moti gitik.

extender based forcing with overlapping extenders and shelah weak hypothesis

let a be a set of regular cardinals s.t. $\min(a) > |a|$. let J be an ideal on a. suppose $f, g \in \prod a$.

$f <_J g$ iff $\{\delta \in a : f(\delta) < g(\delta)\} = a \pmod J$.

$(\prod a, <_J)$. let λ be a regular cardinal. λ is called the <u>true cofinality</u> of $(\prod a, <_J)$ iff there is an $<_J$-increasing seq. $(f_i : i < \lambda)$ of functions in $\prod a$ s.t. f.a. $g \in \prod a$ there is $i < \lambda$ with $f_i >_I g$.

let κ be a singular cardinal. define

$pp(\kappa)$:

true cofinality

$\downarrow$

$$pp(\kappa) = \sup \{ \mathrm{tcf}(\Pi a, <_J) :$$

$a \subset \kappa$, $\bar{\bar{a}} = \mathrm{cf}(\kappa)$, a consists of regular cardinals, J is a ideal on a s.t. $\forall \alpha < \kappa$ $a \cap \alpha \in J$ and

$\mathrm{tcf}(\Pi a, <_J)$ exists $\}$.

let Γ be a set of ideals. define $pp_\Gamma(\kappa)$ as above, only require that $J \in \Gamma$

<u>shelah weak hypothesis:</u>

SWH_1 : for every cardinal λ the set

$$\{ \kappa < \lambda : \kappa > \mathrm{cf}(\kappa),\ pp(\kappa) \geq \lambda \}$$

is at most ctbl.

SWH_2 : for every cardinal λ, the set

$$\{ \kappa < \lambda : \aleph_0 < \mathrm{cf}(\kappa) < \kappa,\ pp(\kappa) \geq \lambda \}$$

is finite.

SWH_3 : for every cardinal λ, the set

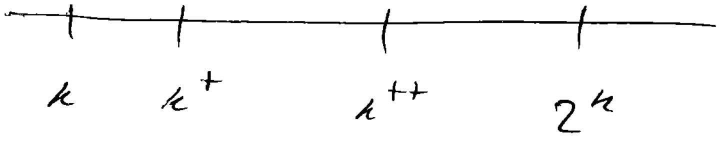
$$\{\kappa < \lambda : \aleph_0 < cf(\kappa) < \kappa, \; pp_{cf(\kappa)-complete}(\kappa) \geq \lambda\}$$ is finite.

Strong hypothesis :

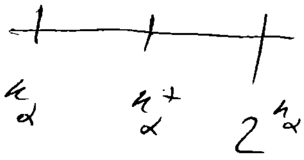
$pp(\kappa) = \kappa^+$ f.a. singular κ.

κ, $\aleph_0 < cf(\kappa) < \kappa$, κ stg lim,

$2^\kappa > \kappa^+$.



$(\kappa_\alpha : \alpha < cf(\kappa)) \nearrow \kappa$



<u>th.</u> (andreas liu) let $\kappa < \lambda$ be singular cardinals, $cf(\kappa) = \aleph_1$, $cf(\lambda) < \beth_3(\aleph_1)$. supp. that for some m, $1 \leq m < \omega$,

$$pp_{NS_{\omega_1}}(\kappa) \geq \lambda^{+m}$$

then there are $\kappa' < \lambda' < \kappa$, $\beth_3(\aleph_1) < \kappa'$, singular, $cf(\lambda') < \beth_3(\aleph_1)$, and $pp(\kappa') \geq \lambda'^{+m}$.

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assume GCH. $\kappa < \lambda$, λ is measurable, U a normal measure on λ. say E is a (κ, λ^+)-extender on κ. we'd like to change cofinality of both κ, λ to ω and make $2^\kappa > \lambda$, κ remains strong limit.

(1) (magidor–gitik) first change cofinality of λ to ω. let $(\lambda_n : n < \omega)$ be the prikry sequence. then use

extender based prikry forcing,

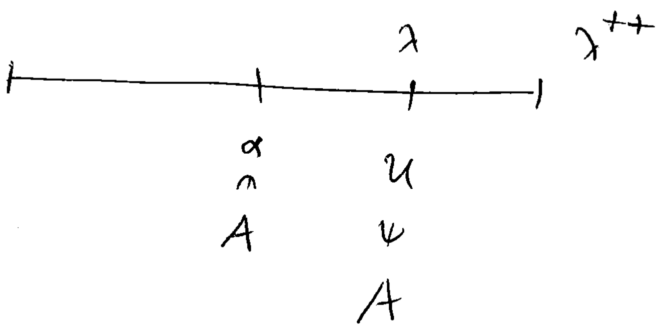
at level n we $E \restriction \lambda_n$.

how to make $2^\kappa > \lambda^+$?

(2) shelah – gitik:

it is possible to force indestructibility of ~~strongly~~ strongness of κ by prikry type forcings $(\mathbb{P}, \leq, \leq^*)$ s.t. $\leq^*$ is κ^+-directed closed.

Supp. now that we have $\kappa(0)_* < \kappa(1)_* < \ldots < \kappa(n) < \ldots$ $(n<\omega)$, $\kappa(\omega) = \bigcup_{n<\omega} \kappa(n)$. we would like to change the cofinality of each $\kappa(n)$ to ω_1 and make $pp(\kappa(n)) \geq \kappa(\omega)$.

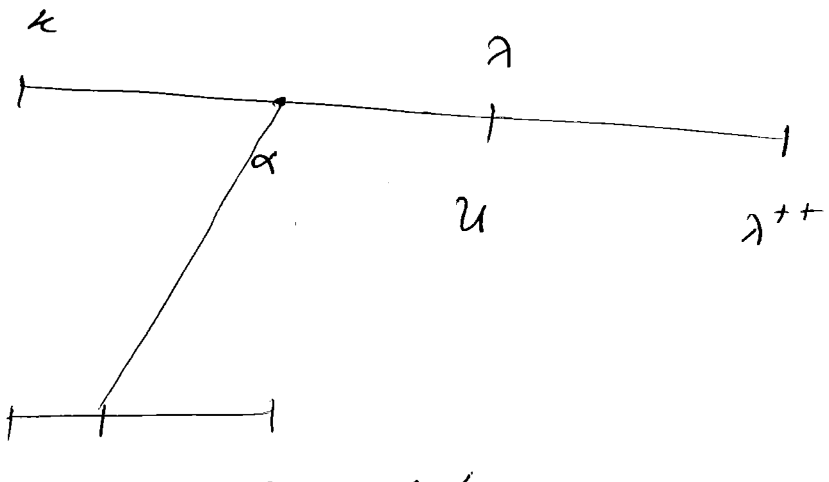


$i_E : V \longrightarrow M_E$

$E_\alpha = \{X \subset \kappa : \alpha \in i_E(X)\}$.

then is $A' \subset A$, $A' \in \mathcal{U}$ s.t. for every $\alpha, \beta \in A'$, $E_\alpha = E_\beta$.

assume that there are $f_\lambda : \kappa \to V$, $f_\mu : \kappa \to V$ s.t. $\lambda = i_E(f_\lambda)(\kappa)$, $\mu = i_E(f_\mu)(\kappa)$.



$\lambda_0 = f_\lambda(\kappa_0)$

$\mu_0 = f_\mu(\kappa_0)$

κ strong limit, $cf(\kappa) = cf(\lambda) = \omega$

$pp(\kappa) = 2^\kappa = \lambda^{++}$, let $(\kappa_n : n < \omega)$ be the prikry seq. E_κ; then each $f_\lambda(\kappa_n)$ change cof. to ω. no other cardinals are effected.

GCH. let $\kappa(0) < \kappa(1) < \ldots < \kappa(n) < \ldots$, $n<\omega$. $\kappa(\omega) = \bigcup_{n<\omega} \kappa(n)$. assume that each $\kappa(n)$ carries a coherent sequence of $(\kappa(n), \kappa(\omega)^+)$ extenders of length ω_1.

change cofinality of each $\kappa(n)$ to ω_1, and simultaneously blow up $pp(\kappa(n))$ to $\kappa(\omega)^+$.

question. ~~is there a cardinal~~ let λ be a cardinal. can the ~~tes~~ set $\{\kappa < \lambda : \aleph_0 = cf(\kappa) < \kappa,\ pp(\kappa) \geq \lambda\}$ have cardinality $\aleph_2$?

or can the set $\{\kappa < \lambda : \aleph_1 = cf(\kappa) < \kappa,\ pp(\kappa) \geq \lambda\}$ have cardinality $\aleph_2$?