Ralf Schindler: Talks#4 on Logic Summer School of Fudan University, 2020

TODAY:

- Show a characterization of preciosity;
- \( V \) is generically iterable with respect to preciosity ideals;
- Discussion of effective counterexamples to \( \text{CH} \);
- Illustrations of Admissible Club Guessing\((\text{ACG}) \Rightarrow u_2 = \omega_2\);
- Prove ACG follows from \( \text{MM} \).

**Theorem 1.** The followings are equivalent:

- \( \text{NS}_{\omega_1} \) is precipitous;
- Let \( x \prec H_\theta \) be countable, and let \( M_x \) be its transitive collapse via embedding \( \sigma \). Define

\[
G_x = \{ X \in P(\omega_1^{M_x}) \cap M_x : \omega_1^{M_x} \in \sigma(X) \}
\]

Then the collection

\[
S = \{ x \prec H_\theta : |x| = \omega \wedge G_x \text{ is } \sigma^{-1}(\text{NS}_{\omega_1}^+)\text{-generic over } M_x \}
\]

is projective stationary.

**Definition.** \( S \) is projective stationary iff f.a. \( T \subset \omega_1 \) stationary, \( \{ x \in S : x \cap \omega_1 \in T \} \) is stationary.

**Proof.** (Sketch) "\( \Rightarrow \)" Fix a stationary set \( T \subset \omega_1 \) and let \( C \subset [H_\theta]^{\omega_1} = \{ x \in [H_\theta]^{\omega_1} : f^n x \subset x, f : H_\theta^{<\omega} \to H_\theta \} \) be a club. Let \( G \) be \( V \)-generic for \( \text{NS}_{\omega_1}^+ \) such that \( T \in G \). This implies the existence of an elementary embedding \( j : V \to \text{Ult}(V; G) = M \).

![Diagram](attachment:image.png)

Since \( T \in G \), \( \omega_1^V \in j(T) \). Now we consider the structure \( j^n H_\theta^V \). It is a substructure of \( H^M_{j(\theta)} \), \( \omega_1^M \cap j^n H_\theta^V = \omega_1^V \), and is closed under \( j(f) \). By reflection and absoluteness between \( V[G] \) and \( M \), it will be true that in \( M \) there is a countable \( x \prec H^M_{j(\theta)} \) such that \( x \cap j(\omega_1^V) = \omega_1^V \in j(T) \), and \( x \) is closed under \( j(f) \). By pulling back the appropriate statement via \( j \), we get that in \( V \) there is a countable \( x \prec H_\theta \) with \( x \cap \omega_1 \in T \) and closed under \( f \). Since \( \theta \) is chosen as large as we want, we now have that \( S \) is in fact projective stationary.

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Assume that $\text{NS}_{\omega_1}$ is not precipitous and let $x \in S$. Then there is a stationary set $T \in \text{NS}_{\omega_1}^+$, such that $T \Vdash “\text{Ult}(V; G) \text{ is ill-founded}”$.

Then by definition $\omega_1^M = \omega_1 \cap M \in T$. Let $\bar{T} = T \cap \omega_1^M \in M$, then the condition $\omega_1^M \in j(\bar{T})$ gives that $\bar{T} \in H$, where $H$ is the generic filter derived from $j$. Thus by downstairs elementarity, $\bar{T} \Vdash “\text{Ult}(M; H) \text{ is ill-founded}”$. This leads to a contradiction since by factoring $j$ by $H$, the diagram on the right commutes, therefore $\text{Ult}(M; H)$ embeds elementarily into $H_\theta$, which is well-founded.

Corollary 2. $\text{NS}_{\omega_1}$ saturated $\implies \text{NS}_{\omega_1}$ precipitous.

How do you obtain models in which $\text{NS}_{\omega_1}$ is saturated/precipitous? We can first pick a measurable cardinal and collapse it to $\omega_1$. Then there is a precipitous ideal of $\omega_1$. Then we can shoot a club through the stationary complements of members of the ideal and make this precipitous ideal to be $\text{NS}_{\omega_1}$. However, this does not work for $\text{NS}_{\omega_1}$ to be saturated. To get it one may need the existence of a Woodin cardinal $\delta$, and some $\delta$-c.c. semi-proper forcing to do that. A proof of this theorem can be found on here*

It is also true that $\text{MM} \implies \text{NS}_{\omega_1}$ is saturated. This theorem is obtained by Foreman-Magidor-Shelah in their original MM paper [2], [3].

1 Generic iterability

Assume $I \subset P(\omega_1)$ is a precipitous normal uniform ideal on $\omega_1$. Work in $V^{\text{Col}(\omega, \theta)}$, where $\theta > \omega_1$ is large enough. We may do the following iteration process:

$V = M_0 \xrightarrow{j_0} M_1 = \text{Ult}(M_0; G_0) \rightarrow ... \rightarrow M_\omega = \lim \text{dir}_{n \rightarrow \omega}(M_n; G_n) \rightarrow ...$

For every $n < \omega$, we let $G_n$ to be the generic filter of $(\text{NS}_{\omega_1}^+)^{M_n}$. By elementarity, $(\text{NS}_{\omega_1}^+)^{M_n}$ is precipitous for every $n < \omega$, so every successor stages are well-founded. We now show that whenever we can construct $(M_\alpha, G_\alpha)$, it is always well-founded.

Remark. Notice that here, the use of $\text{Col}(\omega, \theta)$ is to collapse $H_\theta$ to countable size, so we can always pick $G_n$ (clearly not unique) in $V^{\text{Col}(\omega, \theta)}$. Because of this, we cannot do this iteration to any ordinal stages, but only as much as we want.

*https://ivv5hpp.uni-muenster.de/u/rds/sat_ideal_better_version.pdf
Theorem 3. Let $I \subset P(\omega_1)$ is a precipitous normal uniform ideal on $\omega_1$, then $V$ is generically iterable via $I^+$ and its images.

Here, being generically iterable means that $V$ can be iterated along the sequence without being ill-founded at limit stages.

Proof. A variant of the argument of the iterability of $V$ via a measure in $V$ and its images. Suppose the statement is false. Then we pick the least triple $(\theta, \lambda, \alpha)$ with respect to lexicographic order, such that

- $\theta$ is the least ordinal such that in $V^\Col(\omega, \theta)$, there is a $\lambda < \theta$ such that the $\lambda$-th generic iteration taken inside $V^\Col(\omega, \theta)$ is ill-founded.
- $\lambda$ is the least ordinal such that the $\lambda$-th generic iteration contains an ill-founded sequence of ordinals;
- $\alpha$ is the least ordinal such that there is a ill-founded sequence of ordinals below $j_{0\lambda}(\alpha)$.

Suppose $\gamma < \lambda$ such that there is an $\bar{\alpha}_1$, $j_{\gamma\lambda}(\bar{\alpha}_1)$ is the first element of the infinite descending sequence in $M_\lambda$. By elementarity, $M_\gamma$ sees that $(\theta, \lambda - \gamma, \bar{\alpha}_1)$ is lexicographically smaller, and it satisfies our requirements listed above. Contradiction.

2 Effective counterexamples to CH

In the last lecture, we have proved that $\MM \implies 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. This implies a surjection $f : \mathbb{R} \rightarrow \omega_2$. We now look at the set

$$R_f = \{(x, y) \in \mathbb{R}^2 : f(x) \leq f(y)\}.$$  

What are possible levels of definability of $R_f$? And can we have $f$ such that $R_f \in L(\mathbb{R})$? Or even: Can(in the presence of large cardinals, or under $\MM$) such an $R_f$ be projective?

Definition. $R \subset \mathbb{R}^n$ is projective iff $R$ is definable(with parameters) over $(H_{\omega_1}; \in)$.

This is not the usual definition for projectiveness; however, since every element in $H_{\omega_1}$ is coded by a real, $H_{\omega_1}^{L(\mathbb{R})} = H_{\omega_1}^{\mathbb{R}}$. So if something is definable over $(H_{\omega_1}; \in)$, then it is certainly inside $L(\mathbb{R})$. Equivalently, $R$ is projective iff we can write $\bar{x} \in \bar{R}$ iff

$$\exists x_0 \in \mathbb{R} \forall x_1 \in \mathbb{R} \ldots Q x_k(\bar{x}, x_0, \ldots, x_k) \in C,$$

where $C$ is a Borel set of $\mathbb{R}^{n+k+1}$.

Let us look at $H_{\omega_2}$. A formula $\phi$ is $\Pi^2_2$ is $H_{\omega_2}$ if it is equivalent(in $\ZFC$) to a function of the form:

$$\forall A \in H_{\omega_2} \exists B \in H_{\omega_2} \psi(A, B),$$

where $\psi$ is $\Sigma_0$. It turns out that $\MM$ is complete with respect to $\Pi^2_2$ statements. Important example of $\Pi^2_2$ statements:

$\downarrow L(\mathbb{R}) = \text{the least transitive model of ZF which contains } \mathbb{R} \cup \text{ORD}.$
$u_2 \downarrow = \omega^Y_2$.

- Admissible Club Guessing (ACG).

- $\phi_{AC}$ and $\psi_{AC}$, etc.

$u_2 = \omega_2$ is a $\Pi^1_2$-statement: Since $u_2 = \sup\{ (\omega^V_1)^+L[x] : x \in \mathbb{R} \}$, and $u_2 \leq \omega^Y_2$, we have

- $u_2 \geq \omega^Y_2 \iff \forall \alpha < \omega_2 \exists x \in \mathbb{R} [(\omega^V_1)^+L[x] \geq \alpha]$;

- $(\omega^V_1)^+L[x] \geq \alpha \iff \exists \beta \exists \beta \exists N [N \text{ is a transitive structure of height } \beta \wedge N \models \text{"Everything is at most countable"} \wedge L_\beta[x] \models \text{"} \alpha \leq N \cap \text{ORD}^\downarrow \text{"}]$.

Under the hypothesis $\forall x \in \mathbb{R} (\exists x^\#)(\text{given by MM})$ and $u_2 = \omega_2$, we have

$$f : \mathbb{R} \to \omega_2; \quad \omega \supset x \mapsto \omega^1 L[x].$$

Since $u_2 = \omega_2$, $f$ is cofinal. Now look at $R_f$ we defined above and we want to claim this is projective, actually $\Delta^4_1$. For any $x, y \in \mathbb{R}$, we have

$$(\omega^V_1)^+L[x] \leq (\omega^V_1)^+L[y] \iff \exists z \in \omega \exists \tau \exists (L_\tau[z]; U) \text{ iterable } [\kappa = \text{crit}(j) \wedge (L_\tau[z]; U) \models \kappa^+L[x] \leq \kappa^+L[y]].$$

**Proof.** Note that here, $\kappa^+L[x]$ and $\kappa^+L[y]$ is actually the interpretation inside $L_\tau[z]$, that is: $(\kappa^+L[x])L_\tau[z]$ and $(\kappa^+L[y])L_\tau[z]$. Thus to get $\iff$ direction, we may need to assume $x, y$ are Turing reducible to $z$(or other canonical way), and we want to prove that $(\kappa^+L[x])L_\tau[z] = \kappa^+L[z]$. Let $z, \tau$ be chosen to satisfy:

$$(L_\tau[z]; U) \models \kappa^+L[x] \leq \kappa^+L[y],$$

for $\kappa = \text{crit}(j)$. Then by the amenability of $(L_\tau[z], U)$, we may iterate this structure and see that $\tau = \kappa^+L[z]$. Thus there are no more subset of $\kappa$ beyond $\tau$ and by elementarity, $\kappa^+L[x] = (\kappa^+L[x])L_\tau[z]$.

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$u_2$ is the second uniform indiscernible ordinal: Suppose $x^\#$ exists for all $x \in \mathbb{R}$. Since the $x$-indiscernible ordinal class $C_x$ is a club for every $x \in \mathbb{R}$, $\bigcap_{x \in \mathbb{R}} C_x = (u_i : i \geq 1)$ is another class of indiscernibles called the uniform indiscernibles. Clearly countable ordinals can never be uniform indiscernible, and $\omega_1$ is uniform indiscernible, $u_1 = \omega_1$. 

\[\Box\]
So the complexity of this statement is $\Sigma_2$ over $H_{\omega_1}$. Thus, this statement is $\Delta^1_3$.

**Definition.** ACG is the following statement:

$$\forall C \subset \omega_1 \text{ club } \exists D \subset C \text{ club } \exists x \subset \omega_1 [D \text{ is the set of all countable } x\text{-admissible}].$$

Here, $x$-admissible are ordinals $\tau$ such that $L_\tau[x]$ are model of KP set theory. Or just: $D \in L[y]$ for some $y \subset \omega$.

**Remark.** It can be proved that ACG implies the existence of $x^\#$ for all $x \subset \omega$.

**Definition.** $\psi_{ACG}$ is the following statement:

$$\forall S, T \subset \omega_1 \text{ stationary and co-stationary } \exists \eta < \omega_2 \exists C \subset \omega_1 \text{ club } \forall \xi \in C [\xi \in T \iff f_\eta(\xi) \in S].$$

Here, $f_\eta$ is the function defined by some surjection $g : \omega_1 \to \eta$ such that $f_\eta(\xi)$ is the ordertype of $g''\xi$. It is also called the canonical function of $\eta$.

All the listed $\Pi^H_2$ statements are implied by (*), and $\text{MM}^{++}$ implies (*). Next we want to show ACG implies $u_2 = \omega_2$ and $\text{MM}^{++}$ implies ACG.

**Theorem 4.** $\text{ACG} \implies u_2 = \omega_2$.

**Proof.** (Sketch) Let $\alpha < \omega_2$ and some bijection $f : \omega_1 \to \alpha$. Moreover, fix a continuous tower $(X_i : i < \omega_1)$ of countable substructure of $H_\theta$. Let $N_i$ be the transitive collapse of $X_i$. We then let $f \in X_0$, which gives that there is $\alpha_i > \omega_1^{N_i}$ in $N_i$ such that $\alpha_i$ would be mapped to $\alpha$ in $H_\theta$. We can then modify the tower such that $\alpha_i < \omega_1^{N_i+1}$ for every $i < \omega_1$.

Thus ACG gives a club $D \subset \{\omega_1^{N_i} : i = \omega_1^{N_i}, i < \omega_1\}$, and $D \in L[x]$. We may then assume that $D$ is definable with $\omega_1^Y$ as the only parameter. Thus,

$$\xi \in D \iff L[x] \models \phi(\xi, x, \omega_1^Y);$$

Assuming that $\eta < \omega_1^Y$ is $x$-indiscernible, we have

$$\xi \in D \cap \eta \iff L[x] \models \phi(\xi, x, \eta).$$

So now $\eta \mapsto \omega_1^Y$, and $D \cap \eta \mapsto D$ by the elementary embedding from $L[x]$ to itself. This gives every $x$-indiscernible in $D$ is a limit point of $D$. Thus if $\xi \in D$, then $\xi = \omega_1^{N_\xi}$ and the next $x$-indiscernible $> \xi$ is bigger than $\omega_1^{N_{\xi+1}}$, thus bigger than $\alpha_\xi$.

Now we pick another tower $(Y_i : i < \omega_1)$ such that $x^\# \in Y_0$ (in particular, $D \in Y_0$), and $f \in Y_0$. So there is a club $E \subset D$ such that $Y_i \cap \alpha = X_i \cap \alpha$ for all $i \in E$. Now if $i \in E$, we denote the transitive collapse of $Y_i$ as $M_i$, and thus

$$M_i \models "\text{the next } x\text{-indiscernible } > i \text{ is } > \alpha_i".$$  

By elementarity, this gives

$$H_\theta \models "\text{the next } x\text{-indiscernible } > \omega_1 \text{ is } > \alpha."$$

which gives $u_2 = \omega_2$. 

\[ \square \]
Theorem 5. $\text{MM}^{++} \implies \text{ACG}$. 

Proof. (Sketch, Easier) Let $C \subset \omega_1$ be a club. Now we are going to construct a tower $(X_i : i < \omega_1)$ of countable substructures of $H_\theta$, where $\theta \geq \omega_2$. Let $X_0$ be some countable transitive substructure of $H_\theta$, and $N_0$ its transitive collapse. Let $X_0$ satisfy:

- $\omega_1 \in X_0$ and $\omega_1^{N_0} = \alpha_0$;
- $C \in X_0$ and $C \cap \alpha_0 \in N_0$.

Let $G_0 = \{ s \in P(\alpha_0) \cap N_0 : \alpha_0 \in j_0(x) \}$. Then this filter is $N_0$-generic, since $\text{NS}_{\omega_1}^{+N_0}$ is saturated (by $\text{MM}$, $\text{NS}_{\omega_1}$ is saturated). Now by the precitiousness, we can do the generic iteration:

\[
\begin{array}{cccccc}
H_\theta & \searrow & & \nearrow & & \\
& N_0 & \longrightarrow & N_1 & \longrightarrow & \cdots \\
& \downarrow & & \downarrow & & \\
& N_{<\omega_1} & \longrightarrow & \cdots
\end{array}
\]

where $N_{i+1} = \text{Ult}(N_i, G_i)$. By elementarity, the derived $G_i$ is always saturated, so this iteration process can keep on going before we meet $\omega_1$. Now let $X_i = \text{ran}(j_i)$. Since $N_0$ is countable, we can find some countable $x \subset \omega$ such that $x$ codes $N_0$. Now by the following unproved claim:

Claim. Suppose $\alpha < \omega_1$ is $x$-admissible, then $\alpha$ is the limit point of $\{ \omega_1^{N_i} : i < \omega_1 \}$.

We have that every $x$-admissible ordinal $\alpha$ is inside $C$ since $\omega_1^{N_i}$ is the limit point of $C$ for every $i < \omega_1$. Let $D$ be the set of all limit point of $\alpha$ and hence ACG is proved. 

Now we would like to present a harder proof which can be further modified into a way to prove $\text{MM}^{++} \implies (*)$.

Proof. (Sketch, Harder, [1]) We would like to force the existence of some iterable countable structure $(M; I)$, together with its generic iteration $(M_i; I_i : i < \omega_1)$ such that $M_{\omega_1} = H_{\omega_2}^{V \#}$. We do it via a forcing which preserves stationary subsets of $\omega_1$.

We aim to find a transitive model $N$ in the generic extension such that

$$N \models "\exists \text{generic iteration } (M_i; G_i : i < \omega_1), |M_i| = \omega \text{ s.t. }$$

$$M_0 \text{ iterable } \land M_{\omega_1} = \lim \text{dir}_{i \to \omega_1} M_i = (H_{\omega_2}^V, \in, \text{NS}_{\omega_1}^V)".$$ 

It seems that we only need the $\text{MM}^{++}$ to make $\text{NS}_{\omega_1}$ saturated until this claim. However, to prove this claim we may need a little bit more, say there is a measurable cardinal in $V$, or $P(\omega_1)^\#$ exists. This follows from descriptive set theory, where one can draw the conclusion from the existence of such a $\omega_1$-iterable structure $N_0$.

Clearly, the iteration cannot be performed in $V$, since $|M_{\omega_1}| = \aleph_1$. Moreover, since the generic iteration embedding is cofinal, $M_{\omega_1}$ adds a $\omega$-cofinal sequence of $\omega_2$. 

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\(\text{September 15, 2020 Jiaming Zhang}\)
Think of $N$ as a term model. The forcing will consist of finite sets of sentences in a language describing the full theory of such a model + starting to prove that this model is well-founded by ranking the constants:

$$\phi(c_{i_0}, \ldots, c_{i_k}), \ f: c_{i_0} \mapsto \xi \in \text{ORD}$$

such that in some outer model, this finite piece of information can be extended to a maximal consistent theory + a proof that the model which arises is well-founded.

Our forcing notion will actually have size $2^{\omega_2} \geq \omega_3$. We will need to assume $2^{\omega_2} = \omega_3$, which follows from $\Diamond_{\omega_3}$. We will finish this proof in our next lecture.

References


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$^1$There is a sequence $((Q_\alpha, A_\alpha): \alpha < \omega_3)$ such that $(Q_\alpha: \alpha < \omega_3)$ is a tower of transitive substructures of $H_{\omega_3}$ of size $\mathfrak{u}_2$ with $\bigcup_\alpha Q_\alpha = H_{\omega_3}$; Moreover, for all $A \subset H_{\omega_3}$, $\{\alpha: (Q_\alpha, A_\alpha) \prec (H_{\omega_3}, A)\}$ is stationary.