hugh woodin.

thm. (jensen) one of the following holds.

(1) every uncountable cardinal γ is strongly inaccessible in L, or

(2) every singular cardinal γ is singular in L and $\gamma^+ = \gamma^{+L}$.

th. supp. δ is an extendible cardinal. then one of the following holds.

(1) every regular cardinal $\kappa \geq \delta$ is measurable in HOD, or

(2) if $\gamma \geq \delta$ is singular, then γ is singular in HOD and $\gamma^+ = \gamma^{+HOD}$.

lma. let $\kappa > \omega$ be a regular cardinal, I is a κ-complete uniform ideal on κ, $\lambda < \kappa$, $2^\lambda < \kappa$. then all

(1) there is a partition of κ into λ many I-positive sets.

(2) $\mathcal{P}(\kappa)/I$ is atomic with $< \lambda$ many atoms.

Supp. $\kappa \geq \omega_1$ is regular. $S = \{\alpha < \kappa : cf(\alpha) = \omega\}$.

Let I = the ideal of $A \subset \kappa$ s.t.

$A \cap S$ is nonstationary;

$S \in HOD$, $I \cap HOD \in HOD$.

Supp. $\lambda < \kappa$ is a HOD-cardinal and $(2^\lambda)^{HOD} < \kappa$.

<u>defin</u> $\kappa > \omega$ reg. κ is ω-<u>strongly measurable</u> in HOD iff $\exists \lambda < \kappa$ $\left((2^\lambda)^{HOD} < \kappa \right.$ + there is no partition $(S_\alpha : \alpha < \lambda) \in HOD$ of $S = \{\alpha < \kappa : cf(\alpha) = \omega\}$ into V-stationary sets$\left.\right)$.

HOD hypothesis:

there ex. arbitrary large reg κ which are not ω-strongly measurable in HOD.

thm. Supp. δ is an extendible cardinal. one of the following holds.

(1) every regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.

(2) no regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.

definition. a transitive class N is a weak extender model for δ is supercompact iff f.a. $\gamma > \delta$ there is a δ-complete normal fine ultrapower μ on $P_\delta(\gamma)$ s.t. ~~$\mu \cap$~~ $N \cap P_\delta(\gamma) \in \mu$ and $\mu \cap N \in N$.

Th. Supp. N is a weak extender model for δ is supercompact, then every singular cardinal $\gamma \geq \delta$ is singular in N and $\gamma^+ = \gamma^{+N}$.

Th. Supp. N is a weak extender model for δ is supercompact.

(1) Supp. $\mathcal{U}$ is a δ-complete ultrafilter on some $\gamma \geq \delta$.

then $\mathcal{U} \cap N \in N$.

(2) Supp. E is an N extender of length η s.t. $\mathrm{crit}(E) \geq \delta$ and η is a cardinal of $\mathrm{ult}(N; E)$, then the following are equivalent.

a) for all $A \subset \eta$, if $A \in N$, then $j_E(A) \cap \eta \in N$.

b) $E \in N$.

[a) (or, b)) gives rigidity of N above δ.]

<u>thm.</u> Supp. δ is extendible. One of the following holds.

(1) every regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.
($\Rightarrow$ there is no weak extender model for δ is ~~supp~~ supercompact with $N \subset$ HOD)

(2) HOD is a weak extender model for δ is supercompact.

<u>Corollary.</u> Supp. δ is extendible. then every measurable cardinal $\kappa \geq \delta$ is measurable in HOD.

<u>axiom.</u> $V =$ ultimate $- L$

(1) there is a proper class of Woodin cardinals.

(2) Supp. φ is a Σ_2 sentence which is true in V. then there is

a universally baire set $A \subset \mathbb{R}$ s.t.

$$\mathrm{HOD}^{L(A,\mathbb{R})} \models \varphi .$$

ultimate – L conjecture.

Supp. th is an extendible cardinal. then there exists (N, δ) s.t.

① $N \subset \mathrm{HOD}$ and N is a weak extender model for δ is supercompact.

② $N \models$ "$V =$ ultimate – L."

③ $\exists (E_\alpha : \alpha < \delta)$, a seq. of extenders, s.t. $(E_\alpha \cap N : \alpha < \delta) \in N$ + where δ is woodin in N.

- Supp. κ is huge. then the ultimate – L conjecture holds in $V_{\kappa+1}$.

<u>th.</u> supp. δ is a lim of strong cardinals. let $V[G]$ be a gen. extension where δ is ctbl. then projective gen. absoluteness holds for set forcing.

<u>th.</u> Supp. there is a proper class of woodin cardinals. then projective absoluteness holds.

<u>th.</u> supp. $\exists$ a proper class of woodin cardinals and δ is syncopat. let $V[G]$ be a gen. extension in which $V_{\delta+1}$ is ctbl. then $L(\Gamma^\infty)$ is generically sealed, where $\Gamma^\infty = \{A \subset \mathbb{R} : A$ is universally baire$\}$.

$1^{\underline{st}}$ attack: prove Γ^∞-sealing from some large cardinals.

$2^{\underline{nd}}$ attack:

Supp. $M \models ZFC$, $OR \cap M = \omega_1$.

Supp. $<_M$ is an amenable wellordering of M of type ω_1.

Supp. $\{\alpha : \varphi^M(\alpha) < \alpha\}$ contains a club.

can isolate κ_M.

take $Hull^M(\Gamma) \cap \alpha = X_\alpha$

for Γ thin eg.

$X = \bigcup X_\alpha$ take $Hull^M(X)$.

quest. can it be that $\kappa_M \in L[x]$

for some $x \in \mathbb{R}$ s.t. $x^\#$ ex.