Subversions of forcing classes

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CUNY College of Staten Island and the CUNY Graduate Center

Proper and subproper forcing

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Definition

A forcing notion $\mathbb{P}$ is proper if the following holds: for all sufficiently large θ, if $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \mathrm{ZFC}^-$, $\sigma : \bar{N} \prec N$ with $\mathbb{P} \in \mathrm{ran}(\sigma)$, $\bar{N}$ countable and $p \in \mathbb{P} \cap \mathrm{ran}(\sigma)$, then there is a $q \leqslant p$ such that whenever $G \ni q$ is $\mathbb{P}$-generic, then $\bar{G} = \sigma^{-1}$“G is $\bar{\mathbb{P}} = \sigma^{-1}(\mathbb{P})$-generic over $\bar{N}$.

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It follows in this situation that σ can be extended to an elementary embedding

$$\sigma \subseteq \tilde{\sigma} : \bar{N}[\bar{G}] \prec N[G].$$

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A forcing notion $\mathbb{P}$ is subproper if the following holds: for all sufficiently large θ, if $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \mathsf{ZFC}^-$, $\sigma : \bar{N} \prec N$ with $\mathbb{P} \in \mathrm{ran}(\sigma)$, $\bar{N}$ countable and $p \in \mathbb{P} \cap \mathrm{ran}(\sigma)$,

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2. $\bar{G} = (\sigma')^{-1}``G$ is $\bar{\mathbb{P}}$-generic over $\bar{N}$.
3. $\mathrm{Hull}^N(\delta \cup \mathrm{ran}(\sigma)) = \mathrm{Hull}^N(\delta \cup \mathrm{ran}(\sigma'))$, where $\delta = \delta(\mathbb{P})$ is the minimal size of a dense subset of $\mathbb{P}$.

The problem with this definition is that it is too restrictive. For example, if every element of $\mathrm{ran}(\sigma)$ is definable in N from $a_0, \ldots, a_{n-1}$, then σ' would have to be equal to σ.

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Complete and subcomplete forcing

Completeness

In chapter V of Shelah's *Proper and improper forcing*, he defines the concept of $\mathcal{E}$-completeness, for a family $\mathcal{E} \subseteq [\mu]^\omega$. Jensen calls a special case of this condition just completeness, and one can express it as follows:

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$\mathbb{P}$ is complete if there is a θ with $\mathbb{P} \in H_\theta$ such that the following holds: if

$$\sigma : \bar{N} \prec N$$

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Examples of subcomplete forcings:

- Namba forcing (assuming CH, Jensen)
- Příkrý forcing (Jensen)
- generalized Příkrý forcing (Minden)
- Magidor forcing (to change cofinality to ω_1, F.)
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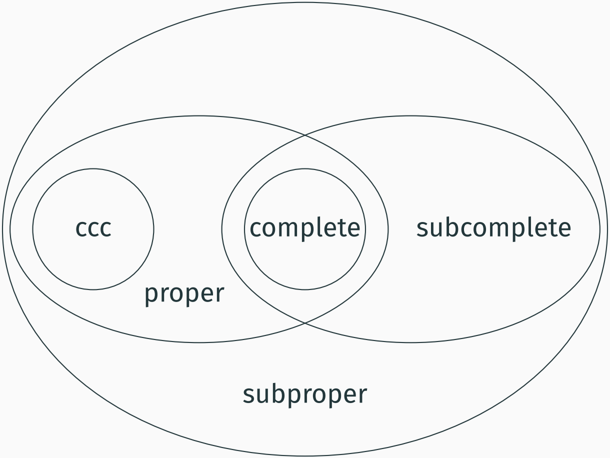
Jensen showed that subcomplete and subproper forcing can be iterated with revised countable support in the style of Donder and Fuchs, with intermediate collapses to ω_1.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Forcing principles

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- Forcing axioms
- Bounded forcing axioms
- Maximality principles

Focus: subcomplete forcing

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- The Subcomplete Forcing Axiom, SCFA ($\Gamma =$the collection of all subcomplete forcings)

- Assuming the existence of a supercompact cardinal κ, one can iterate proper forcings with countable support, with iterands given by a Laver function for the supercompactness of κ, producing a model in which PFA $+ \kappa = \omega_2 = 2^\omega$ holds. (Baumgartner)

Forcing SCFA

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- This can be modified to work for SPFA, by iterating semi-proper forcings with rcs, inserting collapses to ω_1 after each step in the iteration. (Foreman-Magidor-Shelah)
- This can be modified to work for SCFA, by iterating subcomplete forcings. During the iteration, CH, and even $\diamondsuit$ will be forced, and since no reals are added, and $\diamondsuit$ is preserved, the final model will satisfy SCFA $+ \kappa = \omega_2 + \diamondsuit$. (Jensen)

Bounded forcing axioms

For a forcing class Γ, the bounded forcing axiom BFA_Γ, introduced by Goldstern-Shelah, says that if $\mathbb{P} \in \Gamma$ and Δ is a collection of ω_1 many maximal antichains in $\mathbb{B} = r.o.(\mathbb{P})$, each of which has size at most ω_1, then there is a filter in $\mathbb{B}$ that meets them all.

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Bagaria has characterized this principle by saying that whenever $\mathbb{P} \in \Gamma$, then

$$\langle H_{\omega_2}, \in\rangle \prec_{\Sigma_1} \langle H_{\omega_2}, \in\rangle^{V^{\mathbb{P}}}$$

which means that for every $a \in H_{\omega_2}$ and every Σ_1-formula $\varphi(x)$,

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If Γ has the property that whenever $p \in \mathbb{P} \in \Gamma$, then $\mathbb{P}_{\leqslant p}$ is equivalent to some $\mathbb{Q} \in \Gamma$, then this characterization can be equivalently expressed by saying that whenever $\mathbb{P} \in \Gamma$ and G is $\mathbb{P}$-generic, it follows that

$$\langle H_{\omega_2}, \in \rangle \prec_{\Sigma_1} \langle H_{\omega_2}, \in \rangle^{\mathrm{V}[G]}$$

Theorem (Goldstern-Shelah)

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Observation

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Maximality principles

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Definition (Väänänen-Stavi, Hamkins)

Thus, $\mathrm{MP}_\Gamma(X)$ is the scheme expressing that for every formula φ with parameters in X, if φ can be forced to be true by a forcing in Γ in such a way that it stays true in any further forcing extension by a forcing in Γ, then φ is true.

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Depending on Γ, different parameter sets X are reasonable. In the case where Γ is the class of subcomplete forcing, the maximal parameter set possible is H_{ω_2}, and the resulting “boldface” principle is $\mathsf{MP}_{\mathsf{SC}}(H_{\omega_2})$. The lightface principle allows for no parameters and is denoted $\mathsf{MP}_{\mathsf{SC}}$. But parameters from H_{ω_1} are free, it turns out.

Consistency strengths

As with maximality principles for other canonical forcing classes, the consistency strength of $\mathsf{MP}_{\mathsf{SC}}$ is just ZFC, while $\mathsf{MP}_{\mathsf{SC}}(H_{\omega_2})$ is equiconsistent with ZFC, together with the scheme expressing that δ is regular and $\mathrm{V}_\delta \prec \mathrm{V}$ (δ is a constant symbol here).

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It also works for subproper forcing.

Consequences

The principle $\mathsf{MP}_{\mathsf{SC}}$ implies CH, and even $\diamondsuit$, because each can be forced to be true by a subcomplete forcing (for example, by $\mathrm{Add}(\omega_1, 1)$), and once true, it persists to further subcomplete forcing extensions.

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The maximality principles don't have this monotonicity property.

Geology

Motivation

There is a long history of results on the existence of definable well-orders of $\mathcal{P}(\omega)$ or even $\mathcal{P}(\omega_1)$, assuming the (bounded) proper forcing axiom.

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I want to present a result that goes in a similar direction, but assumes $\mathsf{MP}_{\mathsf{SC}}$ instead, together with a (natural?) assumption on the set-theoretic geology of the ambient universe.

Since $\mathsf{MP}_{\mathsf{SC}}$ implies CH, we are in a very different situation than with BPFA. The methods used are also very different.

Definition

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Fact (Ground model enumerability; Laver, F.-Hamkins-Reitz)

There is a formula $\varphi(x, y)$ such that for every $p \in \mathrm{V}$, the class

$$W_p = \{x \mid \varphi(x, p)\}$$

is a ground, and such that every ground is of the form W_p, for some parameter p.

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is a ground, and such that every ground is of the form W_p, for some parameter p.

Hence the following is a natural first order definition:

Definition

The mantle $\mathbb{M}$ is the intersection of all grounds, i.e.,

$$\mathbb{M} = \bigcap_p W_p.$$

In the beginning, it was unclear whether the mantle is a model of ZFC. Together with Hamkins and Reitz, we isolated the following crucial hypothesis:

Strong Downward Directedness of Grounds hypothesis

Any set-sized collection of grounds has a common ground. That is, if a is a set, then there is a q such that $W_q \subseteq \bigcap_{p \in a} W_p$.

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This was a crucial hypothesis, which would answer many initially open questions.

Fact (F.-Hamkins-Reitz)

Assume the strong DDG. Then:

1. *The mantle is constant across the grounds (directedness suffices).*
2. *The mantle is a model of* ZF *(directedness suffices).*
3. *The mantle satisfies* ZFC *(directedness + local set-directedness suffice).*

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Initially, we had defined the "generic mantle" to be the intersection of the mantles of all set-forcing extensions, and proved that it satisfies ZF, and is the largest forcing-invariant class.

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Initially, we had defined the "generic mantle" to be the intersection of the mantles of all set-forcing extensions, and proved that it satisfies ZF, and is the largest forcing-invariant class.

But note that if the strong DDG holds in every set-forcing extension (this is what we called the generic strong DDG), then by 1. above, the generic mantle is equal to the mantle, and the mantle is a forcing-invariant inner model of ZFC.

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Theorem

The mantle is a model of ZFC*, and it is invariant under set-forcing.*

Usuba's contributions

Using a forgotten theorem of Bukovsky ingeniously, Usuba proved:

Theorem (Usuba)

The strong DDG holds.

Thus, the generic strong DDG also holds, and we have:

Theorem

The mantle is a model of ZFC*, and it is invariant under set-forcing.*

Also, the mantle is equal to the generic mantle, and is the intersection of the generic multiverse, so it is the largest forcing-invariant class.

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If there is an extendible cardinal, then there are only set-many grounds.

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Observation

If there are only set-many grounds, then the mantle is a ground.

This is because by the strong DDG, there is a ground that is contained in every ground, in this situation, and this is the minimal ground, and the mantle.

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We have the following properties of the mantle:

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I'll give an application in the following - it seems like there should be many similar uses of these techniques.

The goal theorem

I want to show:

Theorem (F.)

Assume $\mathsf{MP}_{\mathsf{SC}}$ *and there are only set-many grounds. Then there is a well-order of* $\mathcal{P}(\omega_1)$*, definable without parameters, of order type* ω_2*.*

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Before giving a proof of this theorem, I need a crucial coding device, and I should remark:

Observation

If $\mathsf{ZFC} + \mathsf{MP}_{\mathsf{SC}}(H_{\omega_2})$ *is consistent, then so is the theory* $\mathsf{ZFC} + \mathsf{MP}_{\mathsf{SC}}(H_{\omega_2}) +$ *there are only set-many grounds.*

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Theorem (Jensen)

Let κ be an inaccessible cardinal, and assume that GCH *holds below κ. Let $A \subseteq \kappa$ be a set of regular cardinals. Then there is a subcomplete, κ-c.c. forcing $\mathbb{P}$ of size κ such that if G is $\mathbb{P}$-generic, then $\kappa = \omega_2^{\mathrm{V}[G]}$ and for every regular $\tau \in (\omega_1, \kappa)$,*

$$\mathrm{cf}^{V[G]}(\tau) = \begin{cases} \omega_1 & \text{if } \tau \in A \\ \omega & \text{otherwise.} \end{cases}$$

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$$\mathrm{V} = \mathbb{M}[g]$$

g being $\mathbb{P}$-generic over $\mathbb{M}$, for some poset $\mathbb{P} \in \mathbb{M}$.

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This, in turn, implies that there is a proper class of inaccessible cardinals in $\mathbb{M}$, and since V is a set forcing extension of $\mathbb{M}$, also in V.

Let $\alpha > \overline{\overline{\mathbb{P}}}$, and let $\langle \kappa_i \mid i \leqslant \omega_1 \rangle$ enumerate the next $\omega_1 + 1$ inaccessible cardinals above α. We can perform an Easton iteration of at least countably closed forcing notions in order to reach an extension in which GCH holds below κ_{ω_1}, such that if h is generic, then each κ_i is still inaccessible in $\mathrm{V}[h]$.

The proof

Now, given a subset A of ω_1 in V, let $\tilde{A} = \{\kappa_i \mid i \in A\}$.

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Then let

$$\mathbb{Q} = \mathbb{N}^{\mathrm{V}[h]}_{\tilde{A},\kappa_{\omega_1}}$$

be the extended Namba forcing for $\tilde{A}$ in $\mathrm{V}[h]$. Thus, $\mathbb{Q}$ is κ_{ω_1}-c.c., and if G is $\mathbb{Q}$-generic over $\mathrm{V}[h]$, then for every $\mu \in \tilde{A}$, $\mathrm{V}[h][G]$ thinks that $\mathrm{cf}(\mu) = \omega_1$, and for every $\nu \in \kappa_{\omega_1} \setminus \tilde{A}$ which is regular in $\mathrm{V}[h]$, $\mathrm{V}[h][G]$ thinks that $\mathrm{cf}(\nu) = \omega$. In particular, the latter is true for every κ_j with $j < \omega_1$, $j \notin A$, since κ_j remains regular in $\mathrm{V}[h]$. κ_{ω_1} becomes ω_2 in $\mathrm{V}[h][G]$.

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So in $\mathrm{V}[h][G]$, we have:

$$\mathrm{cf}(\kappa_i) = \begin{cases} \omega_1 & \text{if } i \in A \\ \omega & \text{if } i \in \omega_1 \setminus A \end{cases}$$

The proof

Now if $\mathrm{V}[h][G][I]$ is a further subcomplete forcing extension of $\mathrm{V}[h][G]$, then the cofinality of κ_i, for $i < \omega_1$, cannot change, since subcomplete forcing preserves ω_1.

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Moreover, the sequence $\langle \kappa_i \mid i < \omega_1 \rangle$ is definable in $\mathrm{V}[h][G][I]$ from the parameter α, as the enumeration of the next ω_1 many inaccessible cardinals in $\mathbb{M}$ beyond α.

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Hence, A is definable in $\mathrm{V}[h][G][I]$ as the set of $i < \omega_1$ such that $\mathrm{cf}(\kappa_i) = \omega_1$.

Let $\psi(A, \alpha)$ be the statement expressing that if $\langle\lambda_i \mid i < \omega_1\rangle$ enumerates the next ω_1 many inaccessible cardinals of $\mathbb{M}$ beyond α, then for all $i < \omega_1$, $i \in A$ iff $\mathrm{cf}(\lambda_i) = \omega_1$, and $i \notin A$ iff $\mathrm{cf}(\lambda_i) = \omega$. Then the statement $\varphi(A)$, expressing that there is an α with $\psi(A, \alpha)$, holds in $\mathrm{V}[h][G][I]$.

The proof

Since I was generic for an arbitrary subcomplete forcing notion in $\mathrm{V}[h][G]$, this means that $\varphi(A)$ is necessary with respect to subcomplete forcing extensions in $\mathrm{V}[h][G]$, and hence it is forceably necessary with respect to subcomplete forcing in V.

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Given $A \subseteq \omega_1$, let β be such that $\psi(A, \beta)$ holds. In $\mathrm{V}^{\mathrm{Col}(\omega_1,\beta)}$, and in any further subcomplete forcing extension, $\psi(A, \beta)$ continues to hold, and $\beta < \omega_2$ there. Thus, if we let α_A be least such that $\psi(A, \alpha_A)$ holds, it is subcomplete forceably necessary that $\alpha_A < \omega_2$, and so, it is already less than ω_2 in V.

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This shows that if we define $A <^* B$ iff $\alpha_A < \alpha_B$, for $A, B \subseteq \omega_1$, then this is a well-ordering of $\mathcal{P}(\omega_1)$ of order type ω_2, as wished. $\square$

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Theorem (F.)

Assume $\mathsf{MP}_{\mathsf{SC}}(H_{\omega_2})$*, and assume there is no inner model with an inaccessible limit of measurable cardinals. Then there is a well-order of* $\mathcal{P}(\omega_1)$ *of order type* ω_2*, definable from a subset of* ω_1*.*

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Theorem (F.)

Suppose $0^\#$ does not exist, and that BSCFA holds. Then there is a well-order of $\mathcal{P}(\omega_1)$ of order type ω_2, definable without parameters. This well-order is in $\mathsf{L}(\mathcal{P}(\omega_1))$, is Δ_1-definable from a subset I' of ω_1 there, and it is $\Delta_1^{\langle H_{\omega_2},\in\rangle}$-definable in I'.

Thus, we get definable well-orders from subcomplete forcing principles both from large cardinal assumptions and from anti-large cardinal assumptions.

The role of CH

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I will exemplify this by considering the effects of these forcing axioms on (the failure of) square principles.

Squares

Definition (Jensen)

Let κ be a cardinal. $\square_\kappa$ says that there is a $\square_\kappa$-sequence, that is, a sequence $\langle C_\alpha \mid \kappa < \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ such that each C_α is club in α, $\mathrm{otp}(C_\alpha) \leqslant \kappa$ and for each β that is a limit point of C_α, $C_\beta = C_\alpha \cap \beta$.

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If λ is also a cardinal, then $\square_{\kappa,\lambda}$ is the assertion that there is a $\square_{\kappa,\lambda}$-sequence, i.e., a sequence $\langle \mathcal{C}_\alpha \mid \kappa < \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ such that each $\mathcal{C}_\alpha$ has size at most λ, and each $C \in \mathcal{C}_\alpha$ is club in α, has order-type at most κ, and satisfies the coherency condition that if β is a limit point of C, then $C \cap \beta \in \mathcal{C}_\beta$.

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$\square_{\kappa,\kappa^+}$ holds trivially, so $\square^*_\kappa$ is the weakest nontrivial principle here.

Definition (Todorcevic, Jensen (?))

Let λ be a limit of limit ordinals. A sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda,\ \alpha \text{ limit}\rangle$ is coherent if for every limit $\alpha < \lambda$, $\mathcal{C}_\alpha \neq \emptyset$ and for every $C \in \mathcal{C}_\alpha$, C is club in α, and for every limit point β of C, $C \cap \beta \in \mathcal{C}_\beta$.

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A thread through $\vec{\mathcal{C}}$ is a set T such that $\vec{\mathcal{C}}^\frown\{T\}$ is coherent.

A coherent sequence is maximal if it has no thread.

If κ is a cardinal, then the principle $\square(\lambda, {<}\kappa)$ says that there is a maximal coherent sequence of length λ all of whose elements have size less than κ, and such a sequence is called a $\square(\lambda, {<}\kappa)$-sequence.

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The principle $\square(\lambda, 1)$ is denoted $\square(\lambda)$.

CONSEQUENCES OF MM AND SCFA

Conditions on μ, λ	Principle	Under MM	Under SCFA
$\mathrm{cf}(\lambda) \geqslant \omega_2$, $\mu < \mathrm{cf}(\lambda)$	$\square_{\lambda,\mu}$	fails	fails

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Thus, the only instances of square principles that are consistent with SCFA but not with MM are actually consequences of CH.
It is then a natural question to ask whether their negations follow from SCFA $+ \neg$CH.

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All of this may seem to indicate that the theory $\mathsf{SCFA} + \mathsf{CH}$, or $\mathsf{SCFA} + \diamondsuit$ is very natural.

However, searching for ways to produce models of $\mathsf{SCFA} + \neg\mathsf{CH}$ produced a whole range of interesting theories located between SCFA and MM.

Of course, one way to obtain a model of SCFA $+ \neg$CH is to perform a Baumgartner style iteration of semiproper forcing notions up to a supercompact cardinal, resulting in a model of MM.

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More precisely, this produces a model of SPFA, but SPFA implies that every stationary set preserving forcing is semiproper.

It is maybe at first sight a little less obvious how to produce models of BSCFA $+ \neg$CH from just a reflecting cardinal.

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My student Corey Switzer and I analyzed some forcing classes intermediate between subcomplete and subproper.

Iteration theorems

RCS iteration theorems

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Thus, an iteration is a tower $\langle \mathbb{B}_i \mid i \leqslant \lambda \rangle$ of complete Boolean algebras, such that $\mathbb{B}_i$ is completely contained in $\mathbb{B}_{i+1}$. When using revised countable support, the limit stages of this tower are formed using "RCS threads", and in order to be able to prove preservation of subproperness and subcompleteness, it is assumed that $\mathbb{B}_{i+1}$ forces the cardinality of $\delta(\mathbb{B}_i)$ to be equal to ω_1. Thus, in effect, the forcings $\mathrm{Col}(\omega_1, \delta(\mathbb{B}_i))$ have to be interleaved in the iteration.

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It turns out that there are “subversions” of known preservation theorems for semi-proper forcing.

Definition

For $f, g \in {}^{\omega}\omega$ and $n \in \omega$, we write $f \leqslant_n g$ if for all $k \geqslant n$, $f(k) \leqslant g(k)$, and we write $f \leqslant^* g$ if there is an $n \in \omega$ such that $f \leqslant_n g$. A forcing notion $\mathbb{P}$ is ${}^{\omega}\omega$-*bounding* if whenever G is $\mathbb{P}$-generic over V and $f \in ({}^{\omega}\omega)^{\mathrm{V}[G]}$, then there is a $g \in \mathrm{V}$ such that $f \leqslant^* g$, and in fact, in this case, there is then a $g \in \mathrm{V}$ such that $f \leqslant_0 g$.

Theorem (Fuchs-Switzer)

Let $\langle \mathbb{B}_i \mid i \leqslant \delta \rangle$ be an RCS iteration such that for all $i + 1 \leqslant \delta$, the following hold:

1. $\mathbb{B}_i \neq \mathbb{B}_{i+1}$,
2. $\Vdash_{\mathbb{B}_i} (\check{\mathbb{B}}_{i+1}/\dot{G}_{\mathbb{B}_i}$ *is subproper and* ${}^{\omega}\omega$*-bounding*$)$,
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Preserving Souslin trees

A version of the following theorem was originally shown by Miyamoto for his nice iterations of semiproper preorders.

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Let T be a Souslin tree. Let $\langle \mathbb{B}_i \mid i \leqslant \delta \rangle$ be an RCS iteration such that for all $i+1 \leqslant \delta$, the following hold:

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Recall that subcomplete forcings preserve Souslin trees.

Not adding branches to ω_1-trees

Theorem

Let T be an ω_1-tree. Let $\langle \mathbb{B}_i \mid i \leqslant \delta \rangle$ be an RCS iteration such that for all $i + 1 \leqslant \delta$, the following hold:

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Recall that subcomplete forcing does not add a branch to an ω_1-tree.

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   (by iterating dee-subcomplete ω_1*-subproper forcing - but that's another story)*

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Thus, it would be too simplistic to think that the only difference between SCFA and MM stems from CH. SCFA $+ \neg$CH is consistent with many statements that contradict MM.

But the initial question about $\square^*_{\omega_1}$ is still open.

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Corey Switzer observed, and we checked in detail, that one can use this framework for iterating subproper and subcomplete forcing as well, again, without the intermediate collapses.

We then observed further that this framework allows us to simplify the definition of subcompleteness.

Definition (Jensen)

$\mathbb{P}$ is subcomplete if there is a θ with $\mathbb{P} \in H_\theta$ such that the following holds: if

$$\sigma : \bar{N} \prec N$$

where N is of the form $L_\tau[A]$, $H_\theta \subseteq N$, $N \models \mathrm{ZFC}^-$, $\bar{N}$ is countable, transitive and full, $\mathbb{P}, a_0, \ldots, a_{n-1} \in \mathrm{ran}(\sigma)$,

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1. $(\sigma')``\bar{G} \subseteq G$.
2. $\sigma^{-1}(a_i) = (\sigma')^{-1}(a_i)$, for all $i < n$.
3. $\mathrm{Hull}^N(\delta \cup \mathrm{ran}(\sigma)) = \mathrm{Hull}^N(\delta \cup \mathrm{ran}(\sigma'))$, where $\delta = \delta(\mathbb{P})$ is the minimal size of a dense subset of $\mathbb{P}$.

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Condition 3. here is somewhat problematic.

Preservation under forcing equivalence

The main problem with condition 3. is that one may easily construct forcing equivalent posets $\mathbb{P}$ and $\mathbb{Q}$ such that $\delta(\mathbb{P}) \neq \delta(\mathbb{Q})$. For example, given any cardinal κ, we may define $\mathbb{Q}$ to consist of κ many copies of $\mathbb{P}$, side by side (the lottery sum of these copies). Clearly then, $\delta(\mathbb{Q}) = \kappa \cdot \delta(\mathbb{P})$.

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I have experimented with variations of condition 3. that fix this issue. One fairly simple way to do this is to replace $\delta(\mathbb{P})$ with an ordinal ε and call the resulting notion ε-subcompleteness, and say the essentially subcomplete forcings are those that are ε-subcomplete, for some ε.

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The most elegant solution:

Definition (F.-Switzer)

$\mathbb{P}$ is ∞-subcomplete if there is a θ with $\mathbb{P} \in H_\theta$ such that the following holds: if

$$\sigma : \bar{N} \prec N$$

where N is of the form $L_\tau[A]$, $H_\theta \subseteq N$, $N \models \mathrm{ZFC}^-$, $\bar{N}$ is countable, transitive and full, $\mathbb{P} \in \mathrm{ran}(\sigma)$ $a_0, \ldots, a_{n-1} \in \bar{N}$, and if $\bar{G}$ is $\bar{\mathbb{P}} = \sigma^{-1}(\mathbb{P})$-generic over $\bar{N}$, then there is a condition $p \in \mathbb{P}$ such that whenever $G \ni p$ is $\mathbb{P}$-generic, then in $\mathrm{V}[G]$, there is a $\sigma' : \bar{N} \prec N$ with

1. $\sigma'(\bar{\mathbb{P}}) = \mathbb{P}$ and $\sigma(a_i) = \sigma'(a_i)$, for all $i < n$.
2. $(\sigma')``\bar{G} \subseteq G$.

The main joint result with Corey Switzer is that this forcing class is nicely iterable. It is easy to see that it has all the relevant properties of subcomplete forcing: it doesn't add reals, preserves $\diamondsuit$, preserves Souslin trees, adds no branches to ω_1-trees, etc.

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There are many open questions here (this is fairly new). Most obviously: is there a forcing notion that's ∞-subcomplete but not subcomplete?

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...*to be continued*