Bounded Martin’s Maximum and strong cardinals

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Abstract

We show that if Bounded Martin’s Maximum (BMM) holds then for every \( X \in V \) there is an inner model with a strong cardinal containing \( X \). We also discuss various open questions which are related to BMM.

1 Introduction and statement of the result.

This paper strengthens one of the results of [6]. Bounded Martin’s Maximum (BMM, for short) is the statement that whenever \( \mathbb{P} \in V \) is a stationary set preserving forcing notion then

\[
H^V_{\omega_2} \prec_{\omega_1} H^\mathbb{P}_{\omega_2}.
\]

Bounded forcing axioms were introduced in [2] (as weakenings of the “unbounded” forcing axioms PFA and MM). Todorcevic showed (cf. [8]) that BMM implies that \( 2^{\aleph_0} = \aleph_2 \). (This was later improved by J. Moore who showed that already the Bounded Proper Forcing Axiom implies \( 2^{\aleph_0} = \aleph_2 \); cf. [4].) We refer the reader to [9, Section 10.3] for a discussion of BMM. We proved in [6, Theorem 1.3] that if BMM holds then for every \( X \in V \), \( X^# \) exists. The purpose of the present note is to prove the following.

Theorem 1.1 Suppose that BMM holds. Then for every \( X \in V \) there is an inner model with a strong cardinal containing \( X \).

We do not know if Theorem 1.1 gives the optimal lower bound for the consistency strength of BMM. Woodin has shown in unpublished work (cf. [10]) that \( \omega + 1 \) many Woodin cardinals are an upper bound. We refer the reader to [6] for further background information.

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The main result of this note was proven in February 2004 while the author was a guest at the CRM in Barcelona. He would like to thank Neus Portet and Joan Bagaria for their warm hospitality.
Whereas [6] constructs, assuming $\mathbf{BMM} + V$ is not closed under the $\#$ operator, a strictly decreasing sequence of functions from $\omega_1$ to $\omega_1$, the key new idea here is to construct, assuming $\mathbf{BMM}+$ there is some set $X$ which is not in an inner model with a strong cardinal, a strictly decreasing sequence of functions from $\omega_1$ to the set of all countable mice, where “decreasing” means “decreasing in the mouse order.”

2 The proof.

Let us assume that $\mathbf{BMM}$ holds throughout this section. We shall prove that there is an inner model with a strong cardinal. The reader will gladly verify that the argument to follow “relativizes” to any $X \in V$, yielding a proof of Theorem 1.1.

Let us suppose that there is no inner model with a strong cardinal. We may thus let $K$ denote the core model (cf. [3]). If $\mathcal{M}$ is a premouse and $\alpha \leq \mathcal{M} \cap \text{OR}$ then we say that $\alpha$ is overlapped in $\mathcal{M}$ just in case there is some extender $E^{\mathcal{M}}_\alpha$ such that $\text{crit}(E^{\mathcal{M}}_\alpha) < \alpha$ and $\nu \geq \alpha$. As there is no inner model with a strong cardinal, in $K$ there is no $E^{K}_\alpha$ such that $\text{crit}(E^{K}_\alpha)$ is overlapped in $K$.

We shall use the following notation. Let $\mathcal{M} = J_\alpha[E]$ be a premouse, and let $A \subset \bar{\alpha}$ for some $\bar{\alpha} < \alpha$. Then by $\mathcal{M}[A]$ we denote the transitive set (structure) $J_\alpha[E, A]$. Of course, in general $\mathcal{M}[A]$ will not be a premouse (or not even be a model of a reasonable fragment of ZFC). Nevertheless, models of the form $\mathcal{M}[A]$ will play a key rôle in what follows.

By $\leq^*$ we shall denote the pre-well-ordering of mice (cf. for instance [7]). I.e., if $\mathcal{M}$ and $\mathcal{N}$ are mice and $\mathcal{T}$, $\mathcal{U}$ is the coiteration of $\mathcal{M}$, $\mathcal{N}$ then $\mathcal{M} \leq^* \mathcal{N}$ if and only if $\mathcal{M}^T \leq \mathcal{N}^T$ (in which case $[0, \infty)^T$ contains no drop). We shall write $\mathcal{M} <^* \mathcal{N}$ iff $\mathcal{M} \leq^* \mathcal{N}$ and $\neg(\mathcal{N} \leq^* \mathcal{M})$.

Using the Dodd-Jensen Lemma and the fact that there are no degenerate iterations of mice, one can show that $\leq^*$ is indeed a pre-well-ordering. The following Lemma will thus give the desired contradiction.

**Lemma 2.1** ($\mathbf{BMM}+$ there is no inner model with a strong cardinal.) There is a sequence $\mathcal{S} = (A_n, C_n, (\mathcal{N}_{n, \alpha}; \alpha \in C_n) : n < \omega)$ such that for every $n < \omega$, $A_n \subset \omega_1$, $C_n$ is a club subset of $\omega_1$, $C_n \subset C_{n-1}$ if $n > 0$, and for every $\alpha \in C_n$, $\mathcal{N}_{n, \alpha}$ is a sound mouse with $\rho_\alpha(\mathcal{N}_{n, \alpha}) = \kappa \geq \alpha$, where $\kappa$ is the largest cardinal of $\mathcal{N}_{n, \alpha}$, and $\kappa$ is not overlapped in $\mathcal{N}_{n, \alpha}$, $(A_n \cap \alpha)_{\text{odd}}$ codes the mouse $\mathcal{N}_{n, \alpha} \models [\kappa, 2]$ and $\mathcal{N}_{n, \alpha}[A_n \cap \alpha] \models$

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2 If $A$ is a set of ordinals then by $A_{\text{odd}}$ we mean the set $\{\alpha : 2\alpha + 1 \in A\}$. The coding here and in what follows is to be understood as being according to some standard soft coding device. For instance, if $M$ is transitive and of size $\zeta$ then there is some $E \subset \zeta^2$ and some isomorphism $\sigma : (\zeta; E) \cong (M; \in)$, via Gödel’s pairing function, $E$ may be construed as a subset of $\zeta$ which codes $M$. 

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“α is countable”; moreover, $N_{n, \alpha} < T N_{n-1, \alpha}$ if $n > 0$ and $\alpha \in C_n$.

The proof of Lemma 2.1 exploits a version of the “faster reshaping forcing” which we had introduced in [6].

Let $n \in \omega$, and let us assume that $S \upharpoonright n$ has been constructed. We aim to construct $A_n$, $C_n$, and $(N_{n, \alpha}; \alpha \in C_n)$.

By BMM, it suffices to force the existence of these objects with the desired properties by a stationary set preserving forcing notion.

By [5], there is a $\sigma$-closed forcing notion $P$ such that in $V^P$, CH holds, there is some $A^+ \subset \omega_1$ with $H_{\omega_2} = K[\omega_2[A^+]]$, and $K[\omega_2]$ has a largest cardinal $\kappa$ which is not overlapped in $K[\omega_2]$. We may assume that $A^+$ is chosen such that $H_{\omega_1} = K[\omega_1[A^+]]$.

Let us for the rest of this argument assume that $n > 0$. The case $n = 0$ is an easier variant of what is to come. We thus may and shall assume that $A^+_{\text{odd}}$ is the join of $A_{n-1}$ and a code of $K[\kappa]$. Let $C \subset C_{n-1}$ be club and such that there is a sequence $(M_\alpha; \alpha \in C)$ of mice such that for every $\alpha \in C$, $(A^+ \cap \alpha)_{\text{odd}}$ codes the join of $A_{n-1} \cap \alpha$ and a code of $\mathcal{M}_\alpha$. Let $\kappa_\alpha$ denote the height of $\mathcal{M}_\alpha$ for $\alpha \in C$. The following will be crucial.

**Claim 1.** Let $\pi: \tilde{K}[A^+ \cap \alpha] \equiv X \prec (K[\omega_2[A^+]; \in, A^+)$, where $X$ is countable and $\alpha = X \cap \omega_1 \in C$. Then $\mathcal{M}_\alpha \triangleleft \tilde{K}$, $\kappa_\alpha = \pi^{-1}(\kappa)$ is the largest cardinal of $\tilde{K}$, and $\tilde{K} \leq * N_{n-1, \alpha}$.

**Proof of Claim 1.** Everything except for $\tilde{K} \leq * N_{n-1, \alpha}$ easily follows by the elementarity of $\pi$. Let us write $\mathcal{N} = N_{n-1, \alpha}$. Suppose Claim 1 to be false; i.e., $\mathcal{N} \not\prec \tilde{K}$. We shall argue that $\tilde{K}[A^+ \cap \alpha] \models \text{“}\alpha \text{ is countable}, \text{”}$ which is of course nonsense.

Because $\tilde{K}$ does not have any active extenders with indices between $\pi^{-1}(\kappa)$ and its height, it is easy to see that there must be now some $\xi < \tilde{K} \cap \text{OR}$ such that $\mathcal{N} \not\prec \tilde{K}[\xi]$. Let $\mathcal{T}, \mathcal{U}$ denote the coiteration of $\mathcal{N}$, $\tilde{K}[\xi]$. We know that $[0, \infty)_\mathcal{T}$ contains no drops, and that $\mathcal{M}_\infty \leq \mathcal{M}_\infty$.

Let $\rho_\mathcal{N}(\mathcal{N}) = \eta$. Because $(A_{n-1} \cap \alpha)_{\text{odd}}$ codes $\mathcal{N}[\eta, \mathcal{N}] | \eta \in L[A_{n-1} \cap \alpha] \subset L[A^+ \cap \alpha]$.

Let $\tilde{T}, \tilde{U}$ denote the coiteration of $\mathcal{N}[\eta, \tilde{K}[\xi]$. Clearly, $\tilde{T}$ is an “initial segment” of $\mathcal{T}$, and $\tilde{U}$ is an “initial segment” of $\mathcal{U}$. Moreover, $\mathcal{N}[\eta] \leq \tilde{K}[\xi]$, so that $[0, \infty)_\mathcal{T}$ contains no drops and $\mathcal{M}_\infty \leq \mathcal{M}_\infty$. If we had $\text{lh}(\mathcal{T}) > \text{lh}(\tilde{T})$ then, as $\eta$ is not overlapped in $\mathcal{N}$ and $\rho_\mathcal{N}(\mathcal{N}) = \eta$, $\mathcal{M}_\mathcal{T}$ would be non-sound, although $\mathcal{N} \not\prec \tilde{K}$. Therefore, $\text{lh}(\mathcal{T}) = \text{lh}(\tilde{T})$. But then we must also have that $\text{lh}(\mathcal{U}) = \text{lh}(\tilde{U})$, as $\eta$ is the largest cardinal of $\mathcal{N}$.

Because $\mathcal{N}[\eta] \in L[A^+ \cap \alpha]$ and $\tilde{K}[\xi] \in L[\tilde{K}[\xi]]$, we know that $\pi^{-1}_\infty$ as well as $\mathcal{M}_\infty^\mathcal{T} = \mathcal{M}_\infty^\mathcal{U}$ are elements of $L[A^+ \cap \alpha, \tilde{K}[\xi]]$. 

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As $\mathcal{M}_\infty^L \leq \mathcal{M}_\infty^G$, we thus have that $\mathcal{M}_\infty^G \in L[A^+ \cap \alpha, \bar{K}[\xi]]$. But
\[ \mathcal{N} \cong h^{\mathcal{M}_\infty^G}(\text{ran}(\pi^G_0) \cup \{\pi^G_0(p_N)\}), \]
where $p_N$ is the standard parameter of $\mathcal{N}$ and $h^{\mathcal{M}_\infty^G}$ is an appropriate Skolem hull operator. This yields that in fact $\mathcal{N} \in L[A^+ \cap \alpha, \bar{K}[\xi]]$.

Now let $a \in \omega \cap \mathcal{N}[A_n \cap \alpha]$ code a well-order of $\omega$ of order-type $\alpha$. We have shown that $a \in L[A^+ \cap \alpha, \bar{K}[\xi]]$. By [6], $V^\mathcal{P}$ is closed under the $\#^+$ operator. We therefore have that $(A^+ \cap \alpha, \bar{K}[\xi]) \#^+$ exists and $a \in (A^+ \cap \alpha, \bar{K}[\xi]) \#^+$. But $A^+ \cap \alpha$ and $\bar{K}[\xi]$ are both elements of $\bar{K}[A^+ \cap \alpha]$. Due to $\pi$, $\bar{K}[A^+ \cap \alpha]$ is closed under the $\#$ operator, so that in fact $a \in \bar{K}[A^+ \cap \alpha]$. But then $\alpha$ is countable in $\bar{K}[A^+ \cap \alpha]$, a contradiction. \hfill \Box (Claim 1)

In $V^\mathcal{P}$, we shall now consider the following forcing notion, denoted by $\mathcal{Q}$. We let $(c, p) \in \mathcal{Q}$ if and only if $c$ is a countable closed subset of $C$, $p : \text{max}(c) \rightarrow 2$, and for every $\alpha \in c$ there is some sound mouse $\mathcal{N} \supseteq \mathcal{M}_\alpha$ such that $\kappa_\alpha$ is the largest cardinal of $\mathcal{N}$, $\kappa_\alpha$ is not overlapped in $\mathcal{N}$, $\rho^\mathcal{N}(\mathcal{N}) = \kappa_\alpha$, $\mathcal{N}[A^+ \cap \alpha, p \upharpoonright \alpha] \models \text{"}\alpha\text{ is countable,}\"$ and $\mathcal{N} \not< \mathcal{N}_{n-1, \alpha}$. A condition $(c', q)$ is stronger than $(c, p)$ iff $\text{max}(c') \geq \text{max}(c)$, $c' \cap (\text{max}(c) + 1) = c$, and $q \upharpoonright \text{max}(c) = p$.

Claim 2. (“Extendability Lemma”) If $(c, p) \in \mathcal{Q}$ and $\alpha < \omega_1$ then there is some $(c', q) \leq (c, p)$ such that $\text{max}(c') \geq \alpha$.

Proof of Claim 2. Given $(c, p)$ and $\alpha$, let us assume w.l.o.g. that $\alpha \geq \text{max}(c) + \omega$, $\alpha \in C$, and $\alpha = X \cap \omega_1$ for some $X \subset K[\omega_2[A^+]$, so that $X \cong \bar{K}$ for some $\bar{K} \supseteq \mathcal{M}_\alpha$ and $\bar{K} \not< \mathcal{N}_{n-1, \alpha}$ by Claim 1. Pick a code $x \in \omega \omega$ for the ordinal $\alpha$, and let $\text{dom}(q) = \alpha$, where $q \upharpoonright \text{dom}(p) = p$, $q(\text{max}(c) + n) = x(n)$ for all $n < \omega$, and $q(\gamma) = 0$ for all $\gamma \geq \text{dom}(c) + \omega$. There will be some $\mathcal{P}$ with $\bar{K}[\pi^{-1}(\kappa) \triangleleft \mathcal{P} \triangleleft \bar{K}$ such that $\rho_\omega(\mathcal{P}) = \pi^{-1}(\kappa)$, $\pi^{-1}(\kappa)$ is the largest cardinal of $\mathcal{P}$, and $\mathcal{P}[A^+ \cap \alpha, q] \models \text{"}\alpha\text{ is countable,}\"$ Therefore, $(c \cup \{\alpha\}, q)$ is as desired. \hfill \Box (Claim 2)

In order to finish the proof of Lemma 2.1 (and hence of Theorem 1.1) it now obviously suffices to verify the following.

Claim 3. $\mathcal{Q}$ is stationary set preserving.

Proof of Claim 3. Let $S \subset \omega_1$ be stationary, and let $(c, p) \models \text{"}\hat{C} \text{ is a club subset of } \omega_1\text{.}\"$ We aim to construct some $(c', q) \leq (c, p)$ such that $(c', q) \models \text{"}\hat{C} \cap S \neq \emptyset\text{.}\"$
Let $X < (H_{\omega_1}; \in, A^+, \mathbb{Q}, (c, p))$ be transitive and of size $\aleph_1$, and let $(X_i: i \leq \omega_1)$ be a continuous chain of countable elementary submodels of $X$ approaching $X$ (i.e., $X = X_{\omega_1}$). Recall that $H_{\omega_2} = K[|\omega_2[A^+]|]$ (in $V^P$). Let

$$
\pi: \bar{K}[A^+ \cap \alpha] \cong Y < (H_{\omega_2}; \in, A^+, \mathbb{Q}, (c, p), (X_i: i \leq \omega_1)),
$$

where $Y$ is countable and $\alpha = Y \cap \omega_1 = \omega_1^{R[A^+ \cap \alpha]} < \omega_1$. We also may and shall assume that $\alpha \in S$. Let us write $\alpha_i = X_i \cap \omega_1 < \omega_1$ for $i < \omega_1$. Of course $(\alpha_i: i < \alpha) \in \bar{K}[A^+ \cap \alpha]$, where $(\alpha_i: i < \alpha)$ is cofinal in $\alpha$. Let us pick (externally, i.e., in $V^P$) a sequence $(i_n: n < \omega)$ which is cofinal in $\alpha$; hence $(\alpha_{i_n}: n < \omega)$ is cofinal in $\alpha$ as well. As $H_{\omega_1} = K[|\omega_1[A^+]|]$, $(H_{\omega_1})^{\bar{K}[A^+ \cap \alpha]} \subseteq \bigcup_{i < \alpha} X_i$, so that we may assume that in fact $(c, p) \in X_{i_0}$.

We shall now recursively construct sequences $((c_n, p_n): n < \omega)$ and $((c_n', p_n'): n < \omega)$ of conditions in $\mathbb{Q}$ with the following properties.

1. $(c_0, p_0) = (c, p), (c_n', p_n') \in X_{i_{n+1}}$,
2. for all $\xi \in \text{dom}(p_n) \setminus \text{dom}(p_{n+1}) \cap \{\alpha_i: i < \alpha\}$, $p_n'(\xi) = 1$ if and only if $\xi = \alpha_{i_n}, (c_{n+1}, p_{n+1}) \leq (c_n', p_n'), and \ (c_{n+1}, p_{n+1}) \in X_{i_{n+1}}$,
3. there is some $\beta > \alpha_{i_n}$ such that $(c_{n+1}, p_{n+1}) \models \" \beta \in \dot{C}, \"$
4. $\max(c_{n+1}) > \alpha_{i_n}$.

In the light of (the proof of) Claim 2, there is no problem with this recursion.

Let us now set $c^* = \bigcup_{n<\omega} c_n \cup \{\alpha\}$ and $p^* = \bigcup_{n<\omega} p_n$. We’re done if we can show that $(c^*, p^*)$ is a condition, because then $(c^*, p^*) \models \" \alpha \in \dot{S} \cap \dot{C}, \\"$

Well, we have that $\bar{K}[A^+ \cap \alpha, p^*] \models \" \alpha \text{ is countable}, \"$ because for $\xi \in \{\alpha_i: i_0 \leq i < \alpha\}$, $p^*(\xi) = 1$ if and only if $\xi = \alpha_{i_n}$ for some $n < \omega$, so that $(\alpha_{i_n}: n < \omega) \in \bar{K}[A^+ \cap \alpha, p^*].$

By Claim 1, $\bar{K} \vDash M_{\alpha} = \bar{K}[|\pi^{-1}(\kappa)|]$. Moreover, there will again certainly be some $\mathcal{P}$ with $\bar{K}[|\pi^{-1}(\kappa)|] \subseteq \mathcal{P} \subseteq \bar{K}$ such that $\rho_{\omega}(\mathcal{P}) = \pi^{-1}(\kappa)$, $\pi^{-1}(\kappa)$ is the largest cardinal of $\mathcal{P}$, and $\bar{K}[A^+ \cap \alpha, p^*] \models \" \alpha \text{ is countable}, \\"$ By Claim 1, $\mathcal{P} < ^* \mathcal{N}_{\alpha} \cap \alpha$, so that $(c^*, p^*)$ is really a condition.

\[ \square \] (Claim 3)

\[ \square \] (Lemma 2.1, Theorem 1.1)

3 Some problems.

Let $(f_\alpha: \alpha < \omega_2)$ denote “the” sequence of canonical functions from $\omega_1$ to $\omega_1$. I.e., $f_\alpha(\nu) = \text{otp} g_\alpha(\nu)$, where $g_\alpha: \omega_1 \rightarrow \alpha$ is bijective. The Club Bounding Principle, CBP for short, says that for every $f: \omega_1 \rightarrow \omega_1$ there is some $\alpha < \omega_2$ such that $f < f_\alpha$ on a club, i.e., such that $\{\nu: f(\nu) < f_\alpha(\nu)\}$ contains a club subset of $\omega_1$. 5
P. Larson has shown that CBP implies $P_2(\omega_1)^+$ (cf. [1, Fact 3.4]; cf. [1, Definition 3.1 (b)] on the definition of $P_2(\omega_1)^+$). On the other hand one has:

**Lemma 3.1** If BMM and $P_2(\omega_1)^+$ hold then CBP holds.

**Proof Sketch.** Let $f: \omega_1 \to \omega_1$. If $P_2(\omega_1)^+$ holds then the natural forcing $\mathbb{P}_f$ for adding a canonical function above $f$ is easily seen to be stationary set preserving. $\mathbb{P}_f$ is just the collection of all $(c, p)$ such that $c$ is a closed countable subset of $\omega_1$ and $p: \max(c) \to \omega_2$ is such that for all $\nu \in c$, $f(\nu) < \otp p'' \nu$. \hfill $\square$ (Lemma 3.1)

We do not know if BMM alone implies the Club Bounding Principle. In order to answer this question we need a better understanding of how to construct models of BMM. The same remark applies to the following questions.

Woodin introduced the principle $\psi_{AC}$ and showed that $\psi_{AC}$ implies that $2^{\aleph_0} = \aleph_2$. (Cf. [9, Definition 5.12 and Lemma 5.15].) We do not know if BMM implies that $\psi_{AC}$ holds (we know that BMM implies that $2^{\aleph_0} = \aleph_2$, though, cf. [8]). But we have:

**Lemma 3.2** If BMM and $P_1(\omega_1)^+$ hold then $\psi_{AC}$ holds.

**Proof Sketch.** Cf. [1, Definition 3.1 (d)] on the definition of $P_1(\omega_1)^+$. Let $S, T \subseteq \omega_1$ both be stationary and costationary. By $P_1(\omega_1)^+$, the canonical forcing $\mathbb{P}_{S, T}$ for adding a witness to $\psi_{AC}$ is easily seen to be stationary set preserving. $\mathbb{P}_{S, T}$ consists of all $(c, p)$ such that $c$ is a closed countable subset of $\omega_1$, $p: \max(c) \to \omega_2$, and for all $\nu \in c$, $\otp p'' \nu \in T \iff \nu \in S$. \hfill $\square$ (Lemma 3.2)

It is not known if $\psi_{AC}$ implies $P_1(\omega_1)^+$. By [1, Fact 3.1], $\psi_{AC}$ implies CBP.

Another interesting question is whether BMM implies that $\omega_2 = \omega_2$. This question makes sense by [6]. If BMM holds then every real has a $\#$, and hence we may ask if $\omega_2$, the second uniform indiscernible, is equal to $\omega_2$.

**References**


