$\mathcal{L}$- forcing, III

today: pf. of double-sch + furth analyses.

lem. $\mathbb{P}$ is not proper.

pf.: supp. it were prop. let $X \prec H_\Omega$, $\Omega \gg \theta$.
may find G $\mathbb{P}$-gen./V s.t.

$$X[G] \cap H_\theta = X \cap H_\theta .$$

with $\alpha = X \cap \omega_1$.
So if

$$N[g] \overset{\sigma}{\cong} X[G] \prec H_\Omega[G]$$

+ if $(M_i, \pi_{ij} : i \le j \le \omega_1)$ is the gen. iter. giv by G, then

$$\begin{aligned} M_\alpha &= \cancel{H_\theta} \, \sigma^{-1}(H_\theta) \\ &= \left(H_{\sigma^{-1}(\theta)}\right)^N \in V . \end{aligned}$$

So if $H_\theta^\#$ ex., $M_\alpha^\# \in V$ gen. iterates to $H_\theta^\#$ in ω_1 steps which is nonsense by boundedness. ⊣

recall :

<u>th.</u> (double-sh) TFAE .

(1) NS_{ω_1} is precipitous + for any G, $\mathbb{P}$ is semi-proper

(2) $(\dagger)$.

pf. : (2) $\Rightarrow$ (1) (magidor - foreman - shelah)

let $\mathfrak{A}$ be an antichain in $(NS_{\omega_1})^+ / NS_{\omega_1}$,

and let $\mathbb{S} = \mathbb{S}(\mathfrak{A})$ be the corresponding sealing forcing.

let

$$X \prec H_\Omega$$

with $\mathfrak{A} \in X$. let $p \in \mathbb{S}$ be semi-p., and let $G \ni p$ be $\mathbb{S}$- gen. $/V$.

$$X[G] \prec H_\Omega[G] .$$

(also $X[G] \cap H_\Omega \prec H_\Omega$).

where $\alpha = X \cap \omega_1 = X[G] \cap \omega_1$.

let $(C, e : \omega_1 \to \mathcal{A})$ be gen by G

where $C \subset \omega_1$ is club, and for all $\beta \in C$

there is some $\gamma < \beta$ s.t. $\beta \in e(\gamma)$.

here $\alpha \in C$, so $\alpha \in e(\gamma)$, some $\gamma < \alpha$.

so $e(\gamma) \in X[G]$.

by absoluteness, <u>in V</u> there is some

$Y \supset X$, $X \prec Y \prec H_\Omega$, $Y \cap \omega_1 = X \cap \omega_1 = \alpha$,

$e(\gamma) \in Y$, i.e.,

$\exists S \in \mathcal{A} \cap Y \ \alpha \in S$.

using this we may build a tower of substructures

whose union is some

$$Z \prec H_\Omega, \quad \alpha = Z \cap \omega_1$$

s.t. if $\mathcal{A} \in Z$ is an antichain, then

$\exists S \in \mathcal{A} \cap Z \ \alpha \in S$. hence Z is "self

generic," i.e., if

$$N \overset{\sigma}{\cong} Z \prec H_{\Omega}$$

and u is the N-ultrapower on ω_1^N derived from σ, then u is gen. / N for $(NS_{\omega_1})^N$.

may have fixed $S \in (NS_{\omega_1})^+$ at the outset, picked X s.t. $\alpha = X \cap \omega_1 \in S$.

then cannot have had $S \Vdash_{NS_{\omega_1}} \mathrm{ult}(V; \dot{G})$ is ill-fdd,

$\Downarrow$

as $S \cap \alpha \Vdash_{(NS_{\omega_1})^N} \mathrm{ult}(N; \dot{G})$ is well-fdd. $\dashv$

(2) $\Rightarrow$ (1).

remark:

the last steps of the argument given also shows that

$$(+) \Rightarrow \quad \{X \prec H_{\Omega} : X \text{ is self-gen.}\}$$

is stat.

it is easy to verify that this conclusion is <u>equivalent</u> to the precipitousness of NS_{ω_1}

Compare: NS_{ω_1} sat. $\Leftrightarrow$ $\{\ldots\}$ is club.

yields:

<u>Observation.</u> TFAE

(1) NS_{ω_1} is $\left\{\begin{matrix}\text{precipitous}\\ \text{saturated}\end{matrix}\right\}$

(2) $\{X \prec H_\Omega : X \text{ self-gen.}\}$ is $\left\{\begin{matrix}\text{stationary}\\ \text{club}\end{matrix}\right\}$.

pf of (1) ⟹ (2) of double-arch:

we need a strengthening of Chang's conjecture:

<u>def.</u> CC^{**}:

f.a. $\Omega > \lambda \geq \omega_2$, if

$$X \prec H_\Omega \quad + \quad a \in [\lambda]^{\omega_1}$$

then there is some Y,

$$X \prec Y \prec H_\Omega$$

$Y \cap \omega_1 = X \cap \omega_1$, and $\exists b \in Y \cap [\lambda]^{\omega_1}$ $b \supset a$.

rmk.: for $\lambda = \omega_2$, this is todorcevic's CC^*.

(1) $\Rightarrow$ CC^{**}:

fix $\cancel{\theta = \Omega} > \lambda \geq \omega_2$. fix $X \prec H_\Omega$, $a \in [\lambda]^{\omega_1}$

we have a sub-proper forcing for adding $(M_i, \pi_{ij} : i \leq j \leq \omega_1)$, given by G.

so all M_i, $i < \omega_1$ are ctble., $M_{\omega_1} = H_\theta$.

in part., $a \in M_{\omega_1} = \mathrm{Hull}^{H_\theta}(\omega_1 \cup \mathrm{ran}(\pi_{0\omega_1}))$

so $a = \tau^{H_\theta}(\vec{\eta}_0, \vec{y})$, $\vec{\eta}_0 < \omega_1$, $y \in \mathrm{ran}(\pi_{0\omega_1})$.

set $b = \left(\bigcup \{ \tau^{H_\theta}(\vec{\eta}, \vec{y}) : \vec{\eta} < \omega_1 \wedge \overline{\overline{\tau^{H_\theta}(\vec{\eta}, \vec{y})}} \leq \aleph_1 \}\right)$

$\overline{\overline{b}} \leq \aleph_1$, $b \in V$. ~~$b \in [\lambda]^{\omega_1}$~~. in fact,

$b \in \mathrm{ran}(\pi_{0\omega_1})$, as it's defined from parameters in $\mathrm{ran}(\pi_{0\omega_1})$.

$$M_\alpha \overset{\pi_{\alpha\omega_1}}{\cong} X[G] \cap H_\Omega \prec H_\Omega .$$

if $\lambda \in X$, then we now have

$$b \cap \lambda \in [\lambda]^{\omega_1} \cap \operatorname{ran}(\pi_{\alpha\omega_1}) .$$

now by absoluteness, in V there is Y,

$$X \prec Y \prec H_\Omega , \quad Y \cap \omega_1 = X \cap \omega_1$$

$$\exists b \in Y \cap [\lambda]^{\omega_1} , \quad b \supset a .$$

this shows $(1) \Rightarrow CC^{**}$.

def. $S \subset [\lambda]^\omega$ is semi-stat. iff

$\left(\bigcup_{x \in S} \{ y \in [\lambda]^\omega : y \sqsupseteq x \} \right)$ is stat. $\quad$ $x \sqsubseteq y$ means $x \cap \omega_1 = y \cap \omega_1$.

$SSR([\lambda]^\omega) \equiv$ semi-stat.-refl. $\equiv$

$S \subset [\lambda]^\omega$ semi-stat. $\Rightarrow \exists W \subset \lambda \, (\overline{\overline{W}} = \aleph_1 \wedge$

$S \cap [W]^{\omega}$ is semi-stat. in $[W]^\omega)$.

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$$CC^{**} \Longrightarrow SSR([\lambda]^{\omega}) \quad \forall \lambda \geq \omega_2 .$$

pf.: let $S \in [\lambda]^{\omega}$ semi-stat.,

$$T = \{ y \supset x : y \in [\lambda]^{\omega},\ x \in S \} \text{ stat.}$$

supp. f.a. $W \subset \lambda$, $\overline{\overline{W}} = \aleph_1$,

$$S_W = \{ y \supset x : y \in [W]^{\omega},\ x \in S \cap [W]^{\omega} \}$$

is not stat., witnessed by some $f_W : [W]^{<\omega} \to W$.

let Ω be big. pick

$$X \prec H_\Omega, \quad (f_W : W \subset \lambda, \overline{\overline{W}} = \aleph_1) \in X \text{ etc.}$$

s.t. $X \cap \lambda \in T$.

let $a = \omega_1 \cup (X \cap \lambda)$.

here

$$X \prec Y \prec H_\Omega, \quad Y \supset X$$

s.t. $b \supset a$ for some $b \in Y \cap [\lambda]^{\omega_1}$.

by the def. of T, there is some $x \in S$,

$$x \subset X \cap \lambda \subset Y \cap \lambda$$

by the choice of $f_b : [b]^{<\omega} \to b$,

$\{z \in [b]^\omega : z \text{ closed under } f_b\} \cap S_b = \emptyset$.

but $Y \cap b \in [b]^\omega$, $Y \cap b \sqsupseteq x \in S$, $x \in [b]^\omega$

$((Y \cap b) \cap \omega_1 = Y \cap \omega_1 = X \cap \omega_1 = x \cap \omega_1$,

if $\xi \in x$, $\xi \in X \cap \lambda$, so $\xi \in b \supset X \cap \lambda$ and $\xi \in Y$)

so $\underline{Y \cap b \in S_b}$.

but also $f_b \in Y$ (as $b \in Y$), so

$Y \cap b$ is closed under f_b.

contradiction! $\dashv$ $CC^{**} \Rightarrow SSR([\lambda]^\omega)$

lem (Shelah): $\forall \lambda \; SSR([\lambda]^\omega) \Rightarrow (\dagger)$

pf.: let $\mathbb{P}$ preserve stationary sets.

fix Ω. suppose the set of X s.t.
for some $p \in \mathbb{P}$,

$$X \prec V_\Omega$$

+ there is no $q \leq p$ which is semi-gen./X

is stationary.

So there is a $W \subset V_\Omega$ of size $\aleph_1$ s.t. $W \prec V_\Omega$ +

the set of $X \prec V_\Omega$, $X \subset W$, s.t. there is
no $q \leq p$ semi-g./X is semi-stat. in $[W]^\omega$.

let $f: \omega_1 \to W$ onto, let G be $\mathbb{P}$-g./V.

pick α s.t. $f''\alpha \cap \omega_1 = \alpha$,
$f''\alpha \supset X$, ~~some~~ X s.t.
$X \prec V_\Omega$, $X \subset W$,
no $q \leq p$ semi-g./X etc.

~~f''α~~ $f''\alpha \prec W \prec V_\Omega$.

$f''\alpha[G] \cap \omega_1 = \alpha$

but then some $q \in G$ is X-semigeneric $\not{\!\!\;\!\,}$ ↯

⊣