Some recent results in descriptive inner model theory

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A personal story

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Woodin's Ultimate *L* is an axiom that not only says what the reals are but what $\mathcal{P}(\kappa)$ is for all κ.

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7. Jackson's analysis of measures,
8. there are 4+ Cabal volumes, each about 300+ pages, they knew a lot.

The starting point

Theorem (Steel, 1995)

Assume $\mathcal{M}_\omega^\#$ exists. Let $\mu = (\delta_1^2)^{L(\mathbb{R})}$ and set

$$\mathcal{M} = \mathrm{HOD}^{L(\mathbb{R})} | (\mu^+)^{\mathrm{HOD}^{L(\mathbb{R})}}.$$

Let $\mathcal{H}$ be the direct limit of all countable iterates of $\mathcal{P} =_{def} \mathcal{M}_\omega | (\delta^+)$ where δ is the least Woodin of $\mathcal{M}_\omega$, and let $i : \mathcal{P} \to \mathcal{H}$ be the iteration embedding. Let λ be the least $< \delta$-strong cardinal of $\mathcal{P}$ and let κ be its successor in $\mathcal{P}$. Then

$$\mathcal{M} = \mathcal{H} | i(\kappa).$$

Consequences

Corollary

Assume $\mathcal{M}_\omega^\#$ exists. Then $V_\Theta^{\mathrm{HOD}^{L(\mathbb{R})}} \models GCH$.

Theorem (Steel)

Assume $\mathcal{M}_\omega^\#$ exists. Then $L(\mathbb{R}) \models$ "$\kappa \in (\omega, \Theta)$ is regular if and only if κ is measurable".

Theorem (Steel)

$L[T_{2n}] = L[\mathcal{M}]$ where $\mathcal{M}$ is the direct limit of all countable iterates of $\mathcal{M}_{2n}$ cut at the least cardinal that is strong up to the least Woodin of the aformentioned direct limit.

Full HOD

Theorem (Woodin)

Assume $\mathcal{M}_\omega^\#$ exists and let Σ be its strategy. Then $\mathrm{HOD}^{L(\mathbb{R})}$ has the form $L[\mathcal{M}, \Lambda]$ where

1. *$\mathcal{M}$ is the direct limit of all countable iterates of $\mathcal{M}_\omega$,*
2. *$\Theta^{L(\mathbb{R})} =_{def} \delta$ is the least Woodin cardinal of $\mathcal{M}$,*
3. *Λ is the fragment of $\Sigma_{\mathcal{M}}$ that acts on trees belonging to $\mathcal{M}|\lambda$ where λ is the sup of the Woodin cardinals of $\mathcal{M}$.*

Alternative representations of HOD

1. $\mathrm{HOD}^{L(\mathbb{R})} = L[\mathcal{M}, \pi]$ where π is the iteration embedding via $\Sigma_{\mathcal{M}|\delta}$ from $\mathcal{M}|\delta$ into HOD of the derived model of $\mathcal{M}$.

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2. $\mathrm{HOD}^{L(\mathbb{R})}$ has the form $L[\mathcal{M}, \rho \to \rho^*]$ where $\rho \to \rho^*$ is the restriction of π to the ordinals.

HOD of $L[x][g]$

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Theorem (Woodin)

Suppose $x \in \mathbb{R}$ is such that $\mathcal{M}_1^{\#} \in L[x]$, and $g \subseteq Coll(\omega, < \kappa_x)$ is $L[x]$-generic. Then $\mathrm{HOD}^{L[x][g]} = L[\mathcal{M}_x, \Lambda_x] = L[\mathcal{M}_x, \pi_x]$.

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What is HOD *of* $L[x]$ *where* $x \in \mathbb{R}$ *codes* $\mathcal{M}_1^{\#}$.

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There are partial results due to Schlutzenberg, Steel, Woodin and Zhu.

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Remark

1. *This is probably the most central project of DIMT.*
2. *This would give GCH in* HOD.
3. *To complete the goal one would need some form of capturing.*
4. *NLE stands for "no mouse with a long extender".*

HOD analysis: Mouse Capturing

Definition

MC is the statement that given $x, y \in \mathbb{R}$, $x \in OD_y$ if and only if x is in a y-mouse.

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Conjecture (Mouse Set Conjecture)

Assume $AD^{++} + NLE$. Then MC holds.

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Assume $AD^{++} + NLE$. *Then HPC holds.*

HOD analysis: Some global results

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HOD analysis: Some global results

1. In the presence of large cardinals, Steel reduced *HPC* to unique iterability of V, or *UBH*.
2. Under $AD_{\mathbb{R}}$, Steel reduced HOD analysis to *HPC*.
3. It has been known before that under AD^+, *HPC* reduces to *MC*.
4. So proving *MC* from AD^+ is probably the best route to take. However, it is not likely that one can prove *MC* without proving *HPC* simultaneously.

HOD analysis: Hod Pair Capturing

Given a set of reals A and a triple $(\mathcal{P}, \delta, \Sigma)$ such that δ is a Woodin cardinal of $\mathcal{P}$ and Σ is an ω_1-strategy, we say $(\mathcal{P}, \delta, \Sigma)$ Suslin, co-Suslin captures A if there are δ-complementing trees $T, S \in \mathcal{P}$ such that whenever $i : \mathcal{P} \to \mathcal{Q}$ is an iteration via Σ and $g \subseteq Coll(\omega, i(\delta))$ is $\mathcal{Q}$ generic,

$$A \cap \mathcal{Q}[g] = (p[i(T)])^{\mathcal{Q}[g]}.$$

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Theorem (Woodin)

Under AD^+, every Suslin, co-Suslin set is captured by some triple as above.

HOD analysis: Hod Pair Capturing

Conjecture (Tentatively: $\dagger$)

Suppose A is a set of reals and $(\mathcal{P}, \delta, \Sigma)$ Suslin, co-Suslin captures A. Let Λ be the induced strategy of the fully backgrounded construction of $\mathcal{P}|\delta$. Then $A \leq_w$ Code(Λ).

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Remark

1. † *implies HPC and MC.*

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Remark

1. *† implies HPC and MC.*
2. *It is probably more likely that one would first show that the hod pair construction of $\mathcal{P}|\delta$ inherits complicated strategy.*

HOD analysis: Some partial results

The Largest Suslin Axiom: *AD*$^+$ $+$ "there is a largest Suslin cardinal κ such that for any $\alpha < \kappa$, κ is not a surjective image of an *OD* function with domain α^ω".

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Theorem

Both HPC and MC hold in the minimal model of LSA.

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Theorem

Both HPC and MC hold in the minimal model of LSA.

Theorem

Assume AD$^+$ $+$ *"there is no largest Suslin cardinal", and suppose that there is no hod mouse with a non-domestic cardinal. Then HPC holds.*

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5. Basically we have been working inside the region where the Chang$^+$ model is a $\mathcal{Q}$-structure, and now we are about to leave it, and be where?

HOD analysis: $C^+(\Gamma)$

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3. Say Γ resists C^+ if $C^+(\Gamma) \cap \mathcal{P}(\mathbb{R}) = \Gamma$.

Conjecture

Assume AD^+ and suppose that there is $\Gamma \subset \mathcal{P}(\mathbb{R})$ consisting of Suslin, co-Suslin sets that resists C^+. Then there is an iteration strategy for a mouse with a Woodin cardinal that is a limit of Woodin cardinals.

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Basically it seems that the methods we have been using to build hod mice from AD^+ work in models of AD^+ whose initial segments do not resist C^+.

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Theorem (Steel)

Suppose $\mathcal{M}$ is a hod mouse with a Woodin limit of Woodins and ω more Woodins above. Then some initial segment of the derived model of $\mathcal{M}$ resists C^+

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4. Expected Answer: Yes.
5. Known (work in progress): Below strong reflecting a strong that reflects a strong (alternating chain of length 3) and is also a limit of Woodins.
6. Is it always true ? or is it false beyond Woodin limit of Woodins or perhaps below "alternating chain of length 3"?

HOD analysis: Questions on $C^+(\Gamma)$

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1. Are there models in which $\mathcal{P}(\mathbb{R})$ is not contained in C, the Chang model?
2. Probably yes. Let $M \subseteq N$ be the minimal pair of models of $AD^+ + V = L(\mathcal{P}(\mathbb{R})) + \theta_0 = \Theta$ with the same Δ^2_1 and such that $N \vDash \mathrm{cf}(\Theta^M) = \omega_1$. Is it the case that $\mathcal{P}(\mathbb{R}) \cap C^N \subseteq M$?

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3. What is the large cardinal strength of the above theory?

Large Cradinals $\rightarrow$ Determinacy: the derived model theorem

Theorem (Woodin, The New DMT)

Suppose λ is a limit of Woodin cardinals and let $g \subseteq Coll(\omega, < \lambda)$. Set $\mathbb{R}^* = \cup_{\alpha<\lambda}\mathbb{R}^{V[g\cap Coll(\omega,<\alpha)]}$, and let, in $V(\mathbb{R}^*)$, $\Gamma = \{A \subseteq \mathbb{R}^* : L(A, \mathbb{R}^*) \vDash AD^+\}$. Then $L(\Gamma, \mathbb{R}) \vDash AD^+$.

Large Cradinals $\rightarrow$ Determinacy: the derived model theorem

Theorem (Woodin, The New DMT)

Suppose λ is a limit of Woodin cardinals and let $g \subseteq Coll(\omega, < \lambda)$. Set $\mathbb{R}^* = \cup_{\alpha<\lambda}\mathbb{R}^{V[g\cap Coll(\omega,<\alpha)]}$, and let, in $V(\mathbb{R}^*)$, $\Gamma = \{A \subseteq \mathbb{R}^* : L(A, \mathbb{R}^*) \vDash AD^+\}$. Then $L(\Gamma, \mathbb{R}) \vDash AD^+$.

Theorem (Woodin, The Old DMT)

Working in $V(\mathbb{R}^*)$, let Hom^* be the set of reals A that are λ-uB along the way. Then $L(Hom^*, \mathbb{R}) \vDash AD^+$ and $Hom^* = \{$ Suslin, co-Suslin sets of $L(\Gamma, \mathbb{R})\}$.

Large Cradinals $\rightarrow$ Determinacy: all sets are uB

Question

What predicates can be added to the derived model and preserve AD^+?

Large Cradinals $\to$ Determinacy: all sets are uB

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Theorem (Larson, s., Wilson)

Suppose λ is a limit of Woodins and strongs. Let $g \subseteq Coll(\omega, < \lambda)$. Then in $V(\mathbb{R}^*)$, there is a definable class F such that $L^F(Hom^*) \models AD^+ +$ "Every set of reals is uB".

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Remark

Using AD^+, it is not hard to build a model of "all sets of reals are uB". Suppose $\theta_{\alpha+1} = \Theta$ and α is limit. Let $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$. Then $HOD_\Gamma | \Theta \vDash$" All sets of reals are hom Suslin".

Large Cradinals → Determinacy: all sets are uB

Question

Is there a derived model theorem for "all sets of reals are homogenously Suslin".

Large Cradinals $\to$ Determinacy: all sets are uB

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Problem

Show that AD^+ + "All sets of reals are uB" is equiconsistent with ZFC + "There is λ that is a limit of Woodins and strongs."

Large Cradinals $\rightarrow$ Determinacy: all sets are uB

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Is there a derived model theorem for "all sets of reals are homogenously Suslin".

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Show that AD^+ + *"All sets of reals are uB" is equiconsistent with ZFC* + *"There is* λ *that is a limit of Woodins and strongs."*

Question

Assume UBH. Does $L[\vec{E}](Hom^*, \mathbb{R}^*) \models AD^+$*?*

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Question

Does $C^+(Hom^*, \mathbb{R}^*) \vDash AD^+$*?*

Large Cradinals $\rightarrow$ Determinacy: upper bound for LSA

Theorem

Assume $\mathcal{M}_{wlw}$ exists (there is class size iterable mouse with a Woodin cardinal that is a limit of Woodin cardinals). Then some initial segment of the derived model of $\mathcal{M}_{wlw}$ satisfies LSA.

Large Cradinals $\to$ Determinacy: upper bound for LSA

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Conjecture

The following are equiconsistent.

1. *There are divergent models of AD^+.*
2. *There is an inner model with a Woodin cardinal that is a limit of Woodin cardinals.*

Remark

Known to follow from MSC.

Large Cradinals $\to$ Determinacy: derived models of mice

Question

Is the strategy of a mouse the next new set beyond the derived model?

Large Cradinals $\to$ Determinacy: derived models of mice

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Is the strategy of a mouse the next new set beyond the derived model?

More specifically

Problem

Suppose $x \to \mathcal{M}(x)$ is a tractable mouse operator such that for every x there is λ_x with the property that $\mathcal{M}(x) = L[\mathcal{M}(x)|\lambda_x]$ and $\mathcal{M}_x \vDash$ "λ_x is a limit of Woodin cardinals". Let Σ_x be the unique strategy of $\mathcal{M}(x)$. Let M be the derived model of $\mathcal{M}(\emptyset)$, and suppose N is a model of determinacy such that $\mathcal{P}(\mathbb{R}) \cap M \subset N$. Must $\Sigma_x \in N$?

Determinacy $\to$ Large Cardinals

Assume AD^+. The Solovay sequence is a closed sequence of cardinals $(\theta_\alpha : \alpha \leq \Omega)$ such that

1. $\theta_0 = \sup\{\gamma :$ there is an OD surjection $f : \omega^\omega \to \gamma\}$,
2. if $\theta_\alpha < \Theta$, then $\theta_{\alpha+1} = \sup\{\gamma :$ there is an OD surjection $f : \theta_\alpha^\omega \to \gamma\}$,
3. if α is limit then $\theta_\alpha = \sup_{\beta<\alpha} \theta_\beta$,
4. $\Theta = \theta_\Omega$.

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3. (Closson, Neeman, s., Steel) $AD^+ + \theta_{\omega_1+1} = \Theta$ is equiconsistent with λ is a limit of Woodins and there is a $< \lambda$-hyperstrong.

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4. (Adolf-s.) $AD^+ + \theta_{\omega_2} \leq \Theta$ holds in the derived model of a mouse in which there is λ that is a limit of Woodins and $\kappa < \lambda$ whose degree of hyperstrongness is u_2 for sets in V_λ.

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5. Adolf and s. have what they believe are the optimal hypothesis for each of $\theta_n \leq \Theta$ and $\Theta_\Theta = \Theta$ but the reversals have not been verified.

Determinacy $\rightarrow$ Large Cardinals

Problem

1. *Determine the large cardinal strength of $AD_{\mathbb{R}}$ + "Θ is regular" and of LSA.*

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2. *Is there a mouse whose new derived model satisfies LSA?*

Determinacy $\to$ Large Cardinals

Problem

1. *Determine the large cardinal strength of $AD_{\mathbb{R}}$ + "Θ is regular" and of LSA.*
2. *Is there a mouse whose new derived model satisfies LSA?*

Theorem

The following are equiconsistent.

1. *LSA.*
2. *ZFC+" there are ω Woodins with limit λ such that the old derived model at λ is a model of $AD_{\mathbb{R}}$ but the new and old derived models are different".*

Determinacy $\to$ Large Cardinals

Theorem (Zhu)

Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ *and there is no inner model of* $AD_{\mathbb{R}}$ + "Θ *is regular". Then* V *is either a derived model of a mouse or embeds into the derived model of a mouse.*

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Remark

Woodin showed that any model of $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ is either a derived model or embeds into a derived model.

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Problem

Generalize Zhu's result to models of LSA.

Forcing Axioms $\rightarrow$ Determinacy

Let $\Gamma_{max} = \{A \subseteq \mathbb{R}$: there is a hod pair $(\mathcal{P}, \Sigma)$ such that $A \leq_w Code(\Sigma)\}$

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3. (Conjecture) Assume *PFA* and let $g \subseteq Coll(\omega, \omega_1)$ be generic. Then $L(\Gamma_{max}) \vDash AD^+$ and $\mathrm{HOD}^{L(\Gamma_{max})} \vDash$ "there is a superstrong cardinal".

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4. (Trang and s.) Assume *PFA* and let $g \subseteq Coll(\omega, \omega_1)$ be generic. Then in $V[g]$ there is $A \in \Gamma_{max}$ such that $L(A, \mathbb{R}) \vDash LSA$.

Forcing Axioms $\to$ Determinacy

Let $\Gamma_{max} = \{A \subseteq \mathbb{R}$: there is a hod pair $(\mathcal{P}, \Sigma)$ such that $A \leq_w Code(\Sigma)\}$

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4. (Trang and s.) Assume PFA and let $g \subseteq Coll(\omega, \omega_1)$ be generic. Then in $V[g]$ there is $A \in \Gamma_{max}$ such that $L(A, \mathbb{R}) \vDash LSA$.
5. By an absoluteness argument, we also get models as above in V.

$\neg\Box \rightarrow$ Determinacy

Assume κ is a singular strong limit cardinal such that $\neg\Box_\kappa$ holds. Let $\mu < \kappa$ be a countably closed regular cardinal $< \kappa$. Let *g* $\subseteq$ *Coll*(ω, μ) be generic.

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1. (Steel) $V[g] \models AD^{L(\mathbb{R})}$ (by absoluteness, $AD^{L(\mathbb{R})}$).

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1. (Steel) $V[g] \models AD^{L(\mathbb{R})}$ (by absoluteness, $AD^{L(\mathbb{R})}$).
2. (s.) $V[g] \models \exists A \subseteq \Gamma_{max}, L(A, \mathbb{R}) \models AD^+ + \theta_0 < \Theta$ (by absoluteness the same holds in V).

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3. (Adolf) $V[g] \models \exists \Gamma \subseteq \Gamma_{max}, L(\Gamma, \mathbb{R}) \models AD_{\mathbb{R}} +$ "Θ is regular". (by absoluteness the same holds in V).

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3. (Adolf) $V[g] \models \exists \Gamma \subseteq \Gamma_{max}, L(\Gamma, \mathbb{R}) \models AD_{\mathbb{R}} +$ "Θ is regular". (by absoluteness the same holds in V).
4. (Open Problem) Does $V[g] \models \exists \Gamma \subseteq \Gamma_{max}, L(\Gamma, \mathbb{R}) \models LSA$?

Determinacy $\to \neg\square_{\omega_2} + \neg\square(\omega_2)$

1. (Woodin) Assume $V = L(\mathcal{P}(\mathbb{R}))$ and $V \models AD_{\mathbb{R}} +$ "Θ is regular". Then $V^{\mathbb{P}_{max} * Add(\omega_3, 1)} \models MM(c)$. Hence, $\neg\square(\omega_2)$ holds.

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2. (Caicedo-Larson-s.-Schindler-Steel-Zeman) Assume *LSA*. Then for some $\Gamma \subseteq \mathcal{P}(\mathbb{R})$, letting $W = L(\Gamma, \mathbb{R})$, $W^{\mathbb{P}_{max} * Add(\omega_3, 1)} \models MM(c) + \neg\Box_{\omega_2}$.

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3. Thus, $MM(c) + \neg\square_{\omega_2}$ is weaker than a Woodin limit of Woodins.

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3. Thus, $MM(c) + \neg\Box_{\omega_2}$ is weaker than a Woodin limit of Woodins.
4. (Open Problem) Can one force $\neg\Box_{\omega_3} + \neg\Box(\omega_3)$ over models of determinacy?

Determinacy $\rightarrow \neg\square_{\omega_2} + \neg\square(\omega_2)$

1. (Woodin) Assume $V = L(\mathcal{P}(\mathbb{R}))$ and $V \models AD_{\mathbb{R}}$ + "Θ is regular". Then $V^{\mathbb{P}_{max} * Add(\omega_3, 1)} \models MM(c)$. Hence, $\neg\square(\omega_2)$ holds.
2. (Caicedo-Larson-s.-Schindler-Steel-Zeman) Assume *LSA*. Then for some $\Gamma \subseteq \mathcal{P}(\mathbb{R})$, letting $W = L(\Gamma, \mathbb{R})$, $W^{\mathbb{P}_{max} * Add(\omega_3, 1)} \models MM(c) + \neg\square_{\omega_2}$.
3. Thus, $MM(c) + \neg\square_{\omega_2}$ is weaker than a Woodin limit of Woodins.
4. (Open Problem) Can one force $\neg\square_{\omega_3} + \neg\square(\omega_3)$ over models of determinacy?
5. A more doable project is to force failure of "maximal model covering" over models of determinacy.

Consecutive failures of square $\rightarrow$ Determinacy

1. (Open Problem, Woodin) What is the strength of $MM(c)$? (Guess: probably $AD_{\mathbb{R}}$ + "Θ is regular", but we can only get $AD^{L(\mathbb{R})}$ and its neighborhoods).

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2. (Open Problem) What is the strength of $\neg\square_{\omega_2} + \neg\square(\omega_2)$? Not much is known as above.

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2. (Open Problem) What is the strength of $\neg\square_{\omega_2} + \neg\square(\omega_2)$? Not much is known as above.
3. (Trang) Suppose $2^{\omega} = \omega_1$, $2^{\omega_1} = \omega_2$ and $2^{\omega_2} = \omega_3$. Suppose further $\neg\square_{\omega_3} + \neg\square(\omega_3) + \neg\square(\omega_4)$. Let $g \subseteq Coll(\omega, \omega_1)$ be generic. Then there is $\Gamma \subseteq \Gamma_{max}$ such that $L(\Gamma, \mathbb{R}) \models AD_{\mathbb{R}}$ + "Θ is regular".

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4. (Trang-s.) Can push the above theorem to LSA.

All cardinals are singular $\to$ Determinacy

Theorem (Schindler-Busche)

Assume all uncountable cardinals are singular. Then $AD^{L(\mathbb{R})}$.

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Assume all uncountable cardinals are singular. Then there is a model of $AD_{\mathbb{R}} +$ *"Θ is regular".*

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Problem

Can one get a model of LSA?

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Problem

Can one get a model of LSA?

Remark

Gitik showed that the hypo is consistent relative to proper class of strongly compacts.

Strongcompactness and determinacy

Theorem (Trang-Wilson)

The following theories are equiconsistent:

1. $ZF + DC + AD$.
2. $ZF + DC + \omega_1$ *is* $\mathbb{R}$*-strongly compact and* $\neg\Box_{\omega_1}$

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Supercompactness

Theorem (Woodin)

Assume there is a proper class of Woodin limit of Woodins. Then $C^+ \models AD^+$ and C^+ has universally Baire sharp.

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Assume AD^+ and suppose ω_1 is supercompact. Then there is a model of LSA.

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Theorem (Trang-s.)

Assume AD^+ and suppose ω_1 is supercompact. Then there is a model of LSA.

Question

Can one get more? We are close to an equiconsistency here!

Generic absoluteness $\rightarrow$ determinacy

Theorem (Wilson)

If κ is a measurable limit of Woodin cardinals and two-step $\exists^{\mathbb{R}}(\Pi^2_1)^{uB_\kappa}$ generic absoluteness holds below κ, then $L(\mathbb{R}^*_\kappa, Hom^*_\kappa)$ satisfies $\theta_0 < \Theta$.

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Problem

Find natural parameter free generic absoluteness results corresponding to $\theta_1 < \Theta$, $\theta_2 < \Theta$,..., $AD_{\mathbb{R}}$ + "Θ is regular" and etc?

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Theorem (Wilson)

If κ is a measurable limit of Woodin cardinals and the (lightface) theory of $L(\mathbb{R}, uB_\kappa)$ is generically absolute below κ, then $L(\mathbb{R}, uB_\kappa)$ and $L(\mathbb{R}^*_\kappa, Hom^*_\kappa)$ both satisfy $AD_\mathbb{R}$.

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Question

Do they satisfy more?

Generic absoluteness $\rightarrow$ determinacy

Problem

What is the large cardinal strength of uB-sealing+proper class of Woodin cardinals? What is the large cardinal strength of Σ^2_1-absoluteness (modulo CH).

Generic absoluteness $\to$ determinacy

Theorem (Woodin, s.-Wilson, s.)

The following theories are equiconsistent.

1. *There is a proper class of Woodin cardinals and a strong cardinal.*
2. *There is a proper class of Woodin cardinals and two-step $\exists^{\mathbb{R}}(\Pi^2_1)^{uB}$ generic absoluteness holds.*
3. *There is a proper class of Woodin cardinals and no generic extension has a $(\Delta^2_1)^{uB}$ wellordering of its reals.*
4. *There is a proper class of Woodin cardinals and for a stationary class of λ, $L(\mathbb{R}^*_\lambda, Hom^*_\lambda)$ satisfies $\theta_0 < \Theta$.*

Generic absoluteness $\to$ determinacy

Theorem (Woodin, s.-Wilson, s.)

The following theories are equiconsistent.

1. *There is a proper class of Woodin cardinals and a strong cardinal.*
2. *There is a proper class of Woodin cardinals and two-step $\exists^{\mathbb{R}}(\Pi^2_1)^{uB}$ generic absoluteness holds.*
3. *There is a proper class of Woodin cardinals and no generic extension has a $(\Delta^2_1)^{uB}$ wellordering of its reals.*
4. *There is a proper class of Woodin cardinals and for a stationary class of λ, $L(\mathbb{R}^*_\lambda, Hom^*_\lambda)$ satisfies $\theta_0 < \Theta$.*

Problem

Determine the large cardinal strength of "for a stationary class of λ, the old derived model at λ satisfies $\theta_1 < \Theta$" and etc

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9. Set $\mathcal{V} = L[\mathcal{M}, \Lambda]$ ($\mathcal{V}$ is called Varsovian model).

Varsovian Models

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What is the mantle of the minimal class size mouse with proper class of strongs and Woodins? Guess: just the Lp stack.

Varsovian Models

Problem

1. *Assume* $AD^+ + V = L(\mathcal{P}(\mathbb{R})) + \theta_1 = \Theta$*.*
2. *We know that* V_Θ^{HOD} *is a hod mouse with 2 Woodins.*

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1. *Assume* $AD^+ + V = L(\mathcal{P}(\mathbb{R})) + \theta_1 = \Theta$.
2. *We know that* V_Θ^{HOD} *is a hod mouse with 2 Woodins.*
3. *Is it also a varsovian model?*
4. *i.e., of the form* $L_\Theta[\mathcal{M}, \pi]$ *where* $\mathcal{M}$ *is an* $L[\vec{E}]$*-model of height* Θ *with a strong cardinal, its least strong is a limit of Woodins and* $\pi : \mathcal{M}|\theta_0 \to \mathcal{N}$ *is the iteration embedding of* $\mathcal{M}|\theta_0$ *into the* HOD *of the derived model of* $\mathcal{M}$ *at the least strong of* $\mathcal{M}$*.*

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1. *Assume* $AD^+ + V = L(\mathcal{P}(\mathbb{R})) + \theta_1 = \Theta$.
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5. *Can we take* $\mathcal{M}$ *to be* $K^{V_\Theta^{\mathrm{HOD}}}$*?*

Varsovian Models

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Is there a way of making sense of the mantle of a determinacy world? Guess: W is a ground if V is a symmetric extension of W. The mantle is the intersection of all grounds. Is it true that HOD *is the mantle?*

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Question

Can we have core model theory in models that are saturated?

First attempt

Theorem (s.-Zeman)

Assume $(\mathcal{P}, \Sigma)$ is a hod pair such that $\mathcal{P}$ is a mouse (i.e. has a single Woodin and etc) and that Σ is a fullness preserving (Ord, Ord)-iteration strategy with branch condensation. Suppose further that $\Sigma^{\#}$ doesn't exist. Then the core model exists and it has a Woodin cardinal.

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Funny Fact: If the core model has 2 Woodins then it has ω Woodins.

Questions and problems

Problem

Suppose $\mathcal{M}$ is the minimal mouse with a strong that is a limit of Woodins. Let κ be the strong and let $g \subseteq Coll(\omega, \kappa)$ be $\mathcal{M}$-generic. Show that in $\mathcal{M}[g]$, K exists and has a strong cardinal that is a limit of Woodins.

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Are there core models that have strongs past Woodins?

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Are there core models that have strongs past Woodins? More specifically, suppose $\mathcal{M} = \mathcal{M}_{wsws}$ and g collapses the second strong to ω. Does $K^{\mathcal{M}[g]}$ exist? If yes, is it an iterate of $\mathcal{M}$?

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Are there core models in universes that are completely saturated?

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Question

Are there core models in universes that are completely saturated? What does this even mean?

An approach to the K^c problem

Goal: Show the following: there is a K^c construction that either

1. converges or
2. it reaches a model $\mathcal{N}$ with a measurable cardinal κ that is a limit of Woodins, $(\kappa^+)^{\mathcal{N}}$ exists, $\mathrm{cf}((\kappa^+)^{\mathcal{N}}) \geq \omega_2$ and the square sequence of $\mathcal{N}|(\kappa^+)^{\mathcal{N}}$ is not threadable.

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Theorem (s.-Zeman)

Suppose the goal fails. Then $Lp(\mathbb{R}) \vDash AD^+$.

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1. Suppose $\mathcal{N}$ is a model appearing in the K^c-construction and it doesn't have the properties we want.
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3. This gives rise to a mouse $\mathcal{S}_\kappa$ extending $\mathcal{N}|(\kappa^+)^{\mathcal{N}}$ and projecting to or across κ.

Problem

Show that if $\mathcal{S}_\kappa$ is defined for all κ as above then countable submodels of $\mathcal{N}$ are iterable (Idea: $\mathcal{S}_\kappa$ determines extenders with critical point κ).

An approach to the K^c problem

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In the case we have $\mathrm{cf}((\kappa^+)^{\mathcal{N}}) \geq \omega_2$*, show that there are collapsing structures coming from* HOD *analysis (recall covering with derived models), and use them to repeat the above proof.*

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Conjecture

Assume there is no inner model of LSA. Then there is a K^c construction such that either

1. *It converges or*
2. *It produces a model $\mathcal{N}$ in which there is a measurable cardinal κ such that κ is a limit of Woodins,* $\mathrm{cf}((\kappa^+)^{\mathcal{N}}) \geq \omega_2$ *and the square sequence of* $\mathcal{N}|(\kappa^+)^{\mathcal{N}}$ *is not threadable.*

Thank you Ronald!