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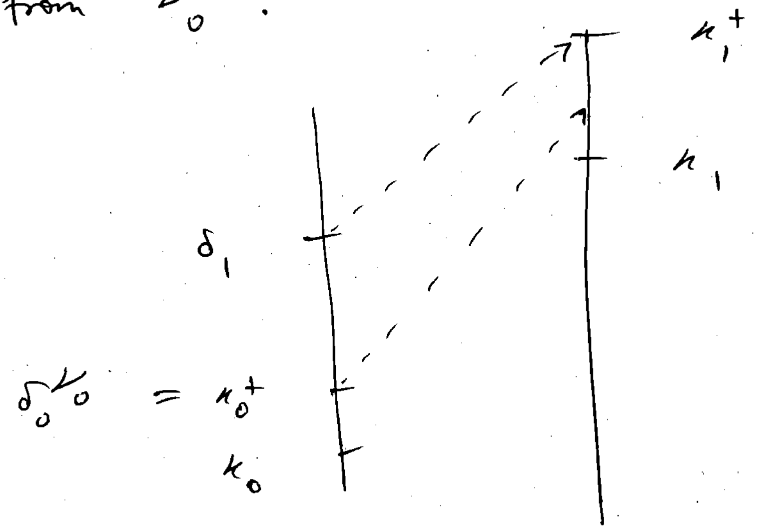
Varsovian models, II, cont'd   Nov 2017

Lemma 27. $\mathcal{M}^1_\infty$ is fully iterable inside M.

Proof: We here think of $\mathcal{V}_0 = L[\mathcal{M}^0_\infty, p \mapsto p^*] = \mathrm{HOD}_{\mathcal{E}}^{M^{Col(\omega,<\kappa_0)}}$ as being represented as $\mathbb{P}$, $\mathbb{P}$ as on pp. 83 – 86. In V, $\mathcal{M}^1_\infty$ is an iterate of $\mathcal{V}_0$; in particular, $\mathcal{M}^1_\infty$ inherits this particular representation from $\mathcal{V}_0$.



Let $\mathcal{T} \in M$ be a normal tree on $\mathcal{M}^1_\infty$.

Let α be s.t. $\mathcal{T} \upharpoonright (\alpha+1)$ is the part of $\mathcal{T}$ which lives below $\delta_0^{\mathcal{V}_0}$.

If $\mathcal{I}\restriction\alpha+1 = \mathcal{I}$, then there is nothing to prove.

Otherwise, there is no drop along $[0,\alpha]_{\mathcal{I}}$.

If $E^{\mathcal{I}}_{\alpha} \in M^{\mathcal{I}}_{\alpha} \mid \pi^{\mathcal{I}}_{0\alpha}(\kappa_0^{+M^0_\infty})$, then $E^{\mathcal{I}}_{\alpha}$ will be applied to an initial segment of

$$(*) \qquad \pi^{\mathcal{I}}_{0\alpha}\left(J_\beta[M^0_\infty \mid \kappa_0^{+M^0_\infty}, (p \mapsto p^*) \restriction \delta_0^{M^0_\infty}] \right),$$

where β is least s.t. $J_\beta[M \mid \kappa_0^+]$ is admissible, cf. p. 83. In this case, $\mathcal{I}\restriction[\alpha,\infty)$ starts with a drop and in fact all of $\mathcal{I}\restriction[\alpha,\infty)$ may be construed as an iteration of $(*)$.

By Claim 3, p. 12, there is then nothing to prove.

We may thus assume that $\mathcal{I}\restriction[\alpha,\infty)$ is entirely above $\pi^{\mathcal{I}}_{0\alpha}(\kappa_0^{+M^0_\infty})$.

Let $\alpha<\beta$ be s.t. $\mathcal{I}\restriction[\alpha,\beta)$ lives in the interval $\left(\pi^{\mathcal{I}}_{0\alpha}(\kappa_0^{+M^0_\infty}),\ \pi^{\mathcal{I}}_{0\alpha}(\delta_1)\right)$.

Let us assume that $\mathcal{I} = \mathcal{I} \upharpoonright \beta$, as otherwise $\mathcal{I} \upharpoonright [\beta+1, \infty)$ is simple enough s.t. M can handle it.

Also let us assume that

- $\mathcal{I}$ is of limit length, and
- if b is the correct branch thru $\mathcal{I}$, then b does not drop.

We want to compute b inside M.

In V, there is a normal tree $\mathcal{U}$ on V_0 of length $\lambda+1$ s.t. $\mathcal{M}^{\mathcal{U}}_{\lambda} = \mathcal{M}^1_{\infty}$. Let $\mathcal{U}^0$ be the part of $\mathcal{U}$ which lives below $\delta_0^{V_0}$, and let $\mathcal{U}^1$ be the part of $\mathcal{U}$ which lives above the image of $\delta_0^{V_0}$, which is $= \delta_0^{\mathcal{M}^1_{\infty}}$.

Let $\mathcal{M}_{\infty}^{\mathcal{U}^0 \frown \mathcal{I} \upharpoonright (\alpha+1)}$ be the last model of $\mathcal{U}^0 \frown \mathcal{I} \upharpoonright \alpha+1$, where we let $\mathcal{I} \upharpoonright \alpha+1$ act on the last model $\mathcal{M}_{\infty}^{\mathcal{U}^0}$ (rather than on $\mathcal{M}^1_{\infty}$, which agrees with $\mathcal{M}_{\infty}^{\mathcal{U}^0}$ up thru its bottom Woodin cardinal). $\mathcal{U}^0 \frown \mathcal{I} \upharpoonright (\alpha+1)$ and $\mathcal{M}_{\infty}^{\mathcal{U}^0 \frown \mathcal{I} \upharpoonright \alpha+1}$ are in M, by Claim 3, p. 12.

Inside M, we may construct a tree, $\mathcal{W}^*$, of limit length by starting to coiterate $\mathcal{M}_\infty^{\mathcal{U}^0 \frown \mathcal{I}\restriction(\alpha+1)}$ with $\mathcal{M}(\mathcal{I})$. In V we can see the reason why $\mathcal{M}(\mathcal{I})$ doesn't move, so that we produced a tree $\mathcal{W}^*$ on $\mathcal{M}_\infty^{\mathcal{U}^0 \frown \mathcal{I}\restriction(\alpha+1)}$ s.t. $\mathcal{M}(\mathcal{W}^*) = \mathcal{M}(\mathcal{I})$ and if c is the correct branch then $\mathcal{W}^*$, $c \in V$, then $\mathcal{M}_c^{\mathcal{W}^*} = \mathcal{M}_b^{\mathcal{I}}$.

Now let G be a total M-extender with $\mathrm{crit}(G) = \kappa_1$ and $\mathcal{I} \in M|\mathrm{lh}(G)$.

Let F be a total M-extender with $\mathrm{crit}(F) = \kappa_0$ and $F \in M|\mathrm{lh}(G)$ and such that if $\mathcal{W}$ is the normal tree on $\mathcal{V}_0$, $\mathcal{W} \in M$, such that

$$\mathcal{M}_\infty^{\mathcal{W}} = \pi_F^M(\mathcal{V}_0),$$

then $\mathcal{W}$ "absorbs" $\mathcal{U}^0 \frown \mathcal{I}\restriction(\alpha+1)$ in that there is a canonical embedding

$$i : \mathcal{M}_\infty^{\mathcal{U}^0 \frown \mathcal{I}\restriction(\alpha+1)} \longrightarrow \mathcal{M}_\infty^{\mathcal{W}}.$$

The embedding i can be seen inside M.

$$\begin{tikzcd}[column sep=large]
\mathcal{V}_0 \arrow[r, dashed, "\mathcal{U}^0 \frown \mathcal{U}^1"] \arrow[rr, dashed, bend right, "\underbrace{\mathcal{U}^0 \frown \mathcal{I}\restriction\alpha+1} \frown \mathcal{W}^*"'] \arrow[d, "M \ni i"'] & \mathcal{M}^1_\infty \arrow[r, dashed, "\mathcal{I}\restriction\alpha+1 \frown \mathcal{I}\restriction[\alpha,\beta)"] & \mathcal{M}^{\mathcal{I}}_b \arrow[d, equal] \\
\underbrace{\mathcal{W} \frown i\mathcal{W}^*} \arrow[d] & & \mathcal{M}_c^{\mathcal{W}^*} \arrow[dl] \\
\mathcal{I}^{ult(M;G)} \ni \rho(\mathcal{I}^*) \arrow[d] & \mathcal{M}_c^{i\mathcal{W}^*} \arrow[d] & \\
{} & \rho(\mathcal{I}^*) \arrow[d] & \\
& {} &
\end{tikzcd}$$

$\rho(\mathcal{I}^*) = \rho(\mathcal{I}^*)$

We may thus use i to copy $\mathcal{W}^*$ onto $\mathcal{M}^{\mathcal{W}}_\infty$, producing a tree $i\mathcal{W}^*$ on $\mathcal{M}^{\mathcal{W}}_\infty$.

c is also the correct branch thru $i\mathcal{W}^*$.

now let η be a cutpoint of $\mathcal{V}_0^{ult(M;G)} = ult(\mathcal{V}_0; G \cap \mathcal{V}_0)$, $\eta > \delta(i\mathcal{W}^*)$, $\eta < \pi^M_G(\kappa_1)$.

We may absorb the tree $\mathcal{W}^\frown i\mathcal{W}^* \in M$ by a tree $\mathcal{I}^*$ of length $\eta^{+ult(M;G)}$ which is one of the trees from the direct limit system producing $\mathcal{M}'_\infty$ of $\mathrm{ult}(M;G)$.

(as a tree on $\mathrm{ult}(V_0; G\cap V_0)$)

$$\begin{tikzcd}
\mathrm{ult}(V_0; G\cap V_0) \arrow[r, squiggly, "\mathcal{I}^*"] & p^{\mathrm{ult}(M;G)}(\mathcal{I}) = \mathcal{M}_d^{\mathcal{I}^*} \arrow[r, dashed] & (\mathcal{M}'_\infty)^{\mathrm{ult}(M;G)} \\
V_0 \arrow[u] \arrow[r, squiggly, "\mathcal{I}^*"'] & \mathcal{M}_d^{\mathcal{I}^*} \arrow[u] \arrow[ur, dotted] &
\end{tikzcd}$$

I.e., if d is the correct branch thru $\mathcal{I}^*$, then there is an embedding from $\mathcal{M}_d^{\mathcal{I}^*}$ into $p^{\mathrm{ult}(M;G)}$ and hence into $(\mathcal{M}'_\infty)^{\mathrm{ult}(M;G)}$.

We now claim that b is the unique branch thru $\mathcal{I}$ s.t. there is an embedding

$$k : \mathcal{M}(\mathcal{I}) \longrightarrow (\mathcal{M}'_\infty)^{\mathrm{ult}(M;G)} \restriction \kappa_1^{+\mathrm{ult}(M;G)}$$

s.t. $$k \circ i_b \restriction \delta(\mathcal{I}) = \pi^M_G \restriction \delta(\mathcal{I}).$$

This is like the proof of [VMI, Lemma 2.8 (a)].

$\dashv$ (Lemma 27)

On pp. 83–86 we reorganized $V_0 = L[M^\circ_\infty, p \mapsto p^*]$ $= \mathrm{HOD}_{\mathcal{E}}^{M^{\mathrm{Col}(\omega,<\kappa_0)}}$ as a "strategic premouse."

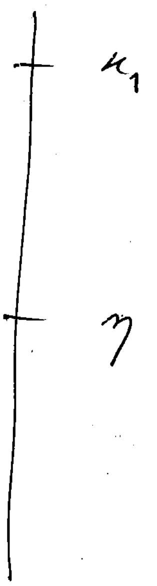
Let us somewhat informally write $\overrightarrow{ES}$ for the predicate from which V_0 is constructed, thus reorganized, cf. p. 86.

Let

$$\mathcal{E}_1 = \{ \overrightarrow{ES^*} : \exists \eta < \kappa_1 \ \overrightarrow{ES^*} \text{ and } \overrightarrow{ES} \text{ are intertranslatable above } \eta \},$$

cf. the definition of $\mathcal{E}$ on pp. 90ff.

In this definition of $\mathcal{E}_1$, by "$\overrightarrow{ES^*}$ and $\overrightarrow{ES}$ are intertranslatable above η" we mean the following.



- $L[\overrightarrow{ES^*}]$ is a "strategic premouse" in much the same fashion as $\mathcal{V}_0 = L[\overrightarrow{ES}]$ is, $L[\overrightarrow{ES^*}] \equiv L[\overrightarrow{ES}]$,*)
- η is a strong cutpoint with respect to both $\overrightarrow{ES^*}$ and $\overrightarrow{ES}$, i.e. if E_ν is an extender from the sequence of $\overrightarrow{ES^*}$ or $\overrightarrow{ES}$ with $\nu > \eta$, then $crit(E_\nu) > \eta$,
- there are g^*, g $Col(\omega, <\eta)$-generic over $L[\overrightarrow{ES^*}]$, $\mathcal{V}_0$, resp., such that $L[\overrightarrow{ES^*}][g^*] = L[\overrightarrow{ES}][g]$,
- if $\underbrace{E_\nu}_{(\nu > \eta)}$ is an extender from the sequence of $\overrightarrow{ES^*}$ ($\overrightarrow{ES}$, resp.), and if $\tilde{E}_\nu$ is the lift of E_ν to $L[\overrightarrow{ES^*}][g^*]$ (to $\mathcal{V}_0[g]$, resp.), then $\tilde{E}_\nu \cap \mathcal{V}_0$ ($\tilde{E}_\nu \cap L[\overrightarrow{ES^*}]$, resp.) is on the sequence of $\overrightarrow{ES}$ ($\overrightarrow{ES^*}$, resp.) and has index ν there, and
- $L[\overrightarrow{ES^*}] | \delta_0^{L[\overrightarrow{ES^*}]}$ ($\delta_0^{L[\overrightarrow{ES^*}]}$ being the least Woodin cardinal of $L[\overrightarrow{ES^*}]$) and $\mathcal{V}_0 | \delta_0^{\mathcal{V}_0}$ may be compared to one Q inside

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*) i.e., both models have the same 1st order properties.

$L[\overrightarrow{ES^*}][g^*]\,|\,\kappa = L[\overrightarrow{ES}][g]\,|\,\kappa$ using the canonical lifts of the respective iteration strategies given by $\overrightarrow{ES^*}$, $\overrightarrow{ES}$, via trees $\mathcal{I}^*$, $\mathcal{I}$,

$$\begin{array}{ccc}
L[\overrightarrow{ES^*}]\,|\,\delta_0^{L[\overrightarrow{ES^*}]} & \xrightarrow[\;i\;]{\mathcal{I}^*} & \\
 & & Q \\
\nu_0\,|\,\delta_0^{\nu_0} & \xrightarrow[\mathcal{I}]{\;j\;} &
\end{array}$$

and if $\Sigma^*_{\mathcal{I}^*,Q}$, $\Sigma_{\mathcal{I},Q}$ are the strategies in

$\underbrace{L[\overrightarrow{ES^*}][g^*] = \nu_0[g]}_{\text{for}}$ Q induced by $\mathcal{I}^*$, $\mathcal{I}$ and the strategies for $L[\overrightarrow{ES^*}]\,|\,\delta_0^{L[\overrightarrow{ES^*}]}$ and $\nu_0\,|\,\delta_0^{\nu_0}$, and if i, j are the iteration maps as in the diagram above, then :

if S_ν activates strategy information on $\overrightarrow{ES^*}$ (on $\overrightarrow{ES}$, resp.), where $\nu > \eta$, then the same is true at ν on $\overrightarrow{ES}$ (or on $\overrightarrow{ES^*}$, resp.), and the two predicates $S_\nu^{\overrightarrow{ES^*}}$, $S_\nu^{\overrightarrow{ES}}$ are computable from each other via the diagram above :

the fragment of the strategy for $\mathcal{L}_0 | \delta_0^{\mathcal{L}_0}$ which $S_{\mathcal{L}}^{\vec{ES}}$ provides is given by $\hat{j}$-pullback of $\Sigma^*_{\mathcal{I}^*, q}$, restricted to trees in $L[\vec{ES}] | \lambda$ (λ the largest cardinal in $L[\vec{ES}] | \mathcal{L}$), and the fragment of the strategy for $L[\vec{ES^*}] | \delta_0^{L[\vec{ES^*}]}$ which $S_{\mathcal{L}}^{\vec{ES^*}}$ provides is given by the i-pullback of $\Sigma_{\mathcal{I}, q}$, restricted to trees in $L[\vec{ES^*}] | \lambda$ (same λ).

<u>Lemma 28.</u> $\mathcal{L}_1 = HOD_{\mathcal{E}_1}^{M^{Col(\omega, <\kappa_1)}}$.

Proof: "$\supset$" : Let $\xi \in X$ iff $M^{Col(\omega, <\kappa_1)} \models \varphi(\xi, \vec{\zeta}, \mathcal{E}_1)$, where $\mathcal{E}_1 = \mathcal{E}_1^M =$ $\mathcal{E}_1$ as being defined over M. Let P be a model from the system giving rise to $\mathcal{M}^1_\infty$ such that $\xi^{**}, \vec{\zeta}^{**} = \pi_{P, \mathcal{M}^1_\infty}(\xi, \vec{\zeta})$. Then $\xi \in X$ iff $P^{Col(\omega, <\kappa_1)} \models \varphi(\xi, \vec{\zeta}, \mathcal{E}_1^P)$, as

$P[h] = M$, where h is generic for a forcing of size $< \kappa_1$, hence $P^{\mathrm{Col}(\omega, <\kappa_1)} = M^{\mathrm{Col}(\omega, <\kappa_1)}$, and $\mathcal{E}_1^P = \mathcal{E}_1^M$. This gives $\xi \in X$ iff

$$(\mathcal{M}^1_\infty)^{\mathrm{Col}(\omega, <\kappa_1^{\mathcal{M}^1_\infty})} \models \varphi(\xi^{**}, \vec{\zeta}^{**}, \mathcal{E}_1^{\mathcal{M}^1_\infty}).$$

This shows $X \in \mathcal{V}_1$.

"$\subset$" $\mathcal{M}^1_\infty$ is clearly definable from $\mathcal{E}_1$, as any $L[\vec{ES}^*]$, for $\vec{ES}^* \in \mathcal{E}_1$, may be used as a base for forming a system which gives rise to $\mathcal{M}^1_\infty$. Similarly, $p \mapsto p^{**}$ is definable from $\mathcal{E}_1$. Thus $\mathcal{V}_1 \subset \mathrm{HOD}_{\mathcal{E}_1}^{M^{\mathrm{Col}(\omega, <\kappa_1)}}$.

<u>Lemma 28.</u> (a) $H^{\mathcal{V}_1}_{\delta_1^{\mathcal{V}_1}} = H^{\mathcal{M}^1_\infty}_{\delta_1^{\mathcal{M}^1_\infty}}$, and

(b) $\delta_1^{\mathcal{V}_1} = \delta_1^{\mathcal{M}^1_\infty} = \kappa_1^{+M}$.

(c) $\delta_1^{\mathcal{V}_1} = \delta_1^{\mathcal{M}^1_\infty}$ is a Woodin cardinal in $\mathcal{V}_1$.

Proof: (a) This is by the above argument for "$\supset$" in the proof of Lemma 28. The point is that for each $\eta < \delta_1^{\mathcal{M}_\infty^1}$, $** \restriction \eta \in \mathcal{M}_\infty^1$.

(b) The argument for (a) above immediately gives $\delta_1^{\mathcal{M}_\infty^1} \leq \kappa_1^{+M}$. But M is a generic extension of $\mathcal{L}_1$ via a forcing which has the $\delta_1^{\mathcal{L}_1}$-c.c., so that we must have that $\delta_1^{\mathcal{M}_\infty^1} = \delta_1^{\mathcal{L}_1} = \kappa_1^{+M}$.

(c) Deny. Let $N \trianglelefteq \mathcal{L}_1$ be s.t. N knows that $\delta_1^{\mathcal{L}}$ is not Woodin. Let F be an extender with critical point κ_1, F a total M-extender, s.t. $N \in \mathrm{ult}(M; F)$. Let $\pi_M^F : M \to \mathrm{ult}(M; F)$ be the ultrapower embedding. Then $\pi_M^F \restriction \mathcal{L}_1 | \delta_1^{\mathcal{L}_1}$ is given by a normal tree on $\mathcal{L}_1 | \delta_1^{\mathcal{L}_1}$, say by $\mathcal{T}^\frown b$, where $\mathcal{T} \in \mathrm{ult}(M; F)$, and b is the correct cofinal branch thru $\mathcal{T}$. But as $N, \mathcal{T} \in \mathrm{ult}(M; F)$, and we may construe $\mathcal{T}$ as acting on N, $\mathrm{ult}(M; F)$ could compute b.

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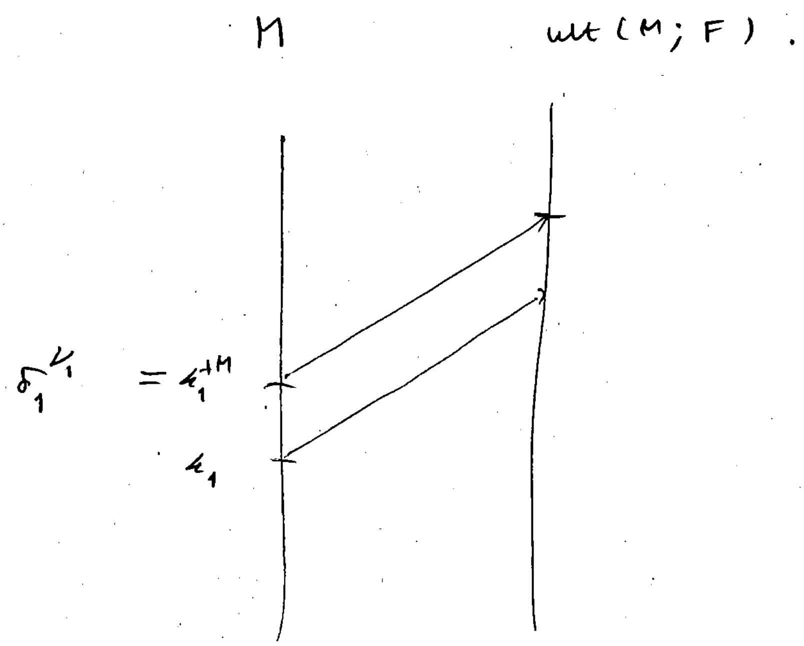

But then $\pi^{\mathcal{I}}_{0,b} \restriction \kappa_1^{+M} : \kappa_1^{+M} \longrightarrow \pi^M_F(\kappa_1)^{+\mathrm{ult}(M;F)}$ is in $\mathrm{ult}(M;F)$. However, $\pi^{\mathcal{I}}_{0,b} \restriction \kappa_1^{+M}$ is cofinal in $\pi^M_F(\kappa_1)^{+\mathrm{ult}(M;F)}$ which is a contradiction!

⊣ (Lemma 28)

We may now reorganize $\mathcal{L}_1$ and show its iterability in M in much the same way as we did similar things for $\mathcal{L}_0$.
A crucial ingredient for showing that $\mathcal{L}_1$ is the mantle of M is an initial segment condition for $\mathcal{M}^1_\infty$ which is inherited from the one for $\mathcal{L}_0$.

Let $S^{V_0}_\nu = S_\nu$ be as on p. 86 : $\nu \in \{\kappa_0^{+\mathcal{M}^0_\infty}\} \cup \{\nu' : \mathrm{crit}(E^M_{\nu'}) = \kappa_0,\ E^M_{\nu'} \text{ total}\}$, and

$S_\nu = \{(\mathcal{T}, \Sigma_{\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}}(\mathcal{T})) : \mathcal{T} \in V_0 | \lambda^{V_0 | \nu}$ is a tree of limit length on $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}\}$. Assume $\nu > \kappa_0^{+\mathcal{M}^0_\infty}$.

We may $(\mathcal{M}_\infty)^\nu$ be the direct limit of all non-dropping (on the main branch) iterates of $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$ via trees which are taken care by S_ν. This gives rise to a canonical map

$$\iota^\nu : \mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty} \longrightarrow (\mathcal{M}_\infty)^\nu$$

We have that S_ν and ι^ν are computable from one another. Moreover, if $\nu > \kappa_0^{+\mathcal{M}^0_\infty}$, then

$$\iota^\nu = \pi^M_{E^M_\nu} \restriction \mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty},$$

where E^M_ν is the (total) M-extender with critical point κ_0 and index ν, and $\pi^M_{E^M_\nu}$ is the ultrapower map. The structure $(V_0 | \nu; \in, \iota^\nu)$ is amenable.