Appendix to Varsovian models, II

<u>Defn.</u> Let $W \subset V$ be an inner model, κ a regular V-cardinal.

W has the <u>$<\kappa$-covering property</u> iff f.a. $X \in V$, $X \subset W$, $\overline{\overline{X}} < \kappa$ in V there is some $Y \in W$, $Y \supset X$, $\overline{\overline{W}} < \kappa$.

W has the <u>$<\kappa$-approximation property</u> iff f.a. $X \in V$, $X \subset W$, if $X \cap x \in W$ for all $x \in W$, $\overline{\overline{x}} < \kappa$, then $X \in W$.

<u>Lemma 1</u> (Mitchell ?) Let $V = W[g]$, g $\mathbb{P}$-generic over W, $V \models$ "$\mathbb{P}$ has the κ-c.c." Then W has the $<\kappa$-covering property as well as the $<\kappa$-approximation property.

Proof. "$<\kappa$-cor. property": Easy; only needs $W \models$ "$\mathbb{P}$ has the κ-c.c." Let us show W has the $<\kappa$-appr. property.

Say $g \ni p \Vdash$ "$\tau \subset \check{\alpha}$, $\tau \cap x \in W$ f.a. $x \in W$, $\bar{\bar{x}} < \check{\kappa}$."

Suppose that $p \Vdash \tau \notin W$.

Construct $(p_\xi, \beta_\xi : \xi < \kappa)$ as follows. Given $(p_{\bar\xi}, \beta_{\bar\xi} : \bar\xi < \xi)$, pick $x \in W$, $x \supset \{\beta_{\bar\xi} : \bar\xi < \xi\}$, $\bar{\bar{x}} < \kappa$ by the $<\kappa$-cov. property.

By the hypothesis on p, we may pick $q \leq p$, $q \in g$, s.t. for some $y \in W$, $q \Vdash \tau \cap \check{x} = \check{y}$.

But q can't decide all "$\check{\beta} \in \tau$," as o.w. $q \Vdash \tau \in W$. Let (p_ξ, β_ξ) be s.t. $p_\xi \leq q$,

$p_\xi \Vdash \check{\beta}_\xi \in \tau$ iff $\beta_\xi \notin \tau^g$ and

$p_\xi \Vdash \check{\beta}_\xi \notin \tau$ iff $\beta_\xi \in \tau^g$.

Notice that $(p_\xi : \xi < \kappa)$ is an antichain. However, the construction was done in $W[g]$, so that we need $V \models$ "$\mathbb{P}$ has the κ-c.c." to have arrived at a contradiction. $\dashv$ (Lemma 1)

<u>Lemma 2.</u> Let $W, W' \subset V$ both be inner models s.t. both W, W' have the $<\kappa$-cov. property as well as the $<\kappa$-approx. property, and assume that $H_\kappa^W = H_\kappa^{W'}$ and $\kappa^{+W} = \kappa^{+W'} = \kappa^{+V}$. Then $W = W'$.

Proof. We claim that

<u>Claim 1.</u> $H_{\kappa^+}^W = H_{\kappa^+}^{W'}$.

To see this, let $A \in \mathcal{P}(\kappa) \cap W$. For each $\alpha < \kappa$, $A \cap \alpha \in W'$ by $H_\kappa^{W'} = H_\kappa^W$, so that $A \in \mathcal{P}(\kappa) \cap W'$ by the fact that W' has the $<\kappa$-appr. property. So $\mathcal{P}(\kappa) \cap W \subset W'$. By symmetry, $\mathcal{P}(\kappa) \cap W = \mathcal{P}(\kappa) \cap W'$.

We next show

<u>Claim 2.</u> ${}^{<\kappa}\mathrm{OR} \cap W = {}^{<\kappa}\mathrm{OR} \cap W'$.

Let $X \in {}^{<\kappa}\mathrm{OR} \cap W$. Let $(X_i : i < \kappa)$ be such that $X_0 \supset X$, $X_j \supset X_i$ for $j \geq i$, $\overline{\overline{X_i}} < \kappa$ for all i, and $X_{2i} \in W'$ and $X_{2i+1} \in W$ for all i. There is such a sequence in V by the $<\kappa$-cor. property of both W, W'.

Set $Y = \bigcup \{X_i : i < \kappa\}$.

If $x \in W$, $\bar{\bar{x}} < \kappa$, then $x \cap Y = x \cap X_{2i+1}$, some i, so that $x \cap Y \in W$. By the $<\kappa$-approx. property of W, $Y \in W$.

By symmetry, $Y \in W \cap W'$.

Let $f : \gamma = \mathrm{otp}(Y) \stackrel{\sim}{=} Y$, so that $f \in W \cap W'$, $\gamma < \kappa^+$.

$f^{-1}{}''X \in [\gamma]^{<\kappa} \cap W$, so that $f^{-1}{}''X \in W'$ by Claim 1 and $\kappa^{+W} = \kappa^{+W'} = \kappa^{+V}$. But then $X = f''(f^{-1}{}''X) \in W'$.

By symmetry, this shows claim 2.

We may now easily show $V_\alpha^W = V_\alpha^{W'}$ by induction on α.

Let $X \subset V_\alpha^W = V_\alpha^{W'}$, $X \in W$. Let $x \in W'$, $\bar{\bar{x}} < \kappa$. Then $x \cap V_\alpha^W \in W$ by Claim 2, and then $x \cap X = (x \cap V_\alpha^W) \cap X \in W$, so that $x \cap X \in W'$ by Claim 2 again.

this shows $X \in W'$ by the fact that W' has the $<\kappa$-appr. property.

Symmetrically, $V^W_{\alpha+1} = V^{W'}_{\alpha+1}$. ⊣ (Lemma 2)

The preceeding arguments are due to Hamkins, Laver, Woodin.

<u>Corollary</u>. Let $V = W[g]$, g $\mathbb{P}$-gen. $/W$, $V \models$ "$\mathbb{P}$ has the κ-c.c." Then W is the unique inner model $W' \subset V$ s.t.

- W' has the κ-cov. prop,
- W' has the κ-appr. prop,
- $H^{W'}_\kappa = H^W_\kappa$, and
- $\kappa^{+W'} = \kappa^{+V}$.

W is thus definable inside V from the parameter H^W_κ.