Ralf Schindler: Talks#1 on Logic Summer School of Fudan University, 2020

- How many real numbers are there?
- More specifically: We want to discuss 2 sets of prominent axioms which decides the size of $\beth_0$ the same way;
  - Forcing Axioms: MA, PFA, SPFA = MM;
  - Woodin’s Axiom (⋆)
- MM++ ⇒ (⋆)

Richness: If you have a set of axioms which has a transitive model, then you have a transitive model inside $L$.
"Maximize": If an object can be imagined to exist, then it exists.

TODAY:
- Stationary sets;
- Forcing revisited;
- Forcing Axioms: MA;
- Proper forcing; semi-proper forcing; stationary set preserved forcing;
- PFA, SPFA, MM.

1 Stationary Sets

Definition. $C \subset [X]^\omega$ is a club iff:
$$\exists f : X^{<\omega} \to X, \quad C = \{ x \in [X]^\omega : f^{\upharpoonright <\omega} x^{<\omega} \subset x \}.$$  

Remark. We may think $f$ as a set of relations on $X$, and consider $(X; f)$ as a model. Then $C$ is just the collection of every countable substructures of $(X; f)$.

Definition. $S \subset [X]^\omega$ is stationary iff $S \cap C \neq \emptyset$ for all club $C \subseteq [X]^\omega$.

Remark. Hence, $S$ is stationary iff for all models $(X; f)$, there is some $x \in S$, $x \prec (X; f)$.

Lemma 1 (Fodor). Let $S \subseteq [X]^\omega$ be stationary, let $f : S \to V$, $f(x) \in x$ for all $x \in S$ (regressive), then there is a stationary $T \subseteq S$, $f \mid T$ is constant.

Proof. o.w.(otherwise) f.a.(for all) $a \in X$, $S_a = \{ x \in S : f(x) = a \}$ is nonstationary. Thus there is a club $C_a = \{ x \in [X]^\omega : f_a^{\upharpoonright <\omega} x^{<\omega} \subset x \}$ with some function $f_a : X^{<\omega} \to X$ and $C_a \cap S_a = \emptyset$.

Define $f^*(a, u) = f_a(u)$, for $u \in [X]^{<\omega}$. Let $C = \{ x \in [X]^\omega : f^* x^{<\omega} \subset x \}$. For all $a \in x \in C$, $x \in C_a$. Pick $x \in S \cap C$. Let $a = f(x) \in x$, then $x \in C_a$. Contradicts to the choice of $C_a$.

Observation. If $S \subset [\omega_1]^\omega$ is stationary, then so is $\{ \xi \in S : \xi \in \omega_1 \}$.

[Hint. if $C \subset [\omega_1]^\omega$ is a club, then so is $\{ \xi \in S : \xi \in \omega_1 \}$.]
1.1 Splitting stationary sets

**Theorem 2** (Solovay). \( S \subset \omega_1 \) stationary, then we may split \( S = \bigsqcup_{i<\omega_1} S_i \), while all \( S_i \) are stationary.

**Proof.** \( \forall n \exists \alpha \exists \xi \langle \alpha, \xi \rangle \in \omega_1 \) is stationary.

**Claim.** \( \forall \alpha < \omega_1 \exists \beta \geq \alpha \{ \xi \in S : \alpha^\xi_n \geq \alpha \} \) stationary.

Otherwise, \( \forall n \exists \alpha \exists \xi \langle \alpha, \xi \rangle \in \omega_1 \) is stationary.

Therefore we can pick \( \xi \in (\bigcap_{n<\omega} C_n) \cap S \), thus \( \xi > \sup_n \alpha_n \). However, \( \alpha^\xi_n < \sup_m \alpha_m \) for all \( n \). Contradiction.

**Remark.** Improve: Fix \( n \) as in the **Claim.** As a immediate consequence of Fodor’s Lemma, we have

**Claim.** \( \forall \alpha < \omega_1 \exists \beta \geq \alpha \{ \xi \in S : \alpha^\xi_n = \beta \} \) stationary.

Now we only need the pairwise disjoint property. Construct \( (S_i, \beta_i : i < \omega_1) \) as the above **Claim.** **Claim:** Assume \( (S_i, \beta_i : i < j) \) are defined, let \( \alpha = \sup_{i<j} \beta_i + 1 \) and \( \beta_j = \beta \) as in the **Claim.** and let \( S_j \) be the corresponding set defined in the **Claim.**

**Comment.** (Shi.) This statement may be credited to Ulam, since the technique of Ulam matrix proves the statement for all successor ordinal instead of just \( \omega_1 \). This procedure is described in [3], Theorem 6.11.

**Comment.** In fact Solovay has proved that the above statement works for any weakly inaccessible cardinal. See [4]

2 Forcing

\( V \ni P, P = (\mathbb{P}; \leq P) \) a partial order. \( D \subset P \) is dense iff
\[
\forall p \in P \exists q \in D : q \leq P p(q \text{ is stronger than } p)
\]

\( G \subset P \) is \( V \)-generic iff \( G \cap D \neq \emptyset \) f.a. \( D \subset P, D \in V \) dense.

\( V[G] = \{ \tau^G : \tau \in V^P \} \) where \( \tau \) is a \( P \)-name.

**Theorem 3** (Forcing Theorem). If \( V[G] \models \phi(\tau^G, ...) \), then \( \exists \exists p \in G, p \models \phi(\tau, ...) \). If \( p \models \phi(\tau, ...) \), then \( V[G] \models \phi(\tau^G, ...) \) f.a. \( G \ni p \).

3 Forcing Axiom

**Definition.** \( P \) has the c.c.c.(countable chain condition) iff \( P \) does not have any uncountable antichain.

\( A \subset P \) is an antichain iff \( \forall p, q \in A, p \neq q \rightarrow p \perp q(p, q \text{ incompatible } = \text{ no common extension}) \).

\( \mathbb{C} = \text{Cohen forcing} = \omega^{<\omega}, p \leq C q \text{ iff } p \supset q \).
**Definition.** $\text{MA}_\kappa$ (Martin’s Axiom for $\kappa$): $\mathbb{P}$ has the c.c.c., $\mathcal{D} = \{D_i : i < \kappa\}$ a collection of dense sets; then there is a filter $G \subseteq \mathbb{P}$, $G \cap D_i \neq \emptyset$ for all $i < \kappa$.

$\text{MA}_\omega$ is always true: define $\omega$-sequence

$$p_1 \leq p_2 \leq \ldots \leq p_i \leq \ldots, \quad i < \omega$$

while $p_i \in D_i$. Thus the filter $G = \{q \in \mathbb{P} : \exists n \in \omega(q \geq p_n)\}$ is $V$-generic.

**Remark.** This is exactly the diagonal argument, known as the Rasiowa-Sikorski Lemma.

$\text{MA}_{2^{\aleph_0}}$ is false: $\mathbb{C}$ Cohen forcing: Let $(x_i : i < 2^{\aleph_0})$ enumerate all sets of $\omega^\omega$. $D_i = \{p \in \mathbb{C} : \exists n \in \text{dom}(p), p(n) \neq x_i(n)\}$. $\{D_i : i < 2^{\aleph_0}\}$ is a collection of dense sets. If $G \cap D_i \neq \emptyset$ f.a. $i < 2^{\aleph_0}$, then $\bigcup G : \omega \to \omega$, so $\bigcup G = x_i$ for some $i < 2^{\aleph_0}$. However,

$$\exists p \in G \exists n[p(n) \neq x_i(n) \implies x_i(n) \neq \bigcup G(n)]$$

Contradiction.

Using a.d.(almost disjoint) coding, we can prove the *Souslin Hypothesis*:

$$\text{MA}_{\omega_1} \implies 2^{\aleph_0} = 2^{\aleph_1}$$

**Claim.** $\exists$ a.d. sequence $(a_\xi : \xi < \omega_1)$ of subsets $\omega$, i.e., f.a. $\xi, \eta < \omega_1$, $\xi \neq \eta$, $a_\xi \cap a_\eta$ is finite.

**Proof.** Look at $2^{<\omega}$. Let $e : 2^{<\omega} \to \omega$ be bijection. Let $(b_\xi : \xi < \omega_1)$ be a sequence of pairwise different branches of the tree $2^{<\omega}$. Let $a_\xi = \{e(b_\xi \upharpoonright n) : n < \omega\}$. Then $a_\xi$ proves the statement.

**Theorem 4.** $\text{MA}_{\omega_1} \implies 2^{\aleph_0} = 2^{\aleph_1}$.

**Proof.** Let $(a_\xi : \xi < \omega_1)$ be a sequence of pairwise a.d. subsets of $\omega$. Let $X \subset \omega_1$. $p \in \mathbb{P}$ iff $p = (f, x)$:

- $f : n \to 2$, for some $n < \omega$;
- $x \subset X$ finite.

$(f', x') \leq_p (f, x)$ iff $f' \supset f, x' \supset x$, and $\{m \in \text{dom}(f') \setminus \text{dom}(f) : f'(m) = 1\} \cap a_\xi = \emptyset$ for all $\xi \in x$.

One can check that this forcing satisfies c.c.c. since every pair of conditions that shares a common $f$ is compatible. $\{(f, x) : n \in \text{dom}(f)\}$ is dense for all $n$; $\{(f, x) : \xi \in x\}$ is dense for all $\xi \in X$. the generic gives rise to a function $F : \omega \to \omega$ such that f.a. $\xi \in X$, $\{n \in \omega : F(n) = 1\} \cap a_\xi$ is finite. And if $\xi \notin X$, $\{(f, x) : \exists m \geq n(m \in a_\xi \land f(m) = 1)\}$ is dense f.a. $n < \omega$. Thus f.a. $\xi \notin X$, $\{n \in \omega : F(n) = 1\}$ is infinite.

In sum, the generic filter $G$ gives rise to $F : \omega \to \omega$ such that if $a \subset \omega$ such that $F$ is the characteristic function of $a$, then $[a \cap a_\xi$ of finite $\iff \xi \in X]$ f.a. $\xi < \omega_1$. So $a$ codes $X \subseteq \omega_1$ modulo $(a_\xi : \xi < \omega_1)$ in that sense. Thus,

$$\text{MA}_{\omega_1} \to \forall X \subset \omega_1 \exists a \subset \omega \forall \xi < \omega(\xi \in X \iff a \cap a_\xi \text{ is finite}) \quad (1)$$

Define $T : \mathcal{P}(\omega_1) \to \mathcal{P}(\omega)$, $X \mapsto a$ where $a$ satisfies (1). Clearly $T$ is injective.
Thus, \( \text{MA}_{\omega_1} \implies \neg \text{CH} \) since \( 2^{\aleph_0} = 2^{\aleph_1} \leq \aleph_2 \). One can show \( 2^{\aleph_0} > \aleph_1 \) and \( \text{MA}_\kappa \) for all \( \kappa < 2^{\aleph_0} \) is consistent.

We will go ahead and discuss more profound forcing axiom.

Rest of today:
- Proper forcing; (PFA)
- Semi-proper forcing; (SPFA)
- Stationary set preserving forcing. (MM)

4 Proper forcing

Definition. \( P \) is proper iff for all \( X \), if \( S \subseteq [X]^\omega \) is stationary, then \( S \) is still stationary in \( V^P \).

Remark. Here \( V^P \) means all generic extension.

Examples of forcing notions that are NOT proper:
- \( \text{Col}(\omega,\omega_1) \);
- (Shoot a club) Let \( S \subseteq \omega_1 \), \( S \) stationary and \( \omega_1 - S \) is stationary. There is a forcing which adds \( C \subseteq S \) club, every stationary subset of \( S \) remains stationary (In consequence, \( \omega_1 \) is not collapsed). But \( C \) witness the fact that \( \omega_1 - S \) is no longer stationary.

Definition. Let \( x \prec H_\theta \), \( x \) countable, \( p \in P \cap x \). \( q \leq p \) is \( x \)-generic iff f.a. \( \tau \in V^P \cap x \) such that \( \vdash \tau \in \check{\mathcal{H}}_\theta \), we have \( q \vdash \tau \in \check{x} \). (E.g. There is no \( x \) for \( \text{Col}(\omega,\omega_1) \) to be \( x \)-generic.)

Lemma 5. The following statements are equivalent:
1. \( P \) is proper;
2. F.a. \( x \prec H_\theta \), (\( x \) countable, \( \theta \) sufficiently large,) f.a. \( p \in P \cap x \), \( \exists q \leq p \) \( x \)-generic.

Proof. ([2], Theorem 31.7.)
(2) \( \implies \) (1): Let \( S \subseteq [X]^\omega \) be stationary. \( p \Vdash \check{C} \) is a club in\( [X]^\omega \), \( \check{C} = \{ x \in [X]^\omega : \check{f}^x x^{<\omega} \subseteq x \} \). Let \( x \prec H_\theta \), \( x \) countable, and \( p, \check{C}, \check{f} \in x \), \( x \cap X \subseteq S \) (possible, as \( S \) is stationary).

Let \( q \leq p \) be \( x \)-generic.
Claim. \( q \Vdash \check{C} \cap \check{S} \neq \emptyset \); in fact, \( q \Vdash (x \cap X) \subseteq \check{C} \).
This follows from the definition of \( x \)-genericity.
(1) \( \implies \) (2): We may not prove that for all substructures \( x \) (2) holds but, the countable

\[\text{Proper forcing does not collapse } \aleph_1. \text{ See [2], Lemma. 31.4.}\]
substructures satisfying (2) form a club of \([H_\theta]^{\omega_2}\). Towards a contradiction, let
\[
S = \{ x \prec H_\theta : |x| \leq \omega, \exists p \in x \cap P(\exists f \leq p \text{ x-generic}) \}
\]
is stationary. By Fodor's Lemma, let \(g : S \to V\) maps \(x\) to some \(p \in x\) where \(p\) does not have any \(x\)-generic extension. \(g\) is regressive and thus there is a stationary \(T \subset S\) such that \(\exists p \forall x \in T(p \in x \land \exists f \leq p(\exists f \text{ x-generic}))\). Pick a filter \(G\) that is \(V\)-generic, \(p \in G\). \(T\) is still stationary in \(V[G]\). This implies we may pick countable \(x \prec H_{\Omega[G]}\) so that \(x \cap H_\theta \in T\). This implies a contradiction since if \(\tau \in V^P \cap x \cap H_\theta, \models \tau \in H_\theta\), then \(\tau^G \in x \cap H_\theta\). This is forced by some \(q \leq p\).

**Definition.** \(x \prec H_\theta, x\) countable, \(p \in P \cap x, q \leq p\) is \(x\)-semigeneric iff \(f.a. \tau \in V^P \cap x, \models \tau \in \bar{x}(\iff \tau \in (x \cap \omega_1))\). That is, \(q \models \tau \in \alpha\), where \(\alpha = x \cap \omega_1 \in \omega_1\), since \(x \cap \omega_1\) is transitive.

**Definition.** \(P\) is \(x\)-proper iff \(x \prec H_\theta, \text{ countable, } P \in x\), \(f.a. \ p \in x \cap P\) there is \(q \leq p\) such that \(q\) is \(x\)-semigeneric.

**Observation.** \(P\) is proper, then \(P\) is semiproper.

**Definition.** \(P\) preserves stationary subsets (of \(\omega_1\)) iff
\[
\forall S \subset \omega_1(S \text{ stationary in } V \implies S \text{ stationary in } V^P).
\]

**Lemma 6.**
- \(P\) is \(x\)-proper \(\implies P\) preserves stationary subsets of \(\omega_1\);
- \(P\) has the c.c.c., then \(P\) is proper.

**Definition.**
- \(\text{PFA}\): Every \(\omega_1\) family of every proper forcing notion has a generic filter;
- \(\text{SPFA}\): Every \(\omega_1\) family of every semiproper forcing notion has a generic filter;
- \(\text{MM}\): Every \(\omega_1\) family of every stationary preserving forcing notion has a generic filter.

**Remark.** One cannot extend those axioms to \(\kappa\) families like what we do in \(\text{MA}\), since these axioms implies (as we shall later show,) that \(2^{\aleph_0} = \aleph_1 = \aleph_2\).

**Theorem 7.** The followings are equivalent:
- \(\text{MM}\);
- \(f.a. \ models M \in V (\text{signature } \leq \omega_1) \ f.a. P \text{ stationary set preserving f.a. } \phi \Sigma_1\text{-formula,}
  \text{if } V^P \models \phi(M), \text{ then } \exists j : M \to M \text{ elementary, } |M| \leq \omega_1, V \models \phi(M)\).

Proof given by: [1], Theorem 1.3.

\(^2\text{Suppose } \mathcal{C} \text{ is a club of countable } x \in [H_\theta]^{\omega_2} \text{ such that every } p \in P \cap x \text{ has an } x\text{-generic extension. Let } [H_\theta]^{\omega_2} \in H_\Omega, \text{ with } \Omega \text{ sufficiently large, and let some } x \prec H_\Omega \text{ be countable with } P \in x. \text{ Then some } \theta \text{ and } \mathcal{C} \text{ are elements of } x, \text{ but then } x \cap H_\theta \in \mathcal{C}, \text{ from which it follows that every } p \in P \cap x \text{ can be extended to an } x\text{-generic condition. So if f.a. sufficiently large } \theta \text{ there is a club of countable } x \in [H_\theta]^{\omega_2} \text{ s.t. every } p \in P \cap x \text{ can be extended to an } x\text{-generic condition, then for all sufficiently large } \theta \text{ and for every } x \in [H_\theta]^{\omega_2} \text{ with } P \in x, \text{ every } p \in P \cap x \text{ can be extended to an } x\text{-generic condition.}
References


