# Thin Equivalence Relations in Scaled Pointclasses

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**Abstract**

We give a new proof via inner model theory that every thin $\boldsymbol{\Sigma}_1^{J_\alpha(\mathbb{R})}$ equivalence relation is $\boldsymbol{\Delta}_1^{J_\alpha(\mathbb{R})}$, where $\alpha$ begins a $\Sigma_1$ gap and $\Sigma_1^{J_\alpha(\mathbb{R})}$ is closed under number quantification, assuming $\mathsf{AD}^{J_\alpha(\mathbb{R})}$.

In the recent past several results previously proved by direct applications of the axiom of determinacy were shown via an inner model theoretic approach. Here we give an inner model theoretic proof of a result of Harrington and Sami [1] on thin equivalence relations. The proof makes it possible to isolate optimal hypotheses in the case that $\Sigma_1^{J_\alpha(\mathbb{R})}$ is closed under number quantification, where $\alpha$ begins a $\Sigma_1$ gap.

Recall that an equivalence relation $E$ is called *thin* if there is no perfect set of pairwise $E$–inequivalent reals.

**Theorem 0.1.** *Let $\alpha \geq 2$ begin a $\Sigma_1$ gap in $L(\mathbb{R})$. Assume $\mathsf{AD}^{J_\alpha(\mathbb{R})}$. Also, setting $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$, assume $\Gamma$ to be closed under number quantification, i.e., $\forall^\omega \Gamma \subset \Gamma$. Let $E$ be a thin $\Gamma$ equivalence relation. Let $\mathcal{N}$ be an $\alpha$-suitable mouse with a capturing term for the complete $\Gamma$ set. Then $E$ is $\check{\Gamma}$ in any real coding $\mathcal{N}$ as a parameter.*

The notion of $\alpha$–suitable mice with capturing terms (which is due to Woodin), is described in our section 1 and in detail in [6]. Such $\alpha$–suitable mice are in a sense analogues of $M_n^\#$ (capturing $\Sigma^1_{n+2}$) which capture more complicated sets of reals. The pointclass $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$ as in the statement of Theorem 0.1 is scaled under $\mathsf{AD}^{J_\alpha(\mathbb{R})}$ (cf. [7]).

The remaining cases for $\alpha$ which we address in this paper are subsumed in

**Theorem 0.2.** *Let $\Gamma = \mathbf{\Sigma}_n^{J_\alpha(\mathbb{R})}$ where $\alpha \geq 2$ begins a $\Sigma_1$ gap, $n = 1$, and $\alpha$ is a successor ordinal or $cf(\alpha) = \omega$, or else $\alpha$ ends a proper weak $\Sigma_1$ gap and $n$ is least with $\rho_n(J_\alpha(\mathbb{R})) = \mathbb{R}$. Assume $\mathsf{AD}^{L(\mathbb{R})}$. Then every thin $\Gamma$ equivalence relation is $\breve{\Gamma}$.*

In section 1 we collapse a substructure of a suitable premouse and prove upwards absoluteness for the preimages of capturing terms. In section 2 we apply the method of term capturing to prove theorem 0.1 building on an argument of Hjorth [3, lemma 2.5] for $\Sigma^1_2$ equivalence relations. In section 3 we give a proof of theorem 0.2.

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# 1 Weak term condensation

We fix an ordinal $\alpha \geq 2$ beginning a $\Sigma_1$ gap in $L(\mathbb{R})$ (cf. [7]) with the property that $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$ is closed under number quantification. Let us also assume $\mathsf{AD}^{J_\alpha(\mathbb{R})}$ throughout.

**Definition 1.1.** *For any bounded subset $A$ of $\omega_1$, the ($\alpha$–)lower-part closure $Lp^\alpha(A)$ of $A$ is the the model theoretic union of all $A$-premice $\mathcal{N}$ which are sound above $A$, project to $\sup(A)$, and are $\omega_1$-iterable in $J_\alpha(\mathbb{R})$ (i.e., there is an iteration strategy $\Sigma \in J_\alpha(\mathbb{R})$ with respect to countable iteration trees on $\mathcal{N}$).*

Under $\mathsf{AD}^{J_\alpha(\mathbb{R})}$, any two $A$-premice as in definition 1.1 are lined up,[1] so that $Lp^\alpha(A)$ is well–defined.

**Definition 1.2.** *An $A$-premouse $\mathcal{N}$ for bounded $A \subseteq \omega_1$ with a unique Woodin cardinal $\delta = \delta^{\mathcal{N}}$ is called $\alpha$-suitable if*

1. *$\delta$ is minimal such that $\delta$ is Woodin in $Lp^\alpha(\mathcal{N}|\delta)$, and*
2. *$\mathcal{N}$ is the $Lp^\alpha$ closure of $\mathcal{N}|\delta$ up to its $\omega^{th}$ cardinal above $\delta$, i.e. $\mathcal{N} = \bigcup_{k<\omega} \mathcal{N}_k$ where $\mathcal{N}_0 := \mathcal{N}|\delta$ and $\mathcal{N}_{k+1} := Lp^\alpha(\mathcal{N}_k)$ for all $k < \omega$.*

[1] Notice that if $\Sigma$ and $\Sigma' \in J_\alpha(\mathbb{R})$ witness the countable $A$-premice $\mathcal{N}$ and $\mathcal{N}'$ to be $\omega_1$-iterable, respectively, then $\omega_1^{L[\Sigma,\Sigma',\mathcal{N},\mathcal{N}']} < \omega_1^V$ by $\mathsf{AD}^{J_\alpha(\mathbb{R})}$, so that $\mathcal{N}$ and $\mathcal{N}'$ can be successfully compared in $L[\Sigma, \Sigma', \mathcal{N}, \mathcal{N}']$.

In what follows, we let $\delta = \delta^{\mathcal{N}}$ always denote the Woodin cardinal of an $\alpha$-suitable premouse $\mathcal{N}$.

**Definition 1.3.** *An $\omega_1$-iteration strategy $\Sigma$ for an $\alpha$–suitable $\mathcal{N}$ is fullness-preserving if for every iteration tree $\mathcal{T}$ on $\mathcal{N}$ according to $\Sigma$ which lives on $\mathcal{N}|\delta$*

1. *if the branch to the last model $\mathcal{P}$ does not drop, then $\mathcal{P}$ is $\alpha$-suitable, and*
2. *if the branch to $\mathcal{P}$ drops, then $\mathcal{P}$ is $\omega_1$-iterable in $J_\alpha(\mathbb{R})$.*

**Definition 1.4.** *Suppose $\Sigma$ is an $\omega_1$-iteration strategy for a countable $\alpha$-suitable $A$-premouse $\mathcal{N}$ and $\mathbb{Q} \in \mathcal{N}$ is a forcing notion. A $\mathbb{Q}$–name $\dot{E} \in \mathcal{N}^{\mathrm{Col}(\omega,\eta)}$, where $\eta \geq \delta^{\mathcal{N}}$, is said to capture a set $E \subseteq \mathbb{R}$ relative to $\Sigma$ if*

$$\pi(\dot{E})^g = E \cap \mathcal{P}[g]$$

*whenever $\pi : \mathcal{N} \to \mathcal{P}$ is a non-dropping iteration map produced by a countable iteration tree which is according to $\Sigma$ and $g$ is $\pi(\mathbb{Q})$-generic over $\mathcal{P}$. $\dot{E}$ is then also called a ($\mathbb{Q}$–)capturing term for $E$ (relative to $\Sigma$).*

**Theorem 1.5.** *(Woodin, see [6]) Assume $\mathsf{AD}^{J_\alpha(\mathbb{R})}$ holds, where $\alpha \geq 2$ begins a $\Sigma_1$ gap in $L(\mathbb{R})$ and $\Sigma_1^{J_\alpha(\mathbb{R})}$ is closed under number quantification. Let $E \subseteq \mathbb{R}$ be a $\Sigma_1^{J_\alpha(\mathbb{R})}$ set. There is then a countable $\alpha$-suitable $A$-premouse $\mathcal{N}$ and a fullness-preserving $\omega_1$-iteration strategy $\Sigma$ for $\mathcal{N}$ such that for every $\eta \geq \delta$ in $\mathcal{N}$ there is a $Col(\omega,\eta)$–name capturing $E$ relative to $\Sigma$.*

Let us fix such an $A$-premouse $\mathcal{N}$ together with a fullness-preserving $\omega_1$-iteration strategy $\Sigma$. A weak capturing property is retained for $A$-premice which embed into an initial segment of $\mathcal{N}$:

**Lemma 1.6.** *Let $E \subseteq \mathbb{R}$ be a $\Sigma_1^{J_\alpha(\mathbb{R})}$ set and let $\dot{E}, \sigma$ be $Col(\omega,\delta)$-capturing terms for $E$ and its $\Sigma_1^{J_\alpha(\mathbb{R})}$ scale (relative to $\Sigma$). Let $\pi : \mathcal{M} \to \mathcal{N}|(\delta^{+n})^{\mathcal{N}}$ be sufficiently elementary with $\dot{E}, \sigma \in rng(\pi)$ and $n \geq 2$. Let $\bar{E} = \pi^{-1}(\dot{E})$. Then $\bar{E}^g \subseteq E$ for every $Col(\omega, \pi^{-1}(\delta))$-generic filter $g$ over $\mathcal{M}$.*

*Proof.* We argue that it is possible to replace $\dot{E}$ with the name for the projection of a tree and we then use upwards absoluteness for this name. Suppose $g$ is $Col(\omega,\delta)$-generic over $\mathcal{N}$ and $\dot{T} \in \mathcal{N}$ is a $Col(\omega,\delta)$-name for the tree

$$T = \{(x|k, (r_0(x)^{\mathcal{N}[g]}, ..., r_{k-1}(x)^{\mathcal{N}[g]})) : x \in \dot{E}^g = E \cap \mathcal{N}[g], k < \omega\} \tag{1}$$

where the $r_i$ are the ranks in the scale as computed in $\mathcal{N}[g]$ via the capturing term $\sigma$ for the scale. The tree $T$ is the image of a countable subtree $S$ of the tree from the scale on $E$ in $V$ via the map which collapses the set of ordinals occuring in $S$ to a transitive set, so that $p[\dot{T}^g]^{\mathcal{N}[g]} \subseteq E \cap \mathcal{N}[g]$. This implies $\dot{E}^g = E \cap \mathcal{N}[g] = p[\dot{T}^g]^{\mathcal{N}[g]}$.

Notice that $T = \dot{T}^g$ is independent from the choice of the particular generic $g$, and hence $T \in \mathcal{N}$. This is because if $p$, $q \in Col(\omega, \delta)$ are two conditions, then we may pick generics $g_p$ and $g_q$ over $\mathcal{N}$ with $p \in g_p$ and $q \in g_q$ such that $\mathcal{N}[g_p] = \mathcal{N}[g_q]$. As $\dot{E}$ and $\sigma$ capture $E$ and the scale over $\mathcal{N}$, respectively, we get that $\dot{E}^{g_p} = \dot{E}^{g_q}$ and $\sigma^{g_p} = \sigma^{g_q}$, so that $\dot{T}^{g_p} = \dot{T}^{g_q}$, as $\dot{T}$ is defined from $\dot{E}$ and $\sigma$ as in (1).

Now as $p[T]^{\mathcal{N}[g]} = \dot{E}^g$,

$$\Vdash^{\mathcal{N}}_{Col(\omega,\delta)} \; p[\check{T}] = \dot{E},$$

and therefore

$$\Vdash^{\mathcal{M}}_{Col(\omega,\pi^{-1}(\delta))} \; p[\pi^{-1}(\check{T})] = \bar{E}.$$

This yields that $\bar{E}^h = p[\pi^{-1}(T)]^{\mathcal{M}[h]} \subseteq p[T] \subseteq E$ for every $Col(\omega, \pi^{-1}(\delta))$-generic $h$ over $\mathcal{M}$. □

When $\mathcal{M}$ is iterated, the capturing term is still upwards absolute:

**Lemma 1.7.** *(Weak term condensation) Let $E \subseteq \mathbb{R}$ be a $\Sigma_1^{J_\alpha(\mathbb{R})}$ set and $\dot{E}, \sigma$ $Col(\omega, \delta)$-capturing terms for $E$ and its $\Sigma_1^{J_\alpha(\mathbb{R})}$ scale (relative to $\Sigma$). Let, for $n \geq 2$, $\pi : \mathcal{M} \to \mathcal{N}|(\delta^{+n})^{\mathcal{N}}$ be sufficiently elementary with $\dot{E}, \sigma \in rng(\pi)$ and $\bar{E} := \pi^{-1}(\dot{E})$. Let $\rho : \mathcal{M} \to \mathcal{P}$ be a non-dropping iteration map via the pullback strategy. Then $\rho(\bar{E})^g \subseteq E$ for every $Col(\omega, \rho(\pi^{-1}(\delta)))$-generic filter $g$ over $\mathcal{P}$.*

*Proof.* Let $\rho^\pi : \mathcal{N} \to \mathcal{R}$ denote the iteration map of the tree copied onto $\mathcal{N}$. There is an embedding $\pi^* : \mathcal{P} \to \mathcal{R}|\rho^\pi((\delta^{+n})^{\mathcal{N}})$ such that the diagram

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\pi^*} & \mathcal{R}|\rho^\pi((\delta^{+n})^{\mathcal{N}}) \\ \uparrow \rho & & \uparrow \rho^\pi \\ \mathcal{M} & \xrightarrow{\pi} & \mathcal{N}|(\delta^{+n})^{\mathcal{N}} \end{array}$$

commutes. Then $\rho(\bar{E})^g \subseteq E$ by the previous lemma applied to $\mathcal{R}$. □

# 2 At the beginning of a gap

As in the previous section we shall assume that $\alpha$ begins a $\Sigma_1$ gap in $L(\mathbb{R})$, $\Sigma_1^{J_\alpha(\mathbb{R})}$ is closed under number quantification, and $\mathsf{AD}^{J_\alpha(\mathbb{R})}$ holds. Let $\mathcal{N}$ be an $\alpha$-suitable $A$-premouse (for some $A$) as in definition 1.2, and let $\Sigma$ be an $\omega_1$–iteration strategy for $\mathcal{N}$.

**Definition 2.1.** *Let $\mathcal{T}$ be a normal iteration tree of countable length on $\mathcal{N}$, and suppose that $\mathcal{T}$ lives below $\delta^{\mathcal{N}}$.*[2] *We then say that $\mathcal{T}$ is short iff for all limit ordinals $\lambda < \mathrm{lh}(\mathcal{T})$, $Lp^\alpha(\mathcal{M}(\mathcal{T} \restriction \lambda)) \models \delta(\mathcal{T} \restriction \lambda)$ is not Woodin. Otherwise, we say that $\mathcal{T}$ is maximal.*

**Lemma 2.2.** *The restriction of the $\omega_1$-iteration strategy $\Sigma$ to short trees on $\mathcal{N}$ is $\Sigma_1^{J_\alpha(\mathbb{R})}$.*

*Proof.* Let $\mathcal{T}$ be a countable short iteration tree of limit length which is on $\mathcal{N}$ and according to $\Sigma$. We then have that $\Sigma(\mathcal{T}) = b$ if and only if there is a $\mathcal{Q}$–structure $\mathcal{Q} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$ such that $\mathcal{Q}$ is $\omega_1$–iterable in $J_\alpha(\mathbb{R})$. This immediately shows that $\Sigma$, restricted to short trees, is in $\Sigma_1^{J_\alpha(\mathbb{R})}$. □

**Lemma 2.3.** *For all $n \geq 1$ there is $\mathcal{M} \triangleleft \mathcal{N}$ and a fully elementary map $\pi : \mathcal{M} \to \mathcal{N}|(\delta^{+n})^{\mathcal{N}}$ with $\gamma = \pi^{-1}(\delta) < \delta$ and $V_\gamma^{\mathcal{M}} = V_\gamma^{\mathcal{N}}$.*

*Proof.* Let us construct $(H_i \colon i < \omega) \in \mathcal{N}$ as follows. Let $\mathcal{P} = \mathcal{N}|(\delta^{+n})^{\mathcal{N}} + 1$. Set $H_0 = \emptyset$, and given $H_i$ set

$$H_{i+1} = \mathrm{Hull}_{\Sigma_1}^{\mathcal{P}}(V_{\sup(H_i \cap \delta)+1}^{\mathcal{N}})$$

for $i < \omega$. Then $\gamma = \sup(\bigcup_{i<\omega} H_i \cap \delta) < \delta$ since $\delta$ is inaccessible in $\mathcal{N}$. Let

$$\pi^* : \mathcal{M}^* \to \bigcup_{i<\omega} H_i = \mathrm{Hull}_{\Sigma_1}^{\mathcal{P}}(V_\gamma^{\mathcal{N}})$$

be the inverse of the collapsing map. The construction ensures that $V_\gamma^{\mathcal{M}^*} = V_\gamma^{\mathcal{N}}$. We have $crit(\pi) = \gamma$ and $\rho_1(\mathcal{M}^*) = \gamma$. We easily get $\mathcal{M}^* \triangleleft \mathcal{N}$ by the Condensation Lemma (see [8, theorem 5.5.1] or [5, theorem 8.2]). Then

$$\pi^* \restriction (\pi^*)^{-1}(\mathcal{N}|(\delta^{+n})^{\mathcal{N}})$$

[2]That $\mathcal{T}$ lives below $\xi$ means that $\mathcal{T}$ may be construed as an iteration tree on $\mathcal{N}|\xi$.

is as desired. $\square$

Let us fix a notation: given a forcing $\mathbb{P}$ and a $\mathbb{P}$-name $\tau$, let $\tau_i$ for $i = 0, 1$ denote $\mathbb{P} \times \mathbb{P}$-names such that $\tau^{g_i} = \tau_i^g$ for any $\mathbb{P} \times \mathbb{P}$-generic filter $g = g_0 \times g_1$.

**Lemma 2.4.** *Let $E$ be a thin $\Sigma_1^{J_\alpha(\mathbb{R})}$ equivalence relation. Suppose $\dot{E}$ captures $E$ over $\mathcal{N}$ for the forcing $\mathbb{S} = \mathbb{P} \times \mathbb{P}$. Then for every $\mathbb{P}$-name $\tau \in \mathcal{N}$ for a real,*

$$(p, p) \Vdash_{\mathbb{S}}^{\mathcal{N}} \tau_0 \dot{E} \tau_1$$

*holds for a dense set of conditions $p \in \mathbb{P}$.*

*Proof.* The proof is essentially that of [3, lemma 2.2]. Suppose the set is not dense. In this case let $p_\emptyset$ be a condition such that for every $r \leq p_\emptyset$ there are conditions $p, q \leq r$ with

$$(p, q) \Vdash_{\mathbb{S}}^{\mathcal{N}} \neg \tau_0 \dot{E} \tau_1.$$

Let $(D_i : i < \omega)$ enumerate the dense open subsets of $\mathbb{P} \times \mathbb{P}$ in $\mathcal{N}$. We can inductively construct a family $(p_s : s \in 2^{<\omega})$ of conditions in $\mathbb{P}$ so that for all $s, t \in 2^{<\omega}$

1. $p_t \leq p_s$ if $s \subseteq t$,
2. $p_s$ decides $\tau \restriction lh(s)$,
3. $(p_s, p_t) \in D_0 \cap \cdots \cap D_n$ for $s \neq t$ in $2^n$, and
4. $(p_{s^\frown 0}, p_{s^\frown 1}) \Vdash_{\mathbb{S}}^{\mathcal{N}} \neg \tau_0 \dot{E} \tau_1$.

Let further $g_x := \{p \in \mathbb{P} : \exists n < \omega (p_{x \restriction n} \leq p)\}$ for each $x \in \mathbb{R}$. Then $g_x \times g_y$ is $\mathbb{P} \times \mathbb{P}$-generic over $\mathcal{M}$ for any $x, y \in \mathbb{R}$ with $x \neq y$ by condition 3, and we have

$$\dot{E}^{g_x \times g_y} = E \cap \mathcal{N}[g_x \times g_y]$$

since $\dot{E}$ captures $E$ for $\mathbb{S}$. Thus $\neg \tau^{g_x} E \tau^{g_y}$ for $x \neq y$ because $\tau^{g_x} = (\tau_0)^{g_x \times g_y}$ and $\tau^{g_y} = (\tau_1)^{g_x \times g_y}$ and thus

$$\mathcal{N}[g_x \times g_y] \models \neg \tau^{g_x} \dot{E}^{g_x \times g_y} \tau^{g_y}$$

holds by condition 4. Since $\tau^{g_x}$ depends continuously on $x$ by condition 2, the perfect set $\{\tau^{g_x} : x \in \mathbb{R}\}$ would contradict that $E$ is thin. $\square$

By theorem 1.5, we may assume that our $\mathcal{N}$ satisfies the hypothesis of the following theorem.

**Theorem 2.5.** *Let $E$ be a thin $\Sigma_1^{J_\alpha(\mathbb{R})}$ equivalence relation and suppose $\mathcal{N}$ has capturing terms for both $E$ and a $\Sigma_1^{J_\alpha(\mathbb{R})}$ scale on $E$. Then $E$ is $\Pi_1^{J_\alpha(\mathbb{R})}$ in (any real coding) $\mathcal{N}$.*

*Proof.* This is similar to the proof of [3, lemma 2.5]. Let

$$\pi : \mathcal{M} = \mathcal{N}|\beta \to \mathcal{N}|\delta^{++\mathcal{N}}$$

be as in (the proof of) Lemma 2.3 with both $\dot{E}$ and a capturing term for a $\Sigma_1^{J_\alpha(\mathbb{R})}$ scale on $E$ in $rng(\pi)$. This is possible since these capturing terms have size $\delta^{+\mathcal{N}}$. Let $\gamma = \pi^{-1}(\delta)$ and $\bar{E} := \pi^{-1}(\dot{E})$. Let $\dot{r}$ be the preimage under $\pi$ of the $Col(\omega, \delta)$-name for the generic real for the extender algebra at $\delta$. Let further $\sigma, \tau$ be $Col(\omega, \gamma)$-names for reals such that $\Vdash^{\mathcal{M}}_{Col(\omega,\gamma)} \dot{r} = \sigma \oplus \tau$, where $\sigma \oplus \tau$ enumerates the bits of $\sigma$ and $\tau$.

We claim that for $a, b \in \mathbb{R}$ the fact that $\neg aEb$ holds true is equivalent to the following condition.

**Condition 2.6.** *There are $c, d \in \mathbb{R}$ and a non-dropping iteration map $\rho : \mathcal{N} \to \mathcal{P}$ which is produced by an iteration tree $\mathcal{T}$ which lives on $\mathcal{N}|\gamma$ and is according to $\Sigma$ such that*

1. *$aEc$ and $bEd$,*
2. *$c \oplus d = \rho(\dot{r})^g$ for some $Col(\omega, \rho(\gamma))$-generic filter $g$ over $\mathcal{P}$, and*
3. *$\mathbb{1} \Vdash^{\mathcal{P}[g]}_{Col(\omega,\rho(\delta))} \neg \check{c}\rho(\dot{E})\check{d}$.*[3]

Condition 2.6 clearly implies $\neg aEb$ by our hypotheses.

On the other hand, given $a, b \in \mathbb{R}$ with $\neg aEb$, let $\rho : \mathcal{N} \to \mathcal{P}$ be a non-dropping iteration map which is produced by a tree $\mathcal{T}$ which lives on $\mathcal{N}|\gamma$ and is according to $\Sigma$ such that for some $Col(\omega, \rho(\gamma))$-generic filter $g$ over $\rho(\mathcal{M}) = \mathcal{P}|\rho(\beta)$ we have that $a \oplus b = \rho(\dot{r})^g$.

[3]As $Col(\omega, \gamma) \times Col(\omega, \delta) \cong Col(\omega, \delta)$, it is easy to see that $\mathcal{P}[g]$ has a term capturing $E$, which we here confuse with $\rho(\dot{E})$.

By lemma 2.4 and the elementarity of $\pi$, if $\epsilon \in \mathcal{M}^{Col(\omega,\gamma)}$ is a name for a real, then for a dense set $D \in \mathcal{M}$ of $p \in Col(\omega,\gamma)$,

$$(p,p) \Vdash^{\mathcal{M}}_{Col(\omega,\gamma)\times Col(\omega,\gamma)} \epsilon_0 \bar{E} \epsilon_0.$$

Using the elementarity of $\rho$, there is hence some $q \in g$ such that

$$(q,q) \Vdash^{\rho(\mathcal{M})}_{Col(\omega,\rho(\gamma))\times Col(\omega,\rho(\gamma))} \rho(\sigma)_0 \rho(\bar{E}) \rho(\sigma)_1 \ \wedge\ \rho(\tau)_0 \rho(\bar{E}) \rho(\tau)_1. \tag{2}$$

Let $h$ be $Col(\omega,\rho(\gamma))$-generic below $q$ over both $(\mathcal{P}|\rho(\beta))[g]$ and $\mathcal{P}$ and let $c, d \in \mathbb{R}$ with $c \oplus d = \dot{r}^h$. We then have both $aEc$ and $bEd$ by (2) and by lemmas 1.6 and 1.7. As $\neg aEb$, this means that $\neg cEd$, so that $\mathbb{1} \Vdash^{\mathcal{P}[g]}_{Col(\omega,\rho(\delta))} \neg \check{c} \rho(\dot{E}) \check{d}$.

We have shown that condition 2.6 holds.

Finally, it is true that $\Sigma$, restricted to short trees, is $\Sigma_1^{J_\alpha(\mathbb{R})}$ by lemma 2.2, so that the reformulation of $\neg aEb$ given by condition 2.6 shows that $\neg E$ is $\exists^{\mathbb{R}}\ \forall^{\mathbb{N}}\ \Sigma_1^{J_\alpha(\mathbb{R})}$ in $\mathcal{N}$. As we assume $\Sigma_1^{J_\alpha(\mathbb{R})}$ to be closed under number quantification, this shows that $E$ is $\Pi_1^{J_\alpha(\mathbb{R})}$ in $\mathcal{N}$, as desired. □

## 3 $\omega$-cofinal pointclasses

The argument in the last section used that $\Sigma_1^{J_\alpha(\mathbb{R})}$ be closed under number quantification. We do not know how to drop this hypothesis, unless we replace the hypothesis $\mathsf{AD}^{J_\alpha(\mathbb{R})}$ by $\mathsf{AD}^{L(\mathbb{R})}$.

We thus now turn to the case that $\alpha \geq 2$ begins a gap and $\Sigma_1^{J_\alpha(\mathbb{R})}$ is not closed under number quantification. In this case $\alpha = \bar{\alpha}+1$ or $cf(\alpha) = \omega$, since $cf(\alpha) > \omega$ and $J_\alpha(\mathbb{R}) \vDash \forall n\, \varphi(x,n)$ imply that there is some $\bar{\alpha} < \alpha$ with $J_{\bar{\alpha}}(\mathbb{R}) \vDash \forall n\, \varphi(x,n)$. Hence $A \in \Sigma_1^{J_\alpha(\mathbb{R})}$ iff $A$ is a countable union of sets in $J_\alpha(\mathbb{R})$.

**Lemma 3.1.** *Assume* $\mathsf{AD}$. *Let* $\Gamma$ *be a relativized scaled pointclass closed under* $\exists^{\mathbb{R}}$. *Suppose* $\Gamma_k \subseteq \Delta$ *for* $k < \omega$ *are pointclasses such that for every* $A \in \Gamma$ *there are* $A_k \in \Gamma_k$ *with* $A = \bigcup_{k<\omega} A_k$. *Then every thin* $\Gamma$ *equivalence relation is* $\Delta$.

*Proof.* Let $E$ be a thin $\Gamma$ equivalence relation. Then $E$ is co-$\kappa$-Suslin via some tree $T$, since the class of $\kappa$-Suslin sets is closed under countable intersections: trees $T_k$ on $\omega \times \kappa$ with $A_k = p[T_k]$ for $k < \omega$ can be amalgamated into a tree $T$ with $\bigcap_{k<\omega} A_k = p[T]$.

There is no $\omega_1$-sequence of reals under $\mathsf{AD}$ so that $L[T] \cap \mathbb{R}$ is countable. Hence there is a Cohen real in $V$ over $L[T]$. Harrington and Shelah [2] proved that if there is a Cohen real over $L[T]$ (or if the complement of $p[T]$ is transitive in a Cohen generic extension of $L[T]$), then $E$ has at most $\kappa$ equivalence classes and the set of equivalence classes is wellordered. Let $(A_\gamma : \gamma < \delta)$ enumerate the equivalence classes of $E$, where $\delta \leq \kappa$. Now $\Gamma$ is closed under wellordered unions by [4, lemma 2.18], since it is closed under $\exists^{\mathbb{R}}$ and has the prewellordering property. So

$$\mathbb{R}^2 - E = \bigcup_{\gamma \neq \delta < \kappa} (A_\gamma \times A_\delta)$$

is $\Gamma$. □

**Theorem 3.2.** *Let $\Gamma = \mathbf{\Sigma}_n^{J_\alpha(\mathbb{R})}$ where $\alpha \geq 2$ begins a $\Sigma_1$ gap, $n = 1$, and $\alpha$ is a successor ordinal or $cf(\alpha) = \omega$, or else $\alpha$ ends a proper weak $\Sigma_1$ gap and $n$ is least with $\rho_n(J_\alpha(\mathbb{R})) = \mathbb{R}$. Assume $\mathsf{AD}^{L(\mathbb{R})}$. Then every thin $\Gamma$ equivalence relation is $\breve{\Gamma}$.*

*Proof.* If $\alpha$ begins a gap and $\alpha = \bar{\alpha} + 1$, let $\Gamma_k = \mathbf{\Sigma}_k^{J_{\bar{\alpha}}(\mathbb{R})}$. If $cf(\alpha) = \omega$, let $\alpha = \sup \alpha_k$ and $\Gamma_k = J_{\alpha_k}(\mathbb{R})$. Finally let $\Gamma_k = J_\alpha(\mathbb{R})$ for each $k < \omega$ if $\alpha$ ends a gap. The previous lemma applies in each case. □

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