1$ uses only extenders with critical points above $\kappa_0^{+\overline{\mathcal{M}_\infty^1}}$, so that $\mathcal{I}$ only uses extenders with critical points above $\kappa_0^{+\mathcal{L}_0^{\mathrm{ult}(M;E)}}$.

But $\mathrm{ult}(M;E)$ is a generic extension of $\mathcal{L}_0^{\mathrm{ult}(M;E)}$ via Bukowský, so that $\mathcal{I}^\frown b$ lifts to a tree, $\mathcal{I}^{*\frown} b$, on $\mathrm{ult}(M;E)$. We have that $\mathcal{M}_b^{\mathcal{I}^*}$ is a generic extension of $\mathcal{M}_b^{\mathcal{I}}$ via Bukowský.

By $(*)$ and Lemma 27, $\mathcal{M}_b^{\mathcal{I}}$ is fully iterable in M and in fact inside $M[g']$;

this implies that $\mathcal{M}_b^{\mathcal{I}*} = \mathcal{M}_b^{\mathcal{I}}[M|\nu]$ is fully iterable in $M[g']$ with respect to trees which lift trees on $\mathcal{M}_b^{\mathcal{I}}$. Also, $\mathcal{M}_b^{\mathcal{I}} = \mathcal{L}_0^{\mathcal{M}_b^{\mathcal{I}*}}$. $\mathcal{M}_b^{\mathcal{I}*}$ doesn't have any indices for extenders between ν and $\kappa_0^{+\mathcal{L}_0^{\mathrm{ult}(M;E)}}$. Hence:

(**) $\mathcal{M}_b^{\mathcal{I}*}$ is iterable in $M[g']$ with respect to normal trees ~~on~~ which use indices $> \nu$.

If $\mathcal{U}^*$ is a tree as in (**), $\mathcal{U}^*$ on $\mathcal{M}_b^{\mathcal{I}*}$, then we have the following picture:

$$\begin{tikzcd}
\mathcal{M}^1_\infty \arrow[r, "\text{it. in } M", "\text{via } \mathcal{U}"'] & \mathcal{M}_b^{\mathcal{I}} \arrow[r, dashed, "\mathcal{U}^*\text{, constr. on } \mathcal{M}_b^{\mathcal{I}}"] \arrow[d, phantom, "\shortparallel"] & \mathcal{M}_\infty^{\mathcal{U}^*\text{-on-}\mathcal{M}_b^{\mathcal{I}}} \arrow[d, phantom, "\shortparallel"] \\
& \mathcal{L}_0^{\mathcal{M}_b^{\mathcal{I}*}} \arrow[d, phantom, "\cap"] & \mathcal{L}_0^{\mathcal{M}_\infty^{\mathcal{U}^*}} \arrow[d, phantom, "\cap"] \\
& \mathcal{M}_b^{\mathcal{I}*} \arrow[r, "\mathcal{U}^*"'] & \mathcal{M}_\infty^{\mathcal{U}^*}
\end{tikzcd}$$

Hence in $M[g']$, $\mathcal{M}_\infty^{\mathcal{U}^*\text{-on-}\mathcal{M}_b^{\mathcal{I}}} = \mathcal{L}_0^{\mathcal{M}_\infty^{\mathcal{U}^*}}$ is an iterate of $\mathcal{M}^1_\infty$.

Now inside $M[g']$ look at the (set sized) collection of all "candidates" $\bar{M}$ with the same first order properties as M.

For each such $\bar{M}$, we get a version of $\mathcal{M}_b^{\mathcal{I}*}$ as we did above for M itself; let us write $N(\bar{M})$ for this version of $\mathcal{M}_b^{\mathcal{I}*}$ (in particular, $\mathcal{M}_b^{\mathcal{I}*} = N(M)$). For each $\bar{M}$, we get

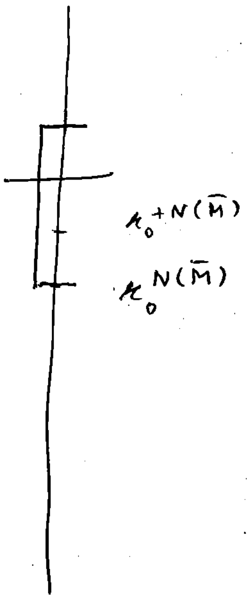
$$\begin{array}{ccc} (\mathcal{M}_\infty^1)^{\bar{M}} & \xrightarrow{\text{it. in } N} & V_0^{N(\bar{M})} \\ & & \cap \\ & & N(\bar{M}) \end{array},$$

cf. the picture on p. 134, and $N(\bar{M})$ is iterable in $M[g']$ with respect to normal trees which use indices $> \nu(\bar{M})$, where $\nu(\bar{M})$ is as ν was for M, cf. $(**)$.

We now want to "coiterate" all $N(\bar{M})$ in a way which will guarantee that the V_0 of all those iterates will be the same.

$E_0 = E_0(\bar{M})$

$\eta = \eta(\bar{M})$



Let $E_0(\bar{M})$ be the least total measure of $N(\bar{M})$ with critical point $\kappa_0^{N(\bar{M})}$, and let $\eta > \kappa_0^{+N(\bar{M})}$, $\eta < lh(E_0(\bar{M}))$ be a Woodin cardinal in $N(\bar{M}) \mid lh(E_0(\bar{M}))$, write $\eta = \eta(\bar{M})$.

By the versions of $(**)$, we may simultaneously iterate all $N(\bar{M})$ in the windows $(\kappa_0^{+N(\bar{M})}, \eta(\bar{M}))$ to make initial segments of all $N(\bar{M})$ generic over all iterates of $N(\bar{M})$. Let us write $\mathcal{U}_0^*(\bar{M})$ for the tree on $N(\bar{M})$ thus produced. I.e., for each pair $\bar{M}, \bar{M}'$ of "candidates," $\bar{M}' \mid \delta(\mathcal{U}_0^*(\bar{M}))$ is generic over

$\cancel{\eta}$

~~image of $q(\bar{E})$ under the it. [illegible]~~

$m_\infty^{\mathcal{U}_0^*(\bar{M})}$ for the extender algebra at

$\delta(\mathcal{U}_0^*(\bar{M}))$, and $\delta(\mathcal{U}_0^*(\bar{M})) = \delta(\mathcal{U}_0^*(\bar{M}'))$.

Once all $\mathcal{U}_0^*(\bar{M})$ are produced, we produce trees $\mathcal{U}_1^*(\bar{M})$ by a comparison process. Having produced all $\mathcal{U}_1^*(\bar{M}) \restriction (\alpha+1)$, we let $E_\alpha^{\mathcal{U}_1^*(\bar{M})} = E_\nu^{m_\alpha^{\mathcal{U}_1^*(\bar{M})}}$, where ν is least such that there is a pair of candidates, $\bar{M}, \bar{M}'$, s.t.

the fattening of $E_\nu^{m_\alpha^{\mathcal{U}_1^*(\bar{M})}}$ to $m_\alpha^{\mathcal{U}_1^*(\bar{M})} |\nu\, [\bar{M}' | \delta(\mathcal{U}_0^*(\bar{M}))]$

$\neq$ the fattening of $E_\nu^{m_\alpha^{\mathcal{U}_1^*(\bar{M}')}}$ to $m_\alpha^{\mathcal{U}_1^*(\bar{M}')} |\nu\, [\bar{M} | \delta(\mathcal{U}_0^*(\bar{M}'))]$,

or there is a candidate $\bar{M}$ s.t.

$E_\nu^{m_\alpha^{\mathcal{U}_1^*(\bar{M})}}$ has critical point $\kappa_0^{N(\bar{M})}$,

AND $\nu > \delta(\mathcal{U}_0^*(\bar{M}))$ (in both cases).

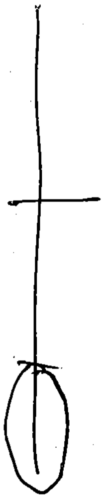
This works and terminates.

Let us write $\mathcal{U}^*(\bar{M}) = \mathcal{U}_0^*(\bar{M})^\frown \mathcal{U}_1^*(\bar{M})$.

By the rules for forming $\mathcal{U}_1^*(\bar{M})$, the common $\delta(\mathcal{U}_0^*(\bar{M}))$ will not be overlapped in any $m_\infty^{u^*(\bar{M})}$, and

$$(***) \qquad m_\infty^{u^*(\bar{M})}[\bar{M}' \mid \delta(u_0^*(\bar{M}))] = m_\infty^{u^*(\bar{M}')}[\bar{M} \mid \delta(u_0^*(\bar{M}'))]$$

for every pair $\bar{M}, \bar{M}'$ of "candidates."



It is now straightforward to verify that by (***),

$$(\dagger) \qquad \nu_0^{m_\infty^{u^*(\bar{M})}} = \nu_0^{m_\infty^{u^*(\bar{M}')}}$$

for every pair $\bar{M}, \bar{M}'$ of "candidates," Let us write $\tilde{\nu}$ for this common value of $\nu_0^{m_\infty^{u^*(M)}}$. By the picture on p. 134,

$M[g']$ can see that $\tilde{\Sigma}$ is an iterate of $(\mathcal{M}^!_\infty)^{\bar{M}}$ for each "candidate" $\bar{M}$.

Moreover, of course, $\tilde{\Sigma}$ "$\in$" $M[g']$.

This gives that

$$\mathcal{M}^!_\infty = (\mathcal{M}^!_\infty)^M = \mathrm{Hull}^{\tilde{\Sigma}}(\Gamma),$$

where Γ is a proper class of ordinals consisting of fixed points of all iteration maps given by all $(\mathcal{M}^!_\infty)^{\bar{M}} \longrightarrow \tilde{\Sigma}$, Γ definable in $M[g']$:

This shows that $\mathcal{M}^!_\infty$ is a definable inner model of $M[g']$, so that $\mathcal{M}^!_\infty \subset W$ (cf. p. 131).

It is now also easy to see that $(\dagger)$ and $(***)$ imply that

$$(\mathcal{M}^!_\infty)^{\mathcal{M}^{u*}_\infty(\bar{M})} = (\mathcal{M}^!_\infty)^{\mathcal{M}^{u*}_\infty(\bar{M}')} \quad \text{and}$$

$$(p \mapsto p^{**})^{\mathcal{M}^{u*}_\infty(\bar{M})} = (p \mapsto p^{**})^{\mathcal{M}^{u*}_\infty(\bar{M}')}$$

for all pairs $\bar{M}, \bar{M}'$ of "candidates."

This then implies that $p \mapsto p^{**}$ is definable in $M[g']$. We verified:

<u>Lemma</u> 29. $\mathcal{L}_1$ is the mantle of M.