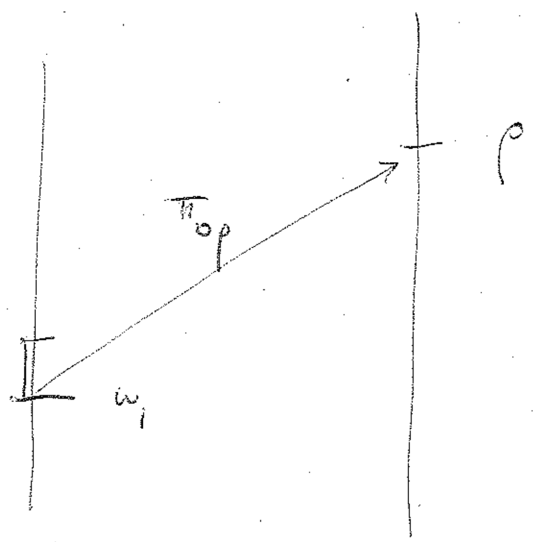
$\mathcal{L}$- forcing, II

today: $\mathcal{L}$- forcing with finite rather than ctbl. conditions.

there is a forcing by Ketchersid-Larson-Zapletal which adds a similar thing to what we'll do
let us suppose I is a (normal) precipitous ideal (+eg.) on ω_1.



let ρ be a "large" reg. cardinal, let G be $Col(\omega, <\rho)$- gen. / V. in $V[G]$, there is an ~~iteration~~ generic iteration

$$(M_i, \pi_{ij} : i \le j \le \rho)$$

of $M_0 = (H_\theta; \in, I)$ (or, of $V = (V; \in, I)$)

s.t.

if $S \in \pi_{0\rho}(I)^+$, then S is stationary in $V[G]$.

(so if $I = NS_{\omega_1}$, every stat. set of M_ρ will be stat. in $V[G]$)

by abs., inside $M_\rho^{\mathrm{Col}(\omega, \pi_{0\rho}(\overline{\overline{H_{2^\theta}}}))}$, there is there a transitive model $\mathfrak{A}$ of ZFC^- s.t. $\pi_{0\rho}(H_{(2^{2^{<\theta}})^+}) \subset \mathfrak{A}$

$\mathfrak{A} \models \exists$ gen. iteration $(M_i', \pi_{ij}' : i \le j \le \rho)$

s.t. $\overline{\overline{M_i'}} = \aleph_0 \quad \forall i < \rho$

$M_\rho' = \pi_{0\rho}((H_\theta; \in, I))$

s.t. if $S \in \pi_{0\rho}(I)^+$, then S is stat. in $\mathfrak{A}$.

using $\pi_{0\rho}$, we get that in $V^{\mathrm{Col}(\omega, \overline{\overline{H_{2^\theta}}})}$, there is a transitive model $\mathfrak{A}$ of ZFC^- s.t. $H_{(2^{2^{<\theta}})^+} \subset \mathfrak{A}$ and

$\mathfrak{A} \models \exists$ gen. iteration $(M_i, \pi_{ij} : i \le j \le \omega_1)$

s.t. $\overline{\overline{M_i}} = \aleph_0 \quad \forall i < \omega_1$

$M_{\omega_1} = (H_\theta; \in, I)$

s.t. if $S \in I^+$, then S is stat. in $\mathfrak{A}$

notice we won't have $M_0^{\mathfrak{A}} \in V$, etc.

We now aim to force this.

$\mathbb{P} \ni p = (\vec{\pi}, \vec{\tau})$ s.t.

there is an $\mathfrak{A}$ as above certifying p, i.e.,

$\vec{\pi}$ gives finitely much information about

$$(M_i^{\mathfrak{A}}, \pi_{ij}^{\mathfrak{A}} : i \le j \le \omega_1)$$

and $\vec{\tau}$ is a finite sequence $(\tau(i) : i \in \mathrm{dom}(p))$

s.t. if $\tau(i)\downarrow$, then

$$\exists x \quad \tau(i) = \text{the } H_\theta\text{-type of } x \text{ over } H_{(2^{2^{<\theta}})^+}$$

$$= \{\varphi(\vec{z}, v) : H_{(2^{2^{<\theta}})^+} \models \varphi(\vec{z}, x), \vec{z} \in H_\theta\}$$

if $i < j$, $\tau(i)\downarrow$, $\tau(j)\downarrow$, then

$\tau(i)$ can be read off from $\tau(j)$ as types

as types: $\tau(i)$ is a child of $\tau(j)$, i.e.

$\exists n_0 \quad (k, \vec{x}) \in \tau(i) \iff (n_0, (k, \vec{x})) \in \tau(j)$.

$+ \quad \mathrm{ran}(\pi_{i\omega_1}^{\mathfrak{A}}) \prec (H_\theta; \in, \tau(i))$.

we will not prove the obvious

facts about this forcing.

lem. ~~$\mathbb{P}$ preserves stat.~~

if $S \in I^+$, then S is stat. in $V^{\mathbb{P}}$.

[so if $I = NS_{\omega_1}$, then $\mathbb{P}$ preserves stat. sets.]

pf. let $p \Vdash \dot{C}$ is club in ω_1, and let

$\mathfrak{N}$ contain p.

point is: $(p, \mathbb{P}, \sigma, \Vdash\!\restriction\ldots) \in H_{(2^{2^{<\theta}})^+} \subset \mathfrak{N}$.

recall $S \in I^+ \Rightarrow S$ stat. in $\mathfrak{N}$.

work in $\mathfrak{N}$ to find $\underline{\alpha \in S}$ s.t., letting

$\tau =$ the H_θ-type of $(p, \mathbb{P}, \sigma, \Vdash\!\restriction\ldots)$ over $H_{(2^{2^{<\theta}})^+}$,

then $\operatorname{ran} \pi^{\mathfrak{N}}_{\alpha\omega_1} \prec (H_\theta ; \tau)$.

let $\tau = \tau(\alpha)$. we claim

$p^\frown \tau(\alpha) \Vdash \check{\alpha} \in \sigma$.

supp. that $q \leq p^\frown \tau(\alpha)$ and $\xi < \alpha$.

it suff. to find $r \leq q$ $r \Vdash \check{\eta} \in \sigma$

for some $\eta < \alpha$, $\eta \geq \xi$.

so fix q, and let $\mathcal{B}$ certify q.

as

$$\operatorname{ran} \pi^{\mathcal{B}}_{\alpha\omega_1} \prec \left(H_\theta ; \in, \tau(\alpha) \right),$$

there is

$$\begin{array}{ccc} H & \xrightarrow{\tilde{\pi}} & H_{(2^{<\theta})^+} \\ \cup & & \cup \\ M^{\mathcal{B}}_\alpha & \xrightarrow[\pi^{\mathcal{B}}_{\alpha\omega_1}]{} & H_\theta \end{array}$$

s.t. $(p; \mathbb{P}, \sigma, \Vdash \restriction \ldots) \in \operatorname{ran}(\tilde{\pi})$.

as the individual components of $\tau(\alpha)$ are

def. th over $\operatorname{ran}(\tilde{\pi})$ + all $\tau(i)$, $i \in \operatorname{dom}\_(q \restriction \alpha)$

must be easily to read off from $\tau(\alpha)$,

all these $\tau(i)$ are in $\operatorname{ran}(\tilde{\pi})$, so that

$$q \restriction \alpha \in \operatorname{ran}(\tilde{\pi}).$$

we then get

$\mathrm{ran}(\tilde{\pi}) \models \exists \eta \geq \xi \;\; \exists r \leq q \restriction \alpha \;\; r \Vdash \check{\eta} \in \sigma$.

this η will of course be $< \alpha$

suff to verify $r \parallel q$.

if $\mathcal{C}$ certifies r, then $\mathcal{C}$ yields

$$(M_i^{\mathcal{C}}, \pi_{ij}^{\mathcal{C}} : i \leq j \leq \omega_1)$$

s.t. if $\tau(i)^r \downarrow$, then $\mathrm{ran}(\pi_{i\omega_1}^{\mathcal{C}}) \prec (H_\theta ; \in, \tau(i))$

by abs. inside $H_{(2^{2^{<\theta}})^+}^{\mathrm{Col}(\omega, \bar{\bar{H}}_\theta)}$, there

is such an iterati (with base model H_θ).

by elementarity, inside $H^{\mathrm{Col}(\omega, \tilde{\pi}^{-1}(\bar{\bar{H}}_\theta))}$,

there is an iterati

$$(M_i , \pi_{ij} : i \leq j \leq \alpha)$$

with $M_\alpha = M_\alpha^{\mathcal{B}}$ and

$$\mathrm{ran}(\pi_{i\alpha}) \prec \left(M_\alpha^{\mathcal{B}} ; \in, \tilde{\pi}^{-1}(\tau(i)) \right)$$

for $i \in \mathrm{dom}_-(r)$.

H is cttn. in $\mathcal{B}$, so that we may produce some g $Col(\omega, \tilde{\pi}^{-1}(\bar{\bar{H}}_\theta))$ – gen. / H, $g \in \mathcal{B}$. i.e., in $\mathcal{B}$, we have such an iteration.

but then

$$(M_i, \pi_{ij} : i \le j \le \alpha) ^\frown (M_i^{\mathcal{B}}, \pi_{ij}^{\mathcal{B}} : \alpha \le i \le j \le \omega_1)$$

to gether with

$$\pi_{ij} = \pi_{\alpha j}^{\mathcal{B}} \circ \pi_{i\alpha} \quad \text{for} \quad i \le \alpha \le j$$

is a gen. iteration in $\mathcal{B}$

$$\operatorname{ran}(\pi_{i\omega_1}) \prec (H_\theta; \in, \tau(i))$$

for $i \in \operatorname{dom}(r)$.

so $\mathcal{B}$ certifies both r and q, with the same iteration,

so $r \parallel q$. $\dashv$

what was important was "~~re~~sectionability" ;

there is a club $C \subset \omega_1$ s.t. if

$\mathcal{O}$ certifies $p \restriction [\alpha, \omega_1)$

$\Downarrow$

$(M_i^{\mathcal{O}}, \pi_{ij}^{\mathcal{O}} : i \leq j \leq \omega_1)$

+ in $\mathcal{O}$ we have

$(N_i, \sigma_{ij} : i \leq j \leq \alpha)$ "certify $p \restriction \alpha$

w/ $N_\alpha = M_\alpha^{\mathcal{O}}$,

then the two threads may be joined

+ ~~this~~ $\mathcal{O}$ certifies p

with this new modification.

~~$\not\exists$~~

this yields to:

<u>th.</u> (classic - ish) BMM + NS_{ω_1} prec. $\Rightarrow$ $\underset{\sim}{\delta}^1_2 = \omega_2$.

Pf.: BMM $\Rightarrow$ V closed under #'s.

by the usual argument, we can force the i.e. of

$$(M_i, \pi_{ij} : i \leq j \leq \omega_1)$$

s.t. $M_{\omega_1} = (H_\theta^\#; \in, NS_{\omega_1})$.

by BMM, in V there is for every $\alpha < \omega_2$

some iteration as above with $M_{\omega_1} \cap OR \geq \alpha$.

but if M_0 iterates to M_{ω_1}, then $M_{\omega_1} \cap OR < \omega_1^{+L[x]}$,

where $x \in \mathbb{R}$ codes M_0.

reason: $\{\xi : \exists$ iteration of M_0 of length $\rho+1$ with $M_\rho \cap OR > \xi\}$

is $= \{\|y\| : y \in A\}$ where A is $\Sigma^1_1(x, p)$ [p code for ρ]

hence $M_{\omega_1} \cap OR < \omega_1^{+L[x]}$.

so $\delta^1_2 = \omega_2$ ⊣

one may produce statements like

ac γ (admissible club guessing)

Ψ_{AC}

with this method.

Ψ_{AC}

open: BMM $\Rightarrow$ $\underset{\sim}{\delta}^1_2 = \omega_2$.

$\mathbb{P}$ is not proper. o.w. if $(M_i, \pi_{ij} : i \leq j \leq \omega_1)$ is obtained by forcing s.t. $M_{\omega_1} = H_\theta$, then

$M_\alpha \in V$ for some α, which is nonsense.

is it semi-proper?

thm. ~~[illegible]~~ (double-sch)

TFAE.

(1) NS_{ω_1} is prec. +
$\forall \theta$ the forcing $\mathbb{P}$ defined above is semi proper.

(2) $(\dagger)$ (i.e., all stat. set pres. forcing are semi proper)

another issue concerning iterability of $\mathbb{P}$;

thm (sh) if there is no i.m. which is a weakly compact cardinal + $\underline{I}$ is a nowhere precip. ideal on κ,

then $\kappa^{+K} = \kappa^{+}$.