Appendix 2 to Varsovian models, II

To the best of our knowledge, Lemma 2 below is due to Hamkins-Woodin.

Lemma 1. Let g be $\mathbb{P}$-generic over V, and let $\mathcal{U} \in V[g]$ be such that $V[g] \models$ "$\mathcal{U}$ is a $<\kappa$-complete filter on κ," where κ is regular in $V[g]$ and $\kappa > \overline{\overline{\mathbb{P}}}$. Then $\mathcal{U} \cap V \in V$.

Proof: Deny. Let $\mathcal{U} = \tau^g$, $\tau \in V^{\mathbb{P}}$. There is then no $p \in g$ s.t. f.a. $X \in \mathcal{U} \cap V$, $p \Vdash \check{X} \in \tau$.

Let us construct $(p_i, X_i : i < \overline{\overline{\mathbb{P}}}^+)$ s.t. $p_i \in g$ and $X_i \in \mathcal{U} \cap V$ f.a. i as follows.

We work in $V[g]$.

Given p_i, X_i, pick $X \in \mathcal{U} \cap V$ s.t. $p_i \not\Vdash \check{X} \in \tau$.

We have, setting $X_{i+1} = X \cap X_i$, $p_i \not\Vdash \check{X}_{i+1} \in \tau$.

Pick $p_{i+1} \in g$ s.t. $p_{i+1} \Vdash \check{X}_{i+1} \in \tau$.

Now suppose that $\lambda < \overline{\overline{\mathbb{P}}}^+$ is a limit, and all p_i, X_i, $i<\lambda$, have already been chosen. Then ~~[illegible]~~ $X = \bigcap\{X_i : i<\lambda\} \in \mathcal{U}$, as $\lambda < \overline{\overline{\mathbb{P}}}^+ \leq \kappa$ and $\mathcal{U}$ is $<\kappa$-complete. (Possibly, $X \notin V$.) Let $X = \sigma^g$, $\sigma \in V^{\mathbb{P}}$. For $p \in g$, let $X^p = \{\xi<\kappa : p \Vdash \check{\xi} \in \sigma\}$. Again, as $\mathcal{U}$ is $<\overline{\overline{\mathbb{P}}}^+$-complete and $X = \bigcup\{X^p : p \in g\}$, one of the X^p, $p\in g$, must be in $\mathcal{U}$.

Set $X_\lambda =$ any such X^p, $p \in g$, ~~and~~ so that $X_\lambda \in \mathcal{U} \cap V$, and $X_\lambda \subset X \subset X_i$ for all $i<\lambda$. Pick $p_\lambda \in g$ s.t. $p_\lambda \Vdash \check{X}_\lambda \in \tau$.

We have that $X_j \subset X_i$ for $i \leq j$, and $p_i \not\Vdash \check{X}_{i+1} \in \tau$ for all $i < \overline{\overline{\mathbb{P}}}^+$.

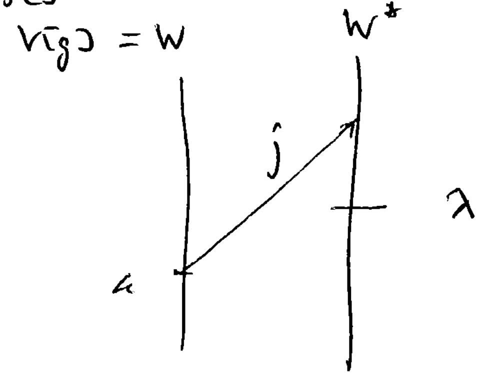
Let $i<j$ be such that $p_i = p_j$. Then $p_j \Vdash \check{X}_j \in \tau$. But $X_{i+1} \supset X_j$, so $p_j \Vdash \check{X}_{i+1} \in \tau$ (we may of course assume w.l.o.g. that every p_k forces τ to be a $<\kappa$-complete filter on κ); but $p_j = p_i$, so $p_i \Vdash \check{X}_{i+1} \in \tau$. Contradiction! $\dashv$

<u>Lemma 2.</u> Let g be $\mathbb{P}$-generic over V, let $j : W = V[g] \longrightarrow W^*$ be definable in W, $\overline{\overline{\mathbb{P}}} = \mu < \kappa = \text{crit}(j) < \lambda \leq j(\kappa)$, $W \models$ "λ is inaccessible", and $V_\lambda^W \subset W^*$. Then

$$E = \{ (a, X) : a \in V_\lambda^V, \; X \in H^V_{\kappa^+}, \; a \in j(X) \}$$

is an element of V, and $V \models$ "E is an extender which witnesses that κ is V_λ-strong."

Proof:



$V_\lambda^V[g] = V_\lambda^W = V_\lambda^{W^*}$, and in V_λ^W, V_λ^V is the unique inner model M s.t. M has the $<\mu^+$ covering and the $<\mu^+$ approximation property, $H^M_{\mu^+} = H^V_{\mu^+}$, and $\mu^{++M} = \mu^{++V} (= \mu^{++W})$. (Cf. Lemma 2 of "Appendix to Varsovian models, II".) Hence V_λ^V is definable over $V_\lambda^{W^*}$ from the parameters $H^V_{\mu^+}$ and μ^{++V}, and thus fore $V_\lambda^V \in W^*$.

Moreover, $j(V)$ is, inside W^*, the unique inner model M s.t. M has the $<\mu^+$ covering and the $<\mu^+$ approximation property, $H^M_{\mu^+} = H^V_{\mu^+}$, and $\mu^{++M} = \mu^{++V}$, so that also $V_\lambda^{j(V)}$ is, inside $V_\lambda^{W^*} = V_\lambda^W$ the unique inner model M s.t. M has the $<\mu^+$ covering and the $<\mu^+$ approximation property, $H^M_{\mu^+} = H^V_{\mu^+}$, and $\mu^{++M} = \mu^{++V}$. Therefore, $V_\lambda^{j(V)} = V_\lambda^V$, and $V_\lambda^V \subset j(V)$.

Now suppose that E, as being defined as in the statement of the lea, is not in V. By $V_\lambda^V \subset j(V)$, E is a V-extender. As $E \notin V$, there is no $p \in g$ s.t. f.a. $a \in V_\lambda^V$, $X \in H^V_{\kappa^+}$, $a \in j(X)$, we have $p \Vdash \check{a} \in \tilde{j}(\check{X})$. Here and in what follows, $\tilde{j}$ is a name for j.

Let us construct $(p_i, a_i, X_i : i < \mu^+)$ s.t. $p_i \in g$ and $(a_i, X_i) \in E$ for all i, as follows. We work in $V[g]$.

Given $(p_j, a_j, X_j : j \leq i)$, pick $(a, X) \in E$ s.t. $p_i \not\Vdash \check{a} \in \tilde{j}(\check{X})$. Let $a_{i+1} : \theta \longrightarrow V_\lambda^V$, $a_{i+1} \in V$, $\theta < \kappa$, $\operatorname{ran}(a_{i+1}) \supset \{a_j : j \leq i\} \cup \{a\}$.

This is possible, as V has the $< \mu^+$ covering property in $V[g]$.

Let $X = \bigtimes_{\xi < \theta} X^\xi$, where

$$X^\xi = \begin{cases} X, & \text{if } a_{i+1}(\xi) = a \\ X_j, & \text{if } a_{i+1}(\xi) = a_j \\ H_{\kappa^+}^V & \text{otherwise} \end{cases}$$

We have that $a_{i+1} \in j(X) = \bigtimes_{\xi < \theta} j(X^\xi)$,

as $a_{i+1}(\xi) \in j(X^\xi)$ for each ξ.

Let $X = \sigma^g$, $\sigma \in V^{\mathbb{P}}$. There is a $p \in g$ such that setting $X^p = \{ f \in {}^\theta H_{\kappa^+}^V \cap V : p \Vdash \check{f} \in \sigma \}$, $a_{i+1} \in j(X^p)$. This is because $a_{i+1} \in j(X) \cap V_\lambda^V$, and $\bigcup \{ j(X^p) : p \in g \} = \cancel{\ldots}\ j(X) \cap j(V) \supset j(X) \cap V_\lambda^V$.

Let X_{i+1} = some such X^ρ, and notice that $X_{i+1} \in V$.

Also $p_i \not\Vdash \check{a}_{i+1} \in \tilde{j}(\check{X}_{i+1})$, as otherwise $p_i \Vdash \check{a}_i \in \tilde{j}(\{f(\xi) : f \in X_{i+1}\}^\vee)$, where $a_{i+1}(\xi) = a$, and $\{f(\xi) : f \in X_{i+1}\} \subset \cancel{X}$ for this ξ (and we may assume w.l.o.g. that $p_i \Vdash \tau$ is an extender).

Pick $p_{i+1} \in g$, $p_{i+1} \Vdash \check{a}_{i+1} \in \tilde{j}(\check{X}_{i+1})$. We have that $p_{i+1} \Vdash \check{a}_j \in \tilde{j}(\check{X}_j)$ for all $j \leq i+1$.

The limit stage is similar; given $(p_j, a_j, X_j : j < \lambda)$ for a limit $\lambda < \mu^+$, we construct a_λ, X_λ with $a_\lambda \in j(X_\lambda)$, $(a_\lambda, X_\lambda) \in E$, and we choose $p_\lambda \in g$ s.t. $p_\lambda \Vdash \check{a}_\lambda \in \tilde{j}(X_\lambda)$; we may arrange that $p_\lambda \Vdash \check{a}_j \in \tilde{j}(\check{X}_j)$ for all $j \leq \lambda$.

But then if $i < j$ are s.t. $p_i = p_j$, $p_i \Vdash \check{a}_{i+1} \in \tilde{j}(\check{X}_{i+1})$.

Contradiction! $\dashv$