Inner model theoretic geology*

Gunter Fuchs† Ralf Schindler

September 9, 2015

Abstract

One of the basic concepts of set theoretic geology is the *mantle* of a model of set theory V: it is the intersection of all grounds of V, that is, of all inner models M of V such that V is a set-forcing extension of M. The main theme of the present paper is to identify situations in which the mantle turns out to be a fine structural extender model. The first main result is that this is the case when the universe is constructible from a set and there is an inner model with a Woodin cardinal. The second situation like that arises if $L[E]$ is an extender model that is iterable in V but not internally iterable, as guided by P-constructions, $L[E]$ has no strong cardinal, and the extender sequence E is ordinal definable in $L[E]$ and its forcing extensions by collapsing a cutpoint to ω (in an appropriate sense). The third main result concerns the *Solid Core* of a model of set theory. This is the collection of all sets that are constructible from a set of ordinals that cannot be added by set-forcing to an inner model. The main result here is that if there is an inner model with a Woodin cardinal, then the solid core is a fine-structural extender model.

1 Introduction

In [FHR15], the authors introduced several types of inner models which are defined following the paradigm of "undoing" forcing. Thus, the mantle $\mathbb{M}$ of a model of set theory V is the intersection of all of its ground models (i.e., the intersection of all inner models of which V is a set-generic forcing extension). By a result of

*AMS MSC 2010: 03E35, 03E40, 03E45, 03E47, 03E55. Keywords: Inner models, mantle, forcing, large cardinals

†Part of the first author's work on this research was supported by PSC CUNY research award # 66572-00 44. He also wishes to thank the Institute for Mathematical Logic at Münster University for travel support and hospitality in the summer of 2013.

Laver and Woodin (independently, see [Lav07], [Woo04]), any such ground model is uniformly definable, by just varying a parameter, which makes the $\mathbb{M}$ a definable class. It is still a fundamental open question whether the mantle necessarily is a model of ZFC. On the positive side, it was shown in [FHR15] that if the universe is constructible from a set, then its mantle is a model of set theory. The mantle of a model of set theory, and other concepts which are arrived at by the idea of "undoing forcing", are the chief objects of study in what was dubbed Set Theoretic Geology in that paper.

One of the main results of [FHR15] was that any model V of set theory has a class forcing extension W such that V is the mantle of W. This result crushed the initial naive hope that the mantle of a model of set theory is somehow a canonical model that one arrives at after stripping away all the artificial layers of forcing that may have been done to it: the mantle of a model of ZFC can basically be anything. The class forcings used to reach W produced models of set theory that are not constructible from a set, though.

The present paper makes three main contributions to set theoretic geology.

Firstly, we show that if $\mathrm{V} = L[x]$ has an inner model with a Woodin cardinal, then its mantle is a fine structural extender model. In particular, the mantle satisfies GCH, has squares and diamonds, etc. This is theorem 3.18. It shows that in general, the proper class forcings leading to models in which the mantle is the original model have to produce models that are not constructible from a set, if the original model had an inner model with a Woodin cardinal. The main tool in proving this result is Woodin's extender algebra, which enables us to make x (the set from which V is constructible) generic over iterates of a certain type of minimal, sufficiently iterable extender model.

The second main result is Theorem 3.33, in which we draw the same conclusion, that the mantle of an extender model of the form $L[E]$ is fine structural, if that model satisfies a set of technical conditions: it has to be tame, it may have no strong cardinal, it may not be internally iterable as guided by $\mathcal{P}$-constructions (a term we will explain), but in V, $L[E]$ is fully iterable, and finally, the extender sequence E is ordinal definable in $L[E]$, and for every cutpoint θ of E, the canonical extension of the extenders on E which have critical points greater than θ to $L[E]^{\mathrm{Col}(\omega,\theta)}$, is also ordinal definable there.

Finally, we analyze a concept which was introduced by the first author, trying to arrive at a canonical inner model by undoing forcing, the solid core. The idea is to undo forcing more locally. So instead of considering only sets that belong to every ground model of the entire universe, call a set *solid* if it cannot be added by set forcing to any inner model. The solid core is then defined to be the union of all solid sets. It is unclear in general whether the solid core is a model of set theory, but we show in Theorem 4.21 that if there is an inner model with a Woodin cardinal, then the solid core is again a fine-structural inner model. So, assuming large cardinals, the solid core is in some sense a canonical model of set theory, and it is a fine structural model, even though its definition does not mention fine structure.

The paper is organized as follows. In section 2, we give an overview of set-theoretic geology, and we generalize a result from [FHR15]. The original result from that paper is that if the universe is constructible from a set, then the grounds are downward directed, and so, the mantle is a model of ZFC. We get the same conclusion, just assuming that the universe satisfies $\mathrm{V} = \mathsf{HOD}_{\{a\}}$, for a set a. Here, $\mathsf{HOD}_{\{a\}}$ is the inner model consisting of all sets that are hereditarily definable from ordinals and the set a (as a parameter).

Section 3 contains the results on calculating the mantle. First, in 3.1, we lay the grounds by isolating, within a model of set theory that has an inner model with a Woodin cardinal, certain sufficiently iterable such models that we call *minimal*. We show that any two such minimal models agree up to their least measurable cardinal. In 3.2, we show that working inside a model of set theory of the form $L[x]$, and assuming the existence of an inner model with a Woodin cardinal, the mantle is the intersection of all linear iterates of a minimal model achieved by applying the first total measure. In 3.3, we begin our analysis of the mantle of an $L[E]$ model under the following assumptions: $L[E]$ is tame, has no strong cardinal, is internally not fully iterable as guided by $\mathcal{P}$-constructions, but is fully iterable in V. The first step is to develop an appropriate variant of minimality. We show that under these assumptions, minimal models exist, and that the mantle is contained in the intersection of all linear iterates of such a minimal model reached by hitting the least measure. We prove the other direction of this inclusion in 3.4, under one extra assumption, which says that E is ordinal definable in $L[E]$, and the canonical extension of E to $\mathrm{Col}(\omega, \theta)$-generic forcing extensions is ordinal definable in these forcing extensions, whenever θ is a cutpoint of E.

Section 4 contains our results on the solid core. First, in 4.1, we assemble the basics (and basic open questions) on the solid core, and 4.2 contains the main results on the solid core: if there is an inner model with a Woodin cardinal, then the solid core is a fine structural extender model, and if there is no such inner model, then the core model may not be equal to the solid core.

Finally, we would like to thank the referee for reading the manuscript very closely, and for providing very useful feedback.

2 Set-theoretic Geology

2.1 Basics on set-theoretic geology

In the paper [FHR15], the authors define new types of inner models, thereby creating a line of research they term "Set-theoretic Geology". Exploiting the discovery that any model of which the set-theoretic universe is a set-forcing extension (such models are called "grounds") is uniformly definable, using a parameter (see [Lav07] (using a proof that's due to Hamkins), [Woo04]) they define the Mantle $\mathbb{M}$ to be the intersection of all grounds, and the Generic Mantle, $\mathsf{g}\mathbb{M}$, to be the intersection of all grounds of all set-forcing extensions, i.e., the intersection of all Mantles of all set-forcing extensions. There are many questions around the Mantle and the generic Mantle, many of which expose disturbing lacks of knowledge concerning very basic questions about forcing. Thus, we don't know in general whether the Mantle is a model of ZFC, or even of ZF – this is only known under appropriate downward directedness of grounds hypotheses: If the grounds are downward directed (meaning that any two grounds have a common ground), then the Mantle is a model of ZF, and if the grounds are downward set-directed (meaning that any collection of set-many grounds has a common ground), then the Mantle is a model of ZFC. It is not known whether there can be a model of set theory the grounds of which are not downward set-directed. The Generic Mantle turns out to be a more robust concept than the Mantle. It is invariant under set-forcing (meaning that the Generic Mantle is the same, whether it is computed in a model of set theory or any set-forcing extension of that model). This has as a consequence that the Generic Mantle is always a model of ZF. Moreover, under an appropriate downward directedness hypothesis, namely that the grounds are downward set-directed in any set-forcing extension (in fact, a local version of downward set-directedness suffices), it can be shown that the Axiom of Choice holds in the Generic Mantle.

A third type of inner model that is investigated in Set-theoretic Geology is the Generic HOD, gHOD. It is the intersection of all HODs of all set-forcing extensions, which is the same as the intersection $\bigcap_{\alpha\in\mathrm{On}}\mathsf{HOD}^{\mathrm{Col}(\omega,\alpha)}$. This model was introduced in [Fuc08], where it was also shown that it satisfies the ZFC axioms. Its relationship to the other protagonists of Set-theoretic Geology was investigated in [FHR15], where it was shown that in general, the Generic HOD is contained in HOD, and that the Generic HOD is contained in the Generic Mantle, which is contained in the Mantle.

It turns out that there is one assumption that conflates these different concepts, and hence simplifies Set-theoretic Geology considerably: it was shown in [FHR15] that if the universe is constructible from a set, then the generic Mantle, the Mantle and the generic HOD coincide, and hence they satisfy ZFC. Here, we prove a slight generalization of this fact, viz. Theorem 2.4. This generalization was observed by the first author while attending a talk by Hugh Woodin at the Apalachian Set Theory meeting 2012 at Cornell, where he made use of a strong form of Vopěnka's theorem in order to show that HOD is a forcing extension of $\mathsf{HOD}^{\mathrm{V}^{\mathrm{Col}(\omega,\alpha)}}$. The form of Vopěnka's theorem needed is as follows:

Theorem 2.1. *For every ordinal κ, there exists a $\mathbb{B} \in \mathsf{HOD}$ such that*

$$\mathsf{HOD} \models \mathbb{B} \textit{ is a complete Boolean algebra}$$

and for every $a \subseteq \kappa$, there exists a HOD-generic filter G on $\mathbb{B}$ such that

$$\mathsf{HOD}[a] \subseteq \mathsf{HOD}_{\{G\}} = \mathsf{HOD}_{\{a\}} = \mathsf{HOD}[G].$$

For a proof of this theorem, see [WDR12, Theorem 6].

A consequence of this theorem is that $\mathsf{HOD}_{\{a\}}$ is a set-forcing extension of HOD. To streamline some arguments to follow, let's note a lemma which is implicit in [FHR15]:

Lemma 2.2. *If M is a ground of V, then there is an α such that $\mathsf{HOD}^{\mathrm{V}^{\mathrm{Col}(\omega,\alpha)}} \subseteq M$.*

Proof. This is an argument from [FHR15]. Let $\mathrm{V} = M[g]$ via $\mathbb{P}$, and let α be the cardinality of $\mathbb{P}$. Let $G \subseteq \mathrm{Col}(\omega, \alpha)$ be V-generic. Then

$$\mathrm{V}[G] = M[g][G] = M[G'],$$

for some G' which is M-generic for $\mathrm{Col}(\omega, \alpha)$, by the absorption property of the collapse. So $\mathsf{HOD}^{V^{\mathrm{Col}(\omega,\alpha)}} = \mathsf{HOD}^{\mathrm{V}[G]} = \mathsf{HOD}^{M[G']} \subseteq M$, by the homogeneity of $\mathrm{Col}(\omega, \alpha)$. $\square$

Let us note as a corollary a result which was shown in [FHR15] as well:

Corollary 2.3. $\mathsf{gHOD} \subseteq \mathbb{M}$.

The following theorem is the above-mentioned generalization of a result from [FHR15].

Theorem 2.4. *If there is a set a such that $\mathrm{V} = \mathsf{HOD}_{\{a\}}$, then $\mathbb{M} = \mathsf{gHOD}$, and $\{\mathsf{HOD}^{\mathrm{V}^{\mathrm{Col}(\omega,\alpha)}} \mid \alpha \in \mathrm{On}\}$ is a collection of grounds which is dense in the grounds. In particular, the grounds are downward set-directed, and $\mathbb{M} = \mathsf{g}\mathbb{M} = \mathsf{gHOD} \models \mathsf{ZFC}$.*

Proof. Let $\mathrm{V} = \mathsf{HOD}_{\{a\}}$, where we may assume that a is a set of ordinals. In view of Lemma 2.2, it suffices to prove:

For every α, $\mathsf{HOD}^{\mathrm{V}^{\mathrm{Col}(\omega,\alpha)}}$ is a ground of V.

Let G be $\mathrm{Col}(\omega, \alpha)$-generic over V. In V$[G]$, V is definable, using a parameter, say b, being a ground model of V$[G]$. It follows then that $\mathrm{V} \subseteq \mathsf{HOD}^{\mathrm{V}[G]}_{\{a,b\}}$, for given an element $x \in \mathrm{V}$, x is definable in V from an ordinal and the set a, say $x = \{y \mid \varphi(y, \beta, a)\}$ in V. But since V is definable in V$[G]$, using b, say $\mathrm{V} = \{z \mid \psi(z, b)\}$, the first definition can be relativized by the second, and we get, in V$[G]$: $x = \{y \mid \varphi(y, \beta, a)\}^{\{z \mid \psi(z,b)\}}$. So altogether, x is defined in V$[G]$, using parameters a, b and β.

By the Vopěnka Theorem 2.1, $\mathsf{HOD}^{\mathrm{V}[G]}_{\{a,b\}} = \mathsf{HOD}^{\mathrm{V}[G]}[g]$, for some g which is set-generic over $\mathsf{HOD}^{\mathrm{V}[G]}$. And by homogeneity of the collapse, $\mathsf{HOD}^{V[G]} \subseteq \mathrm{V}$. So we have

$$\mathsf{HOD}^{V[G]} \subseteq \mathrm{V} \subseteq \mathsf{HOD}^{\mathrm{V}[G]}_{\{a,b\}} = \mathsf{HOD}^{\mathrm{V}[G]}[g],$$

so that V is wedged in between $\mathsf{HOD}^{\mathrm{V}[G]}$ and $\mathsf{HOD}^{\mathrm{V}[G]}[g]$, and hence is a forcing extension of $\mathsf{HOD}^{\mathrm{V}[G]} = \mathsf{HOD}^{\mathrm{V}^{\mathrm{Col}(\omega,\alpha)}}$. $\square$

This can be pushed even further.

Theorem 2.5. *If* V *has a set-forcing extension* V$[G]$ *such that for some* $a \in \mathrm{V}[G]$, $\mathrm{V} \subseteq \mathsf{HOD}^{\mathrm{V}[G]}_{\{a\}}$*, then for* α *greater than the size of the forcing for which* G *is generic,* $\mathsf{HOD}^{\mathrm{V}^{\mathrm{Col}(\omega,\alpha)}}$ *is a ground of* V*, and for every ground* M*, there is an* α *such that* $\mathsf{HOD}^{\mathrm{V}^{\mathrm{Col}(\omega,\alpha)}} \subseteq M$*. So again, the grounds are downward set-directed, and* $\mathbb{M} = \mathsf{g}\mathbb{M} = \mathsf{gHOD} \models \mathsf{ZFC}$.

Proof. Let $G \subseteq \mathbb{P}$ be V-generic, so that the assumptions are satisfied, as witnessed by a, and let α be greater than the size of $\mathbb{P}$. Let H be $\mathrm{Col}(\omega, \alpha)$-generic over V$[G]$. In V$[G][H]$, V$[G]$ is definable, using a parameter b. It follows then that $\mathrm{V} \subseteq \mathsf{HOD}^{\mathrm{V}[G][H]}_{\{a,b\}}$, for given an element $x \in \mathrm{V}$, x is definable in V$[G]$ from an ordinal and the set a, say $x = \{y \mid \varphi(y, \beta, a)\}$ in V$[G]$. But since V$[G]$ is definable in V$[G][H]$, using b, say $\mathrm{V}[G] = \{z \mid \psi(z, b)\}$, it follows that $x = \{y \mid \varphi(y, \beta, a)\}^{\{z \mid \psi(z,b)\}}$ in V$[G][H]$. So altogether, x is defined in V$[G][H]$, using the parameters a, b and β.

By the Vopěnka Theorem 2.1, $\mathsf{HOD}^{\mathrm{V}[G][H]}_{\{a,b\}} = \mathsf{HOD}^{\mathrm{V}[G][H]}[g]$, for some g which is set-generic over $\mathsf{HOD}^{\mathrm{V}[G][H]}$. By the absorption property of the collapse (since α is greater than the size of $\mathbb{P}$), there is H^* generic for $\mathrm{Col}(\omega, \alpha)$ such that $\mathrm{V}[G][H] = \mathrm{V}[H^*]$, and by the homogeneity of the collapse, $\mathsf{HOD}^{V[H^*]} \subseteq \mathrm{V}$. So we have

$$\mathsf{HOD}^{V[G][H]} = \mathsf{HOD}^{V[H^*]} \subseteq \mathrm{V} \subseteq \mathsf{HOD}^{\mathrm{V}[G][H]}_{\{a,b\}} = \mathsf{HOD}^{\mathrm{V}[G][H]}[g],$$

so that V is wedged in between $\mathsf{HOD}^{\mathrm{V}[G][H]}$ and $\mathsf{HOD}^{\mathrm{V}[G][H]}[g]$, and hence is a forcing extension of $\mathsf{HOD}^{\mathrm{V}[G][H]} = \mathsf{HOD}^{\mathrm{V}[H^*]} = \mathsf{HOD}^{\mathrm{V}^{\mathrm{Col}(\omega,\alpha)}}$. So $\mathsf{HOD}^{\mathrm{V}^{\mathrm{Col}(\omega,\alpha)}}$ is a ground of V, as claimed. The other claims follow from Lemma 2.2. $\square$

3 Calculating the Mantle

In this section, we will calculate the mantle of V in certain situations. The most striking results will work under the assumption that there is an inner model with a Woodin cardinal. Before heading in this direction, let us make a simple observation about the situation where there is no inner model with a Woodin cardinal.

Observation 3.1. *When there is no inner model with a Woodin cardinal, then K exists, and $K \subseteq \mathbb{M}$, even $K \subseteq \mathrm{g}\mathbb{M}$.*

3.1 Minimal Models

In this subsection we will recall some facts on inner models with Woodin cardinals, many of which are part of the folklore. We will need these in the next subsection. We refer the reader to [Ste09] and [**?**] for inner model theoretic background.

For the purpose of this and the next subsection, we will define "shortness" and "maximality" as follows.

Definition 3.2. A normal iteration tree $\mathcal{T}$ on a premouse $\mathcal{M}$ is called 0–*short* if for every limit ordinal $\lambda \leq \mathrm{lh}(\mathcal{T})$,[1]

$$L[\mathcal{M}(\mathcal{T} \restriction \lambda)] \models \text{"}\delta(\mathcal{T} \restriction \lambda) \text{ is not a Woodin cardinal."}$$

$\mathcal{T}$ is called 0–*maximal* iff $\mathcal{T}$ has limit length, $\mathcal{T} \restriction \lambda$ is 0–short for every limit ordinal $\lambda < \mathrm{lh}(\mathcal{T})$, but $\mathcal{T}$ is not 0–short.

More generally, a tree $\mathcal{T}$ on $\mathcal{M}$ would be called n–*short*, where $n < \omega$, if for every limit ordinal $\lambda \leq \mathrm{lh}(\mathcal{T})$,

$$M_n[\mathcal{M}(\mathcal{T} \restriction \lambda)] \models \text{"}\delta(\mathcal{T} \restriction \lambda) \text{ is not a Woodin cardinal."}^2$$

Generalizing this even further, we may consider trees whose cofinal branches at limit stages are determined by $\mathcal{Q}$–structures. As for now we won't have a use for trees guided by arbitrary $\mathcal{Q}$–structures, we shall just say "short" instead of "0–short," and we shall also say "maximal" instead of "0–maximal." The moral of Definition 3.2 is that for now our $\mathcal{Q}$–structures will be provided by initial segments of $L[\mathcal{M}(\mathcal{T}\restriction\lambda)]$. This will change in subsection 3.3.

All of our iteration trees will be finite stacks of normal trees.

[1]Here, $\mathcal{M}(\mathcal{T})$ denotes the common part model of $\mathcal{T}$, and $\delta(\mathcal{T})$ denotes the supremum of the lengths of extenders used in $\mathcal{T}$.

[2]M_n is the least sufficiently iterable inner model with n Woodin cardinals.

Definition 3.3. Let $\mathcal{M}$ be a premouse. We call $\mathcal{M}$ 0–pseudo–iterable iff $\mathcal{M}$ is iterable with respect to (0–)short trees. For $n \in \omega$, we call $\mathcal{M}$ $(n+1)$–pseudo–iterable iff $\mathcal{M}$ is iterable with respect to (0–)short trees and whenever $\mathcal{T}$ is a maximal tree on $\mathcal{M}$, then $L[\mathcal{M}(\mathcal{T})]$ is n–pseudo–iterable. We call $\mathcal{M}$ *pseudo–iterable* iff $\mathcal{M}$ is n–pseudo–iterable for every $n \in \omega$.

There are 1–small pseudo–iterable premice $\mathcal{M}$ with no Woodin cardinal such that there is a maximal iteration tree $\mathcal{T}$ on $\mathcal{M}$. For instance, let M_1 be the least sufficiently iterable inner model with a Woodin cardinal, let $j\colon M_1 \to W$ be an embedding obtained by forcing with the countable stationary tower over M_1. Hence $\mathrm{crit}(j) = \omega_1^{M_1}$, and we may let $\mathcal{M}$ be the least initial segment of W end–extending $M_1||\omega_1^{M_1}$ which projects to ω. By absoluteness, $\mathcal{M}$ is 0–pseudo–iterable. As there is a subset of ω which is definable over $\mathcal{M}$ but not contained in M_1, it is straightforward to verify that the comparison of $\mathcal{M}$ with M_1 will have to produce a maximal $\mathcal{T}$ on $\mathcal{M}$.

In what follows, by a K^c–construction we shall mean a construction as in [MS04] and by an $L[E]$–construction we shall mean a construction as in [MS94, §11], albeit with no smallness restriction on the initial segments.

Lemma 3.4. *Let W^* be any inner model, and let W be an extender model which is the result of a K^c or of an $L[E]$ construction performed inside W^*. Then in V, W is pseudo–iterable.*

Proof. Suppose W was not pseudo-iterable in V. By definition, this means that for some $N < \omega$, W is not N-pseudo iterable in V. Let N be minimal so that W is not N-iterable. Then there is a sequence $\langle (\mathcal{P}_i, \mathcal{T}_i) \mid i \leq N \rangle$ so that

1. $\mathcal{P}_0 = W$,
2. for $i < N$, $\mathcal{T}_i$ is a maximal, normal iteration tree on $\mathcal{P}_i$,
3. $\mathcal{T}_N$ is a short, normal, putative iteration tree on $\mathcal{P}_N$,
4. for $0 < i \leq N$, $\mathcal{P}_i = L[\mathcal{M}(\mathcal{T}_{i-1})]$, and
5. either $\mathcal{T}_N$ has a last model which is ill-founded, or $\mathcal{T}_N$ has limit length but no cofinal well-founded branch.

By taking a Skolem hull, there is then some elementary $\sigma^* : R \longrightarrow \mathrm{V}_\theta$, where V_θ is a sufficiently elementary submodel of V, such that R is countable and transitive, and such that, letting $\bar{W} = (\sigma^*)^{-1}(W|\theta)$ and $\sigma = \sigma^* \restriction \bar{W}$, we have:

1. $R \models \mathsf{ZFC}^-$,
2. $\sigma : \bar{W} \longrightarrow W$ is sufficiently elementary,

3. there is a sequence $\langle\langle\bar{\mathcal{P}}_i, \bar{\mathcal{T}}_i\rangle \mid i \leq N\rangle = (\sigma^*)^{-1}(\langle\langle\mathcal{P}_i, \mathcal{T}_i\rangle \mid i \leq N\rangle)$ which reflects the properties listed above, so that, in R, the following hold:

   (a) $\bar{\mathcal{P}}_0 = \bar{W}$ and $\bar{W} \cap \mathrm{On} = R \cap \mathrm{On}$,

   (b) for $i < N$, $R \models$ "$\bar{\mathcal{T}}_i$ is a maximal, normal iteration tree on $\bar{\mathcal{P}}_i$,"

   (c) $\bar{\mathcal{T}}_N$ is a short, normal, putative iteration tree on $\bar{\mathcal{P}}_N$,

   (d) for $0 < i \leq N$, $\bar{\mathcal{P}}_i = L_{R\cap\mathrm{On}}[\mathcal{M}(\bar{\mathcal{T}}_{i-1})]$,

   (e) either $\bar{\mathcal{T}}_N$ has a last model which is ill-founded, or $\bar{\mathcal{T}}_N$ has limit length but no cofinal well-founded branch.

There is a tree $U \in W^*$ of height ω searching for such objects $R, \sigma, \bar{W}, \langle\langle\bar{\mathcal{P}}_i, \bar{\mathcal{T}}_i\rangle \mid i \leq N\rangle$. Since the search is successful in V, U is ill-founded. So by absoluteness, the search is successful in W^* as well, which means that such objects also exist in W^*. Let us denote them again by $R, \sigma, \bar{W}, \langle\langle\bar{\mathcal{P}}_i, \bar{\mathcal{T}}_i\rangle \mid i \leq N\rangle$.

The iterability proof from [Ste96] for K^c or $L[E]$, run inside W^*, then allows us to show inductively that

1. every tree $\bar{\mathcal{T}}_i$ is formed according to the realization strategy, i.e., every model occurring in $\bar{\mathcal{T}}_i$ can be embedded into a model $\mathcal{M}_\xi$ from the K^c- or $L[E]$-construction of W^*, and

2. for every $i < N$, there is a cofinal realizable branch, say b_i, through $\bar{\mathcal{T}}_i$ such that $\bar{W}_{i+1} \trianglelefteq \mathcal{M}_{b_i}^{\bar{\mathcal{T}}_i}$, and if $\bar{\mathcal{T}}_N$ has limit length, then there is a cofinal realizable branch, say b_N, through $\bar{\mathcal{T}}_N$.

This gives a contradiction by standard arguments. E.g., if $\bar{\mathcal{T}}_N$ has limit length, then by uniqueness, absoluteness, and homogeneity of $\mathrm{Col}(\omega, \delta(\bar{\mathcal{T}}_N))$, $b_N \in R$, cf. [Ste96, §2]. $\square$

In what follows, we shall call a sequence $\langle(\mathcal{P}_i, \mathcal{T}_i) \mid i \leq N\rangle$ as in the preceding proof a *pseudo*–iteration of P_0.

We will frequently use the following notation.

Definition 3.5. If $\mathcal{M}$ is any model of set theory, we let $\delta^{\mathcal{M}}$ be its least Woodin cardinal, and we let $\kappa^{\mathcal{M}}$ be its least measurable cardinal, if these exist. If $\mathcal{M}$ doesn't have a Woodin cardinal, then we set $\delta^{\mathcal{M}} = \mathcal{M} \cap \mathrm{On}$, and if $\mathcal{M}$ doesn't have a measurable cardinal, then we set $\kappa^{\mathcal{M}} = \mathcal{M} \cap \mathrm{On}$.

If $\mathcal{M}$ is fine-structural, and $\mathcal{T}$ is an iteration tree on $\mathcal{M}$, then we say that $\mathcal{T}$ *lives strictly below* $\delta^{\mathcal{M}}$ iff there is some $\gamma < \delta^{\mathcal{M}}$ such that all extenders used on $\mathcal{T}$ are taken from $\mathcal{M}||\gamma$ and its images, i.e., if $\xi < \mathrm{lh}(\mathcal{T})$ and $[0, \xi]_{\mathcal{T}} \cap \mathcal{D}_{\mathcal{T}} = \emptyset$, then $\mathrm{lh}(E_\xi^{\mathcal{T}}) < \pi_{0\xi}^{\mathcal{T}}(\gamma)$.

Definition 3.6. Let $\mathcal{M}$ be a premouse with a Woodin cardinal. An infinite sequence $(W_i, \mathcal{T}_i \colon i \in \omega)$ is called a *degenerate pseudo–iteration of* $\mathcal{M}$ iff $W_0 = \mathcal{M}$, and for each $i < \omega$, $\mathcal{T}_i$ is a maximal tree on W_i living strictly below δ^{W_i} and $W_{i+1} = L[\mathcal{M}(\mathcal{T}_i)]$.

Lemma 3.7. *Let W^* be an inner model, and let W be an extender model which is the result of a K^c– or an $L[E]$–construction performed inside W^*. In V, there is then no degenerate pseudo–iteration of W.*

Proof. This follows from the proof of Lemma 3.4. As there, inside W^* we may get objects $R, \sigma, \bar{W}, \langle\langle \bar{\mathcal{P}}_i, \bar{\mathcal{T}}_i \rangle \mid i < \omega\rangle$ such that

1. $R \models \mathsf{ZFC}^-$ and R is countable,
2. $\sigma : \bar{W} \longrightarrow W$ is sufficiently elementary,
3. $\bar{\mathcal{P}}_0 = \bar{W}$ and $\bar{W} \cap \mathrm{On} = R \cap \mathrm{On}$,
4. for $i < \omega$, $R \models$ "$\bar{\mathcal{T}}_i$ is a maximal, normal iteration tree on $\bar{\mathcal{P}}_i$ living strictly below $\delta^{\bar{W}_i}$,"
5. for $0 < i \leq N$, $\bar{\mathcal{P}}_i = L_{R \cap \mathrm{On}}[\mathcal{M}(\bar{\mathcal{T}}_{i-1})]$,

We again get that

1. every tree $\bar{\mathcal{T}}_i$ is formed according to the realization strategy, i.e., every model occurring in $\bar{\mathcal{T}}_i$ can be embedded into a model $\mathcal{M}_\xi$ from the K^c- or $L[E]$-construction of W^*, and
2. for every $i < \omega$, there is a cofinal realizable branch, say b_i, through $\bar{\mathcal{T}}_i$ such that $\bar{W}_{i+1} \trianglelefteq \mathcal{M}^{\bar{\mathcal{T}}_i}_{b_i}$.

The point now is that as $\bar{\mathcal{T}}_i$ lives strictly below $\delta^{\bar{W}_i}$, we must in fact have that $\bar{W}_{i+1} \triangleleft \mathcal{M}^{\bar{\mathcal{T}}_i}_{b_i}$, so that if $\sigma_i \colon \mathcal{M}^{\bar{\mathcal{T}}_i}_{b_i} \to \mathcal{M}_{\xi_i}$ are the realization maps, then $\xi_{i+1} < \xi_i$ for every $i < \omega$. Contradiction! $\square$

Lemma 3.8. *Suppose there is an inner model with a Woodin cardinal. There is then a 1–small fine structural inner model W with a Woodin cardinal which is pseudo–iterable and such that there is no degenerate pseudo–iteration of W.*

Proof. We may find such a W as follows. Let W^* be an inner model with a Woodin cardinal, and let $\mathcal{M}_\xi$ and $\mathcal{N}_\xi = \mathrm{core}(\mathcal{M}_\xi)$ be the models from the $L[E]$–construction performed inside W^*.

If there is a ξ such that $\mathcal{M}_\xi$ is undefined, then there is some $\bar{\xi} < \xi$ such that $\mathcal{M}_{\bar{\xi}}$ is defined and not 1–small. If there is some ξ such that $\mathcal{M}_\xi$ is not 1–small and if $\bar{\xi}$ is the least such ξ, then we may let W be the result of iterating the top extender of $\mathcal{M}_{\bar{\xi}}$ out of the universe.

If every $\mathcal{M}_\xi$ is 1–small, then $\mathcal{M}_{\mathrm{On}}$ exists and has a Woodin cardinal, so that we may set $W = \mathcal{M}_{\mathrm{On}}$. Notice that W is as desired by Lemmas 3.4 and 3.7. $\square$

The following Definition 3.9 is crucial.

Definition 3.9. Let W be a fine structural inner model. We call W *minimal* iff W is 1–small, iterable with respect to short trees, $\delta^W \in W$ (and is hence a Woodin cardinal in W), and for all $\gamma < \delta^W$, there is no maximal tree on $W||\gamma$.

If W is minimal, then $W||\gamma$ is fully iterable for every $\gamma < \delta^W$, since every tree on $W||\gamma$ is short and W is iterable with respect to short trees. It is easy to see that the requirement that W be 1–small is redundant in Definition 3.9 and follows from the rest.

The following Lemma 3.10 is an immediate consequence of Lemma 3.8.

Lemma 3.10. *Suppose that there is an inner model with a Woodin cardinal. There is then a minimal fine structural inner model.*

Lemma 3.11. *Let W and W' be minimal fine structural inner models, and set $\kappa = \min(\{\kappa^W, \kappa^{W'}\})$. Then $W||\kappa = W'||\kappa$.*

Proof. Suppose not. Let us start comparing W with W', and let us suppose without loss of generality that $W||\kappa$ moves. By the minimality of W, $W||\kappa$ is fully iterable. As $W||\kappa$ is a lower–part model, the comparison of W with W' may be construed as a comparison of just $W||\kappa$ with W'. Let $\mathcal{T}$ denote the iteration tree on the W– (equivalently, $W||\kappa$–) side of the comparison.

Let $\mathcal{U}$ be the tree produced on the W'–side of the comparison. Then either $\mathcal{U}$ has last model, $\mathcal{M}^{\mathcal{U}}_\infty$, and there is no drop along $[0,\infty]_{\mathcal{U}}$, or $\mathcal{U}$ is maximal. Let us write $\delta = \pi^{\mathcal{U}}_{0\infty}(\delta^{W'})$ and $\mathcal{M} = \mathcal{M}^{\mathcal{U}}_\infty||\delta$ in the first case and $\delta = \delta(\mathcal{U})$ and $\mathcal{M} = \mathcal{M}(\mathcal{U})$ in the second. In both cases we will have that $L[\mathcal{M}] \models$ "δ is Woodin," as either $L[\mathcal{M}] = \mathcal{M}^{\mathcal{U}}_\infty$ or $L[\mathcal{M}] = L[\mathcal{M}(\mathcal{U})]$.

Let $\alpha < \mathrm{lh}(\mathcal{T})$ be minimal such that $\mathcal{M}^{\mathcal{T}}_\alpha \trianglerighteq \mathcal{M}$. Then $\rho_\omega(\mathcal{M}^{\mathcal{T}}_\alpha) < \delta$, so that $\mathcal{M}^{\mathcal{T}}_\alpha$,and hence W, can't be 1–small. Contradiction! $\square$

Lemma 3.12. *Let W be minimal in V. Then W is minimal in every forcig extension of V.*

Proof. Let W be a fine structural inner model. Let α be arbitrary, and let g be $\mathrm{Col}(\omega,\alpha)$–generic over V. Suppose that W is not minimal in $V[g]$. We aim to prove that W is not minimal in V.

By absoluteness, W is still iterable with respect to short trees in $V[g]$.

Let $\gamma < \delta^W$ and $\mathcal{T} \in V[g]$ be such that $\mathcal{T}$ is a maximal tree on $W||\gamma$. Let $\lambda > \max\{\alpha^+, \gamma, \delta(\mathcal{T})\}$ be a cardinal such that $2^{<\lambda} = \lambda$. Let $\mathcal{F}$ be the family of all $L[\mathcal{M}(\mathcal{U})]$, where $\mathcal{U} \in V[g]$ is a maximal tree on $W||\gamma$ with $\delta(\mathcal{U}) < \lambda$.

We may jointly pseudo–coiterate all $\mathcal{M} \in \mathcal{F}$, which produces a common pseudo–coiterate W^∞ with a Woodin cardinal $\leq \lambda^+$. As W^∞ is ordinal–definable, $W^\infty||\delta^{W^\infty} \in V$ by the homogeneity of $\mathrm{Col}(\omega, \alpha)$.

Let us assume towards a contradiction that W is minimal in V. Then $W||\gamma$ is fully iterable in V, which is easily seen to imply that we may successfully coiterate $W||\gamma$ with $W^\infty||\delta^{W^\infty}$ and if $\mathcal{R}$ and $\mathcal{S}$ are the induced iteration trees on $W||\gamma$ and $W^\infty||\delta^{W^\infty}$, respectively, then $\mathcal{M}^{\mathcal{R}}_\infty \lhd \mathcal{M}^{\mathcal{S}}_\infty$.

Let $p \in \mathrm{Col}(\omega, \alpha)$ and $\dot{\mathcal{T}}$, $\dot{\mathcal{W}} \in V^{\mathrm{Col}(\omega,\alpha)}$ be such that $p \Vdash$ "$\dot{\mathcal{T}}$ is a maximal tree on $(W||\gamma)\check{}$, and $\dot{\mathcal{W}}$ is a maximal tree on $\mathcal{M}(\dot{\mathcal{T}})$ such that $\mathcal{M}(\dot{\mathcal{W}}) = (W^\infty||\delta^{W^\infty})\check{}$." Let $X \prec H_{\theta^+}$, where θ is regular and big enough, X is countable, and $\{W||\delta^W$, γ, $W^\infty||\delta^{W^\infty}$, α, p, $\dot{\mathcal{T}}$, $\dot{\mathcal{W}}$, $\mathcal{R}$, $\mathcal{S}\} \subset X$. Let $\sigma \colon \bar{H} \cong X$ be the inverse of the transitive collapse, and let $\bar{W} = \sigma^{-1}(W||\theta)$, $\bar{\gamma} = \sigma^{-1}(\gamma)$, $\bar{W}^\infty = \sigma^{-1}(W^\infty||\delta^{W^\infty})$, $\bar{\mathcal{R}} = \sigma^{-1}(\mathcal{R})$, and $\bar{\mathcal{S}} = \sigma^{-1}(\mathcal{S})$. Let $\bar{g} \in V$ be $\mathrm{Col}(\omega, \sigma^{-1}(\alpha))$–generic over $\bar{H}$, with $\sigma^{-1}(p) \in \bar{g}$, and let $\bar{\mathcal{T}} = (\sigma^{-1}(\dot{\mathcal{T}}))^{\bar{g}}$ and $\bar{\mathcal{W}} = (\sigma^{-1}(\dot{\mathcal{W}}))^{\bar{g}}$. Write $\Omega = \bar{H} \cap \mathrm{On}$.

There is a cofinal branch b through $\bar{\mathcal{T}}$ such that $L_\Omega[\mathcal{M}(\bar{\mathcal{T}})] \lhd \mathcal{M}^{\bar{\mathcal{T}}}_b$. We may then construe $\bar{\mathcal{W}}$ to be a tree on $\mathcal{M}^{\bar{\mathcal{T}}}_b$ which has a cofinal branch c such that $L_\Omega[\bar{W}^\infty] \lhd \mathcal{M}^{\bar{\mathcal{W}}}_c$. I.e., $\mathcal{T}_0 = \bar{\mathcal{T}}^\frown b^\frown \bar{\mathcal{W}}^\frown c$ is a tree on $\bar{W}||\bar{\gamma}$ such that $L_\Omega[\bar{W}^\infty] \lhd \mathcal{M}^{\mathcal{T}_0}_\infty$. On the other hand, we have that $\mathcal{M}^{\bar{\mathcal{R}}}_\infty \lhd \mathcal{M}^{\bar{\mathcal{S}}}_\infty$, where $\bar{\mathcal{R}}$ is a tree on $\bar{W}||\bar{\gamma}$ and $\bar{\mathcal{S}}$ is a tree on $\bar{W}^\infty$. The iteration map $\pi^{\bar{\mathcal{R}}}_{0\infty}$ then embeds $\bar{W}||\bar{\gamma}$ into $\mathcal{M}^{\bar{\mathcal{R}}}_\infty$, where the latter is a strict initial segment of a non–simple iterate of $\bar{W}||\bar{\gamma}$ via $\mathcal{T}_0{}^\frown\bar{\mathcal{S}}$. This contradicts the Dodd–Jensen Lemma! $\square$

A simplified version of the preceding argument yields the following folklore result.

Lemma 3.13. *Forcing cannot add an inner model with a Woodin cardinal, i.e., if V does not have an inner model with a Woodin cardinal, then no generic extension of V does.*

Proof. Let α be arbitrary, and let g be $\mathrm{Col}(\omega, \alpha)$–generic over V. Suppose that $V[g]$ has an inner model with a Woodin cardinal, δ. Let $\mathcal{M}_\infty$ be the joint pseudo–comparison of all short tree iterable fine structural inner models of $V[g]$ which have a Woodin cardinal $\leq \delta$. $\mathcal{M}_\infty$ has a Woodin cardinal $\eta \leq \delta^+$ and $\mathcal{M}_\infty||\eta \in V$, again, because this is ordinal definable in V$[g]$ and $\mathrm{Col}(\omega, \alpha)$ is almost homogeneous. So V also has an inner model with a Woodin cardinal. $\square$

The following will not be used explicitly in this paper, and we state it without proof. (For a proof see [**?**].) It helps understanding the hypothesis of Theorem 3.18, though.

Theorem 3.14. (Woodin) *Suppose that there is an inner model with a Woodin cardinal. Then the following are equivalent.*

(1) There is a fully iterable inner model with a Woodin cardinal.

(2) V is closed under $X \mapsto X^{\#}$.

Minimality in the sense of Definition 3.9 is the best substitute for full iterability in the absence of a fully iterable inner model with a Woodin cardinal:

Lemma 3.15. *Let W be a fully iterable fine structural 1–small inner model with a Woodin cardinal. Then W is minimal.*

3.2 The Mantle of $L[x]$ in the presence of an inner model with a Woodin cardinal

The assumption that the universe is constructible from a set simplifies set-theoretic geology a lot (see Theorem 2.4). We shall assume in this subsection, in addition, that there is an inner model with a Woodin cardinal. The main result of this section is that in this situation, the mantle $\mathbb{M}$ is a fine-structural inner model, see Theorem 3.18. This shows that this is a special situation indeed, since one of the main results of [FHR15] is that in general, the mantle of a model of set theory can be anything (quite literally: every model of set theory has a class forcing extension whose mantle is the original model).

Definition 3.16. If W is an fine structural inner model with a measurable cardinal, then let W^α denote the α^{th} iterate of W which is produced by hitting the least measure of W and its images α times.

Theorem 3.17. *Suppose* $\mathrm{V} = L[x]$*, where x is a set, and assume that there is an inner model with a Woodin cardinal. Let W be any fine structural inner model with a Woodin cardinal which is (normally) iterable with respect to short trees. Then*

$$\mathbb{M} \subseteq \bigcap_{\alpha<\infty} W^\alpha.$$

Proof. Set $\kappa_\alpha = \kappa^{W^\alpha}$ and $\delta_\alpha = \delta^{W^\alpha}$; see Definition 3.5. To see that $\mathbb{M} \subseteq \bigcap_{\alpha<\infty} W^\alpha$, it suffices to show:

$(*)$ *For every α, there is a non-dropping normal iteration tree $\mathcal{T}_\alpha$ on W^α so that $N_\alpha := L[\mathcal{M}(\mathcal{T}_\alpha)]$ is a ground.*

For if we know this, then by definition,

$$\mathbb{M} \subseteq \bigcap_{\alpha<\infty} N_\alpha \subseteq \bigcap_{\alpha<\infty} W^\alpha;$$

for the second inclusion, let $a \in \bigcap_{\alpha<\infty} N_\alpha$. Pick α large enough so that $\mathrm{rnk}(a) < \kappa_\alpha$. Then $a \in N_\alpha||\kappa_\alpha = W^\alpha||\kappa_\alpha$, since the iteration from W^α to N_α is beyond κ_α, and so, $a \in \bigcup_{\beta<\infty} W^\beta||\kappa_\beta = \bigcap_{\beta<\infty} W^\beta$.

The proof of $(*)$ is a routine application of the extender algebra, cf. e.g. [Ste09]. Let α be given. We may assume x to be a set of ordinals. Let ξ be the strict supremum of x. We may also assume that $\kappa_\alpha \geq \xi$, for otherwise, we can start the iteration of W^α which is going to produce N_α by hitting κ_α and its images sufficiently many times. In particular, $\delta_\alpha > \xi$. Now $\mathcal{T}_\alpha$ will be a non–dropping genericity iteration of W^α to make x generic. At each successor stage $\beta+1$, $E_\beta^{\mathcal{T}_\alpha}$ is equal to $E_\nu^{\mathcal{M}_\beta^{\mathcal{T}_\alpha}}$, where ν is minimal such that x does not satisfy one of the axioms of the extender algebra corresponding to $E_\nu^{\mathcal{M}_\beta^{\mathcal{T}_\alpha}}$. At each limit stage λ of this iteration, if $\mathcal{T}_\alpha \restriction \lambda$ is short, then there will be a unique cofinal well-founded branch, by our hypothesis on W. If $\mathcal{T}_\alpha \restriction \lambda$ is not short, then λ will be the length of the iteration $\mathcal{T}_\alpha$.

There will be some $\gamma \leq (\delta^{W^\alpha})^+$ such that either x satisfies all the axioms corresponding to all extenders from $\mathcal{M}_\gamma^{\mathcal{T}_\alpha}$, or else γ, the length of $\mathcal{T}_\alpha$, will be a limit ordinal and $\mathcal{T}_\alpha$ is not short. Set $N_\alpha = \mathcal{M}_\gamma^{\mathcal{T}_\gamma}$ in the first case and $N_\alpha := L[\mathcal{M}(\mathcal{T}_\alpha)]$ in the second. Then x must be generic over N_α. Since V is constructible from x, it follows that N_α is a ground of V. □

We remark that if $\mathcal{T}_\alpha$ is non–trivial, then V knows N_α but V *doesn't* know an elementary embedding from V to N_α, since there is never an elementary embedding of a model to a nontrivial ground model, see [**?**].

Now let's turn to the opposite direction. We would like to point out that under the assumptions of the current section, while there is an inner model with a Woodin cardinal, there is no fully iterable one, since the universe, being constructible from a set, cannot be closed under sharps; see Theorem 3.14. But there is a minimal one – see Lemma 3.10 and the following discussion.

Theorem 3.18. *Suppose $V = L[x]$, where x is a set, and assume that there is an inner model with a Woodin cardinal. Let W be a minimal fine structural inner model. Then*

$$\mathbb{M} = \bigcap_{\alpha<\infty} W^\alpha = \bigcup_{\alpha<\infty} W^\alpha||\kappa^{W^\alpha}.$$

Proof. By Theorem 3.17, we only need to verify that $\bigcup_{\alpha<\infty} W^\alpha||\kappa^{W^\alpha} \subseteq \mathbb{M}$. To this end, let W^* be a ground, and let $\alpha < \infty$. We need to show that $W^\alpha||\kappa^{W^\alpha} \subset W^*$.

As W^* is a ground of V, Lemma 3.13 readily implies the following.

(∗∗) *W^* has an inner model with a Woodin cardinal.*

By (∗∗), W^* will thus have a minimal fine structural inner model, W', say, by Lemma 3.10. W' will still be minimal in V by Lemma 3.12. As $(W')^{\kappa^{W^\alpha}}$ is then also minimal in V, $(W')^{\kappa^{W^\alpha}}||\kappa^{W^\alpha} = W^\alpha||\kappa^{W^\alpha}$ by Lemma 3.11. But of course $(W')^{\kappa^{W^\alpha}}||\kappa^{W^\alpha} \in W^*$. □

3.3 Grounds and cores of an $L[E]$ model

We are now going to analyze the mantle of a fine structural inner model.

For this subsection, we shall make the following assumption:

Assumption 3.19. $M = L[E]$ is an extender model such that

(A1) M is tame.

(A2) M does not have a strong cardinal.

(A3) Inside M, M is *not* fully iterable as guided by $\mathcal{P}$–constructions.

(A4) M is fully iterable in V, say via the iteration strategy Σ.

We will explain (A3) in what follows, cf. Definition 3.24 on p. 18. (A3) is an apparent strengthening of the hypothesis that M does not know how to iterate itself, and it implies that M has an inner model with a Woodin cardinal which is "full" at the Woodin cardinal with respect to mice which exist (and can be certified via $\mathcal{P}$–constructions) inside M.

We should point out that we don't know if (A2) is necessary or not, it may well be that the presence of a strong cardinal in $L[E]$ changes the picture substantially. As a test question, one might try to analyze the mantle of the least $L[E]$ which has a Woodin cardinal strictly above a strong cardinal; we don't know how to do that.

We shall use methods and some notation from [SS09]. For details concerning Definition 3.20, see [SS09], and for the proof of Lemma 3.21, see [SS09, Lemma 1.3].

Definition 3.20. ([SS09, Lemma 1.3]) If $\mathcal{M}$ is a premouse and δ is a Woodin cardinal of $\mathcal{M}$, then we write $\mathbb{P}^{\mathcal{M}|\delta}$ for the "δ generator" version of Woodin's extender algebra.

In this situation, $\mathbb{P}^{\mathcal{M}|\delta} \subseteq \mathcal{M}|\delta$ is definable over $\mathcal{M}|\delta$, and $\mathbb{P}^{\mathcal{M}|\delta}$ has the δ-c.c. in $\mathcal{M}$.

Lemma 3.21. *Let $\mathcal{M}$ be a normally $(\omega, \kappa^+ + 1)$-iterable premouse, and let δ be a Woodin-cardinal of $\mathcal{M}$ such that $\delta < \kappa^+$. Let $A \subseteq \kappa^+$. There is then a normal non-dropping iteration tree $\mathcal{U}$ on $\mathcal{M}$ of length $< \kappa$ and with last model $\mathcal{M}^{\mathcal{U}}_\infty$ such that $A \cap \pi^{\mathcal{U}}_{0\infty}(\delta)$ is $\pi^{\mathcal{U}}_{0\infty}(\mathbb{P}^{\mathcal{M}|\delta})$–generic over $\mathcal{M}^{\mathcal{U}}_\infty$.*

We will employ the "maximal $\mathcal{P}$-construction" introduced in [SS09, pp. 757ff.]. This construction works in a setting where $\mathcal{M}$ is a premouse of height $\gamma > \delta$, δ is a cutpoint of $\mathcal{M}$, $\bar{\mathcal{P}}$ is a premouse with $\bar{\mathcal{P}} \cap \mathrm{On} = \delta + \omega$, δ is a Woodin cardinal in $\bar{\mathcal{P}}$, $\bar{\mathcal{P}}|\delta \subseteq \mathcal{M}|\delta$ is definable over $\mathcal{M}|\delta$, and $\bar{\mathcal{P}}[G] = \mathcal{M}|(\delta+1)$ for some $\bar{\mathcal{P}}$-generic $G \subseteq \mathbb{P}^{\bar{\mathcal{P}}|\delta}$. The maximal $\mathcal{P}$-construction produces a sequence $(\mathcal{P}_i \mid \delta + 1 \leq i \leq \bar{\gamma})$, for some $\bar{\gamma} \leq \gamma$. First, $\mathcal{P}_{\delta+1} = \bar{\mathcal{P}}$. It will be maintained that δ is a Woodin cardinal in the premouse $\mathcal{P}_i$ and that $\mathcal{P}_i[G] = \mathcal{M}|i$. At limit stages λ, $\mathcal{P}_\lambda$ will be the union of the previous stages of the construction, augmented by the restriction of the top extender of $\mathcal{M}||\lambda$, if there is one. Successor stages $\mathcal{P}_{i+1}$ will be defined if $i + 1 \leq \gamma$, δ is Woodin in $\mathcal{P}_i$ with respect to definable subsets of $\mathcal{P}_i$ and the ultimate projectum of $\mathcal{P}_i$ is not less than δ. In that case, $\mathcal{P}_{i+1}$ is the result of constructing one step further (i.e., taking the rudimentary closure of $\mathcal{P}_i$). The construction terminates at stage $i \leq \gamma$ if δ is not definably Woodin over $\mathcal{P}_i$ or $\rho_\omega(\mathcal{P}_i) < \delta$ or $i = \gamma$, and we then set $\bar{\gamma} = i$. The *maximal $\mathcal{P}$–construction* is then the final model $\mathcal{P}_{\bar{\gamma}}$. We write $\mathcal{P}(\mathcal{M}, \bar{\mathcal{P}}, \delta)$ for this model, cf. [SS09, p. 757].

The key idea of this section is that if $\mathcal{P} = \mathcal{P}(L[E], \bar{\mathcal{P}}, \delta)$ is proper class sized for some $\bar{\mathcal{P}} \in L[E] = M$, where δ is a cutpoint of $L[E]$ and for some $G \subseteq \mathbb{P}^{\bar{\mathcal{P}}|\delta}$ which is $\bar{\mathcal{P}}$-generic, $M|(\delta+1) = \bar{\mathcal{P}}[G]$, then $\mathcal{P}$ is a ground of $L[E]$, in fact $\mathcal{P}[G] = L[E]$.

One may relax the definition of $\mathcal{P}(\mathcal{M}, \bar{\mathcal{P}}, \delta)$ and not demand that δ be a cutpoint of $\mathcal{M}$, cf. [SS09, pp. 759f.]. Namely, if δ is not a cutpoint of $\mathcal{M}$, then let $\alpha \geq \delta$ be least such that $E^{\mathcal{M}}_\alpha \neq \emptyset$ and $\kappa = \mathrm{crit}(E^{\mathcal{M}}_\alpha) \leq \delta$. Let $\alpha \leq \zeta \leq \gamma$ be maximal such that $\kappa^{+\mathcal{M}|\alpha} = \kappa^{+\mathcal{M}|\zeta}$. Then $\mathcal{P}(\mathcal{M}, \bar{\mathcal{P}}, \delta) = \mathcal{P}(\mathrm{ult}_n(\mathcal{M}||\zeta; E^{\mathcal{M}}_\alpha), \bar{\mathcal{P}}, \delta)$, where $n < \omega$ is least such that $\rho_{n+1}(\mathcal{M}||\zeta) \leq \kappa$ (if it exists, otherwise $n = 0$).

[SS09, Lemmas 1.5 and 1.6] give important information on $\mathcal{P}(\mathcal{M}, \bar{\mathcal{P}}, \delta)$.

In what follows we shall make use of the "$+\omega$" notation from [SS09, p. 759]: if $\mathcal{R}$ is a sound premouse, then $\mathcal{R} + \omega$ is the premouse which end-extends $\mathcal{R}$ and is obtained from $\mathcal{R}$ by constructing over $\mathcal{R}$ one step further.

Definition 3.22. Let $\mathcal{M}$ be an extender model, and let $\mathcal{T}$ be an iteration tree on $\mathcal{M}^{\mathcal{T}}_0$ (where possibly $\mathcal{M}^{\mathcal{T}}_0 \neq \mathcal{M}$). Then set:

$$\mathcal{P}^{\mathcal{M}}(\mathcal{T}) := \begin{cases} \mathcal{P}(\mathcal{M}, \mathcal{M}(\mathcal{T}) + \omega, \delta(\mathcal{T})) & \text{if } \mathcal{T} \text{ has limit length} \\ & \text{and this is defined,} \\ \mathcal{P}(\mathcal{M}, \mathcal{M}^{\mathcal{T}}_\infty||\delta^{\mathcal{M}^{\mathcal{T}}_\infty} + \omega, \delta^{\mathcal{M}^{\mathcal{T}}_\infty}) & \text{if } \mathcal{T} \text{ has successor length} \\ & \text{and this is defined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

All our iteration trees will be finite stacks of normal iteration trees.

Let W be an extender model which is definable in $M = L[E]$. Let $\mathcal{T} \in L[E]$ be an iteration tree on W which lives strictly below δ^W. We say that $\mathcal{T}$ is *guided by $\mathcal{P}$–constructions in* $L[E]$ provided the following holds true. For every limit ordinal $\lambda < \mathrm{lh}(\mathcal{T})$, there is an iteration tree $\mathcal{U}_\lambda \in L[E]$ on $\mathcal{M}(\mathcal{T} \restriction \lambda)$ of successor length such that for every limit ordinal $\eta < \mathrm{lh}(\mathcal{U}_\lambda)$, $\mathcal{P}^M(\mathcal{U}_\lambda \restriction \eta)$ is well–defined and

$$\mathcal{P}^M(\mathcal{U}_\lambda \restriction \eta) \trianglelefteq \mathcal{M}_\eta^{\mathcal{U}_\lambda} \tag{1}$$

is a $\mathcal{Q}$–structure for $\mathcal{U}_\lambda \restriction \eta$, and there is some $\mathcal{Q}$–structure $\mathcal{Q} \trianglelefteq \mathcal{M}_\lambda^{\mathcal{T}}$ together with some sufficiently elementary embedding $\sigma \colon \mathcal{Q} \to \mathcal{P}^M(\mathcal{U}_\lambda)$ which is to exist inside $M^{\mathrm{Col}(\omega, \mathcal{P}^M(\mathcal{U}_\lambda))}$.

Speaking vaguely, $\mathcal{T}$ is thus guided by $\mathcal{P}$–constructions in $L[E]$ iff the branches which $\mathcal{T}$ picks at limit stages are determined by $\mathcal{Q}$–structures which are in turn pullbacks of $\mathcal{Q}$–structures which have been obtained by maximal $\mathcal{P}$–constructions in $L[E]$. By our hypothesis (A4), if $\mathcal{T}$ is guided by $\mathcal{P}$–constructions in $L[E]$, then $\mathcal{T}$ is in fact according to any iteration strategy for $\mathcal{M}_0^{\mathcal{T}}$.

It is clear how one would canonically find witnesses to show that a given $\mathcal{T}$ is guided by $\mathcal{P}$–constructions in $L[E]$, cf. the construction of $\mathcal{U}$ in [SS09, pp. 763ff.]. For each limit ordinal $\lambda < \mathrm{lh}(\mathcal{T})$ one would start iterating $\mathcal{M}(\mathcal{T} \restriction \lambda) + \omega$ to make an initial segment of E generic over the iterate à la Lemma 3.21; at successor stages of the iteration one would hit the least extender which violates an axiom of the extender algebra with respect to E, and at limit stages one would use the $\mathcal{P}$–construction (1) to find the $\mathcal{Q}$–structure and thus the branch, until by pulling back via some map σ one finds the $\mathcal{Q}$–structure and thus the branch for $\mathcal{T} \restriction \lambda$. More details of such a construction will be presented in the proof of Lemma 3.29 below, cf. p. 20. Of course this recipe need not work out, and it may also be that $\mathcal{U}_\lambda$ may be found by delaying the process of making an initial segment of E generic over the iterate, i.e., that we also hit extenders which don't violate an axiom.

We are now going to work towards identifying the "minimal core" of $L[E]$.

Let W be an extender model which is definable in $L[E]$, and which is fully iterable in V. E.g., $W = L[E]$ by our hypothesis (A4). Let $0 < N \leq \omega$. We call

$$((W^n \colon n < N), (\mathcal{T}^n \colon n + 1 < N))$$

a *W–based sequence of length N* iff the following hold true for all $n + 1 < N$.

1. $W^0 = W$,

2. $\mathcal{T}^n \in L[E]$ is an iteration tree on W^n of limit length which is guided by $\mathcal{P}$–constructions in $L[E]$,

3. $W^{n+1} = \mathcal{P}(L[E], \mathcal{M}(\mathcal{T}^n) + \omega, \delta(\mathcal{T}^n))$ is proper class sized,[3] and

4. $\mathcal{T}^n$ lives strictly below δ^{W^n}.

Lemma 3.23. *Let W be an extender model which is definable in $L[E]$, and which is fully iterable in V. There is then no W–based sequence of length ω.*

Proof. Let $((W^n : n < \omega), (\mathcal{T}^n : n+1 < \omega))$ be a counterexample. Both $M = L[E]$ and W are fully iterable. Let us assume inductively that W^n is fully iterable, which is true for $n = 0$. $\mathcal{T}^n$ is then according to any iteration strategy for W^n, as the iteration strategy Σ for M induces iteration strategies for iterating the $\mathcal{Q}$–structures found by the $\mathcal{P}$–constructions in M above the δs of the respective trees.

We may thus let $b_n \in V$ be the cofinal branch through $\mathcal{T}^n$ given by any iteration strategy for W^n. As $\mathcal{T}^n$ is strictly below δ^{W^n}, $\delta(\mathcal{T}^n)$ is not definably Woodin in $\mathcal{M}^{\mathcal{T}^n}_{b_n}$. Again, the iteration strategy Σ for M induces an iteration strategy for iterating $W^{n+1} = \mathcal{P}^M(\mathcal{T}^n)$ above $\delta(\mathcal{T}^n)$, so that $\mathcal{M}^{\mathcal{T}^n}_{b_n}$ iterates past W^{n+1}, which shows that W^{n+1} is fully iterable and $W^{n+1} <^* W^n$. Here, $<^*$ denotes the prewellordering of mice.

We therefore inductively get that every W^n is fully iterable and $W^{n+1} <^* W^n$ for every $n < \omega$. But there is no infinite descending sequence in the prewellordering of mice. Contradiction! $\square$

Let us now discuss and apply our hypothesis (A3) on M.

Definition 3.24. Let W be a (set or class sized) premouse which is definable in $L[E]$ (from parameters in $L[E]$), and which is fully iterable in V. We say that W is *fully iterable inside M as guided by $\mathcal{P}$–constructions* iff for every tree $\mathcal{T}$ on W of limit length which is guided by $\mathcal{P}$–constructions in $L[E]$ there is an iteration tree $\mathcal{T}^+$ on W of length $\mathrm{lh}(\mathcal{T})+1$, extending $\mathcal{T}$, which is also guided by $\mathcal{P}$–constructions in $L[E]$.

I.e., we may always use $\mathcal{P}$–constructions to find $\mathcal{Q}$–structures and thus branches.

Definition 3.25. Let W be an extender model which is definable in $L[E]$, and which is fully iterable in V. W is called *minimal* (*for* $L[E]$) iff the following hold true.

(1) W has a Woodin cardinal.

(2) There is no W–based sequence of length 2.

[3] W^{n+1} is thus a class sized extender model such that $W^{n+1} \models$ "$\delta(\mathcal{T}^n)$ is a Woodin cardinal."

(3) (a) If $\mathcal{U} \in L[E]$ is an iteration tree on $W|\delta^W$ that is guided by $\mathcal{P}$-constructions in $L[E]$ and $\mathcal{U}$ does not live strictly below δ^W (and is hence of limit length), then $\mathcal{P}^M(\mathcal{U})$ is a proper class, if defined.

(b) If $\mathcal{U} \in L[E]$ is an iteration tree on $W|\delta^W$ of successor length such that $\mathcal{U}$ is guided by $\mathcal{P}$-constructions in $L[E]$, and $[0,\infty]_{\mathcal{T}}$ does not drop, then $\mathcal{P}^M(\mathcal{U})$ is a proper class, if defined.

In particular, if W is minimal, then W is fully iterable inside $L[E]$ with respect to trees which live strictly below δ^W and the relevant iteration strategy is given by $\mathcal{P}$–constructions inside $L[E]$. If W' is an iterate of a minimal W via a tree in $L[E]$ which lives strictly below δ^W, then W' is minimal again.

By Lemma 3.23, it is now an immediate consequence of our hypothesis (A3) that there is an extender model which is minimal for $L[E]$.

Lemma 3.26. *There is a minimal W.*

Proof. Let us start with M itself and an M–based sequence $((W^n : n < N), (\mathcal{T}^n : n+1 < N))$ with $0 < N < \omega$ which may not be extended to an M–based sequence which is strictly longer.

We claim that W^{N-1} is minimal. This is clear if $N > 1$, as the existence of $\mathcal{U}$ and $\mathcal{P}^M(\mathcal{U})$ as in (3) of Definition 3.25 would contradict the choice of $\mathcal{T}^{N-2}$ and the fact that $W^{N-1} = \mathcal{P}(L[E], \mathcal{M}(\mathcal{T}^{N-2}) + \omega, \delta(\mathcal{T}^{N-2}))$ has to be proper class sized, cf. p. 18. Let us then assume that $N = 1$, so that $W^{N-1} = W^0 = W$.

Let $\mathcal{U}$ on W be as in (3) (b) of Definition 3.25. Let Ω be large enough that $\mathcal{P}^M(\mathcal{U}) = \mathcal{P}^{M|\Omega}(\mathcal{U})$. Write $\tilde{W} = \mathcal{M}^{\mathcal{U}}_\infty$. Inside $\tilde{W}^{\mathrm{Col}(\omega,\pi^{\mathcal{U}}_{0\infty}(\Omega))}$, there is a tree S of height ω searching for:

(i) a model $\bar{M}$ together with an elementary embedding $\sigma\colon \bar{M} \to \pi^{\mathcal{U}}_{0\infty}(M|\Omega)$ and

(ii) a model $\mathcal{Q}$ such that either $\delta^{\tilde{W}}$ is not definably Woodin in $\mathcal{Q}$ or $\rho_\omega(\mathcal{Q}) < \delta^{\tilde{W}}$, and $\mathcal{Q}$ is equal to $\mathcal{P}(\bar{M}, \tilde{W}|\delta^{\tilde{W}} + \omega, \delta^{\tilde{W}})$.

S is ill–founded in $V^{\mathrm{Col}(\omega,\pi^{\mathcal{U}}_{0\infty}(\Omega))}$, as witnessed by $\bar{M} = M|\Omega$ and $\sigma = \pi^{\mathcal{U}}_{0\infty}$. S is thus also ill–founded in $\tilde{W}^{\mathrm{Col}(\omega,\pi^{\mathcal{U}}_{0\infty}(\Omega))}$. However, any $\mathcal{Q}$ which is obtained from a branch through S must be iterable-above-$\delta^{\tilde{W}}$ in $V^{\mathrm{Col}(\omega,\pi^{\mathcal{U}}_{0\infty}(\Omega))}$, due to the existence of σ. There can thus be only at most one such $\mathcal{Q}$, so that in fact the unique $\mathcal{Q}$ which is obtained via a branch through S must be in $\tilde{W}$. But this $\mathcal{Q}$ kills the Woodinness of $\delta^{\tilde{W}}$, whereas $\delta^{\tilde{W}}$ is Woodin in $\tilde{W}$. Contradiction!

Let us now assume that $\mathcal{U}$ is as in (3) (a) of Definition 3.25. As $\mathcal{U}$ must be according to any iteration strategy on W, we may let b be the cofinal branch through $\mathcal{U}$ which is given by such a strategy. Let $\mathcal{U}^+$ be the tree of length $\mathrm{lh}(\mathcal{U})+1$ which extends $\mathcal{U}$ by adding the limit model $\mathcal{M}^{\mathcal{U}}_b$ as its last model.

Let us assume that $\mathcal{U}$ is normal. As $\mathcal{U}$ does not live strictly below δ^W, $[0,\infty]_{\mathcal{U}^+}$ cannot drop. Also, $\pi^{\mathcal{U}}_{0b}(\delta^W) = \delta(\mathcal{U})$. We may from now on argue exactly as in the case that $\mathcal{U}$ is as in (3) (b) of Definition 3.25, albeit with $\mathcal{U}^+$ replacing $\mathcal{U}$. If $\mathcal{U}$ is not normal, then we consider the last normal component of $\mathcal{U}$ and argue as before. □

Lemma 3.27. *Let W and W' both be minimal, and set $\kappa = \min\{\kappa^W, \kappa^{W'}\}$. Then $W|\kappa = W'|\kappa$.*

Proof. By the minimality of both W and W', we may successfully coiterate $W|\kappa$ with $W'|\kappa$ inside M. As the latter two models are lower part premice, at most one side can move in the comparison. We aim to show that neither of $W|\kappa$ and $W'|\kappa$ moves, so by symmetry let us assume that $W|\kappa$ moves. Let $\mathcal{T}$ be the resulting iteration tree on $W|\kappa$.

We may extend $\mathcal{T}$ to an iteration tree $\mathcal{T}^*$ of successor length on $W|\kappa$ resulting from the comparison of $W|\kappa$ with $W'|\delta^{W'}$. This comparison will also produce an iteration tree $\mathcal{U}$ on $W'|\delta^{W'}$. As W' is iterable inside $L[E]$ with respect to trees which live strictly below $\delta^{W'}$ and the relevant iteration strategy is given by $\mathcal{P}$–constructions inside $L[E]$, we will get one of the following two options.

(a) $\mathcal{U}$ has successor length, $[0,\infty)_{\mathcal{U}}$ does not drop, $\mathcal{M}^{\mathcal{U}}_\infty \lhd \mathcal{M}^{\mathcal{T}^*}_\infty$, and $\rho_\omega(\mathcal{M}^{\mathcal{T}^*}_\infty) < \mathcal{M}^{\mathcal{U}}_\infty \cap \mathrm{On}$.

(b) $\mathcal{U}$ has limit length, $\mathcal{M}(\mathcal{U}) \lhd \mathcal{M}^{\mathcal{T}^*}_\infty$, and $\rho_\omega(\mathcal{M}^{\mathcal{T}^*}_\infty) < \mathcal{M}(\mathcal{U}) \cap \mathrm{On}$.

It is straightforward to see that in both cases we get a contradiction with clause (3) in Definition 3.25. □

For the notation used in the following theorem, see Definition 3.16.

Definition 3.28. The *minimal core of* $L[E]$ is defined to be

$$\bigcap_{\alpha<\infty} W^\alpha = \bigcup_{\alpha<\infty} W^\alpha|\kappa^{W^\alpha},$$

where W is minimal.

Notice that the minimal core of $L[E]$ exists and by Lemma 3.27 does not depend on the choice of the minimal W.

Lemma 3.29.

$$\mathbb{M}^{L[E]} \subseteq \textit{ the minimal core of } L[E].$$

Proof. Let us fix a minimal W. We will in fact not make any use of hypothesis (2) of Definition 3.25. We shall use the method of [SS09, pp. 763ff.] to perform a genericity iteration on W^α to produce ground models. Let $\alpha < \infty$ be a cutpoint of $L[E]$ which is a regular cardinal in $L[E]$, e.g. $\alpha = \beta^{+L[E]}$, where β is non–measurable in M and there is no strong cardinal in $M|\beta$. (We here use our hypothesis (A2).) We may also assume without loss of generality that $\delta^{W^\alpha} < \alpha^{+L[E]}$.

We are going to describe an iteration tree $\mathcal{T}$ on M^α which will be a member of $L[E]$ and which will be above $\kappa := \kappa^{M^\alpha}$. $\mathcal{T}$ will be a genericity iteration to make an initial segment of E generic over the iterate. When we're done, $\mathcal{P} = \mathcal{P}^M(\mathcal{T})$ will be defined, and it will be a proper class model which is a ground of $L[E]$ that coincides with M^α up to κ. The length of the iteration will be α^+, as calculated in $L[E]$.

Let $\tilde{E}$ code E in some natural way as a class of ordinals.

Successor case. Suppose $\mathcal{T} \restriction (\gamma+1)$ is constructed already. Then choose ν least such that $E_\nu^{\mathcal{M}_\alpha^\mathcal{T}}$ is total on $\mathcal{M}_\alpha^\mathcal{T}$ and violates some axiom of the extender algebra with respect to $\tilde{E} \cap \pi_{0,\alpha}^\mathcal{T}(\delta^M)$. If there is no such ν, then the process terminates: $\alpha + 1 = \mathrm{lh}(\mathcal{T})$.

Limit case. Suppose $\mathcal{T} \restriction \lambda$ is constructed, where λ is a limit ordinal.

Subcase 1. $\mathcal{P}^M(\mathcal{T} \restriction \lambda)$ is a proper class. Then we're done with the construction and set $\mathrm{lh}(\mathcal{T}) = \lambda$.

Subcase 2. $\mathcal{P}^M(\mathcal{T} \restriction \lambda)$ is set-sized.

The case assumption together with [SS09, Lemma 1.6] implies that $\mathcal{P}^M(\mathcal{T} \restriction \lambda)$ is a $\mathcal{Q}$–structure for $\mathcal{M}(\mathcal{T} \restriction \lambda)$, i.e., either $\delta(\mathcal{T} \restriction \lambda)$ is not definably Woodin in $\mathcal{P}^M(\mathcal{T} \restriction \lambda)$ or else $\rho_\omega(\mathcal{P}^M(\mathcal{T} \restriction \lambda)) < \delta(\mathcal{T} \restriction \lambda)$. As $\mathcal{T} \restriction \lambda$ is according to the iteration strategy for W, we may let b be the cofinal branch through $\mathcal{T} \restriction \lambda$ given by that strategy. By hypothesis (3) (a) of Definition 3.25, then, $\mathcal{T}$ must live strictly below δ^{W^α}. This implies that $\mathcal{P}^M(\mathcal{T} \restriction \lambda) \trianglelefteq \mathcal{M}_b^{\mathcal{T}\restriction\lambda}$, and then by absoluteness, $b \in L[E]$. We may thus extend $\mathcal{T} \restriction \lambda)$ to $\mathcal{T} \restriction \lambda + 1)$ by letting $\mathcal{M}_\lambda^\mathcal{T} = \mathcal{M}_b^{\mathcal{T}\restriction\lambda}$ and $[0, \lambda]_\mathcal{T} = b$.

This finishes the construction of $\mathcal{T}$.

By the usual comparison argument, $\mathrm{lh}(\mathcal{T}) \leq \alpha^{+L[E]}$. By [SS09, Lemma 1.6] and hypothesis (3) (b) of Definition 3.25, $\mathcal{T}$ cannot have successor length. This is because otherwise $\alpha < \pi_{0\infty}^\mathcal{T}(\delta^{W^\alpha}) < \alpha^{+L[E]}$. By [SS09, Lemma 1.6 (b)] and hypothesis (3) (b) of Definition 3.25, $\pi_{0\infty}^\mathcal{T}(\delta^{W^\alpha})$ must then be a cutpoint of M, so that in fact $\mathcal{P}^M(\mathcal{T})$ would be a proper class model such that for some generic $G \subseteq \mathbb{P} := \mathbb{P}^{\mathcal{M}_\infty^\mathcal{T}|\pi_{0\infty}^\mathcal{T}(\delta^{W^\alpha})}$, $\mathcal{P}^M(\mathcal{T})[G] = L[E]$. But $\mathbb{P}$ has the $\pi_{0\infty}^\mathcal{T}(\delta^{W^\alpha})$–c.c., so that $\pi_{0\infty}^\mathcal{T}(\delta^{W^\alpha})$ would be a cardinal in $L[E]$. Contradiction!

Therefore, $\mathcal{T}$ must have limit length, and by the argument just given we cannot have that $\mathrm{lh}(\mathcal{T}) < \alpha^{+L[E]}$. Therefore, $\mathrm{lh}(\mathcal{T}) = \alpha^{+L[E]}$ and $\delta(\mathcal{T}) = \alpha^{+L[E]}$. Also, $\mathcal{P}(\mathcal{T})$ is a proper class and for some generic $G \subseteq \mathbb{P}^{\mathcal{M}(\mathcal{T})}$, $\mathcal{P}^M(\mathcal{T})[G] = L[E]$. I.e.,

$\mathcal{P}^M(\mathcal{T})$ is a ground of $L[E]$. But $W^\alpha|\kappa^{W^\alpha} \lhd \mathcal{P}^M(\mathcal{T})$. $\square$

3.4 The mantle of an $L[E]$ model

In order to prove the converse of Lemma 3.29, i.e., that the mantle of $L[E]$ is equal to the minimal core of $L[E]$, we need a different representation of the latter model.

Definition 3.30. We recursively define a sequence $(\mathrm{Lp}_i \colon 1 \leq i \leq \mathrm{On})$ as follows. $\mathrm{Lp}_1 = J_\omega$, $\mathrm{Lp}_\lambda = \bigcup_{i<\lambda} \mathrm{Lp}_i$ for limit ordinals λ, and Lp_{i+1} is the union of all sound premice $\mathcal{M} \rhd \mathrm{Lp}_i$ such that $\rho_\omega(\mathcal{M}) \leq \mathrm{Lp}_i \cap \mathrm{OR}$ and $\mathcal{M}$ is fully iterable inside M as guided by $\mathcal{P}$–constructions. We write $\mathrm{Lp}^{\mathcal{P}}$ for $\mathrm{Lp}_{\mathrm{On}}$ and call it the *maximal lower part model which is certified by $\mathcal{P}$–constructions.*

Lemma 3.31. *The maximal lower part model which is certified by $\mathcal{P}$–constructions is equal to the minimal core.*

Proof. Let K denote the minimal core, and let us write $\mathrm{Lp} = \mathrm{Lp}^{\mathcal{P}}$. As both K and Lp are lower–part models which are fully iterable inside M as guided by $\mathcal{P}$–constructions, it is easy to see that if $K \neq \mathrm{Lp}$, then there is some κ which is a cardinal of both K and Lp such that $K|\kappa^{+K} \lhd \mathrm{Lp}|\kappa^{+\mathrm{Lp}}$. Let $\mathcal{M}$ be least such that $K|\kappa^{+K} \lhd \mathcal{M} \lhd \mathrm{Lp}|\kappa^{+\mathrm{Lp}}$ and $\rho_\omega(\mathcal{M}) \leq \kappa$. $\mathcal{M}$ then iterates past K, and in fact if W is a minimal model with $K|\kappa^{+K} = W|\kappa^{+W}$, then $\mathcal{M}$ iterates past W and we get a contradiction as in the proof of Lemma 3.27. $\square$

In addition to our assumptions 3.19, in this subsection we shall now make the extra

Assumption 3.32. $M = L[E]$ is an extender model such that

(A5) E is OD in M, and there are arbitrarily large cardinal strong cutpoints[4] θ of M such that if G is $\mathrm{Col}(\omega, M|\theta)$–generic over M and $\tilde{E}$ is the sequence of (partial and total) extenders of $M[G]$ which canonically extend extenders from E with critical points above θ to $M[G]$, then the restriction of $\tilde{E}$ to indices greater than $\theta^{+M}$is OD in $M[G]$.

The question when (A5) is satisfied is related to work of Schlutzenberg (see [Sch07]), one focus of which is an analysis of mice that satisfy V = HOD. In private communication, Schlutzenberg pointed out that, in the notation of (A5), if $L[E][G]$, viewed as a G-premouse $L[\tilde{E}](G)$, is internally fully normally iterable above θ (or just internally iterable in the intervals between Woodin cardinals which by (A1) all have to be cutpoints), then the restriction of $\tilde{E}$ to indices greater than θ^{+M} is ordinal definable in $M[G]$.

[4] I.e., θ is a cardinal of M and there is no $E^M_\alpha \neq \emptyset$ with $\alpha > \theta$ and $\mathrm{crit}(E^M_\alpha) \leq \theta$.

Theorem 3.33. *The mantle of $L[E]$ is equal to the minimal core of $L[E]$.*

Proof. In the light of Lemma 3.29, we are left with having to prove that if $\bar{V}$ is a ground of $L[E]$, then every strict initial segment of the minimal core of $L[E]$ is an element of $\bar{V}$. Let us fix $\bar{V}$, let $\mathbb{P} \in \bar{V}$ be a partial order, and let g be $\mathbb{P}$–generic over $\bar{V}$ such that $\bar{V}[g] = L[E]$.

Let $\theta > \mathrm{Card}(\mathbb{P})$ be as in (A5), and let G be $\mathrm{Col}(\omega, M|\theta)$–generic over $L[E]$. Let H be $\mathrm{Col}(\omega, \theta)$–generic over $\bar{V}$ such that $\bar{V}[H] = L[E][G]$.

In the light of Lemma 3.31, it suffices to verify that $\mathrm{Lp}^{\mathcal{P}}$ is definable in $L[E][G] = \bar{V}[H]$, as then every proper initial segment of the minimal core will be an element of $\bar{V}$. In order to show that $\mathrm{Lp}^{\mathcal{P}}$ is definable in $L[E][G]$, we are now going to define a version of Lp inside $L[E][G]$ and verify that this model is equal to $\mathrm{Lp}^{\mathcal{P}}$.

We first define $\mathcal{P}^{M[G]}(\mathcal{U})$ in much the same way as $\mathcal{P}^{M}(\mathcal{U})$ was defined, with a small change in the maximal $\mathcal{P}$-construction.

Let $\mathcal{M} = M[G]|\gamma$, for some $\gamma > \theta^{+M}$, and let $\bar{\mathcal{P}}$ be a premouse with $\bar{\mathcal{P}} \cap \mathrm{On} = \delta + \omega$, δ is a Woodin cardinal in $\bar{\mathcal{P}}$, $\theta^{+M} \leq \delta < \gamma$, $\bar{\mathcal{P}}|\delta \subseteq \mathcal{M}|\delta$ is definable over $\mathcal{M}|\delta$, and

$(*)$ $u = |(\bar{\mathcal{P}}[G^*])| = |\mathcal{M}|(\delta+1)|$,[5] *for some $\bar{\mathcal{P}}$-generic $G^* \subseteq \mathbb{P}^{\bar{\mathcal{P}}|\delta}$.*

Notice that $\mathcal{M}$ may be construed as a $H^{M[G]}_{\theta^{+M}}$-premouse with extender sequence $\tilde{E}|\gamma$, restricted to indices greater than θ^{+M} ($\tilde{E}$ as in (A5)), so $\mathcal{M}|i$ makes sense for all i with $\theta \leq i \leq \gamma$. The modified maximal $\mathcal{P}$-construction produces a sequence $\langle \mathcal{P}'_i \mid \delta + 1 \leq i < \bar{\gamma} \rangle$ of u-premice, for some $\bar{\gamma} \leq \gamma$. First, $\mathcal{P}'_{\delta+1} = \bar{\mathcal{P}}$. It will be maintained that δ is a Woodin cardinal in the premouse $\mathcal{P}'_i$ and that

$(**)$ $\mathcal{P}'_i[G^*] = \mathcal{M}|i$

where G^* is as in $(*)$ and both structures are viewed as u-premice. At limit stages λ, $\mathcal{P}'_\lambda$ will be the union of the previous stages of the construction, augmented by the restriction of the top extender of $\mathcal{M}||\lambda$, if there is one. Successor stages $\mathcal{P}'_{i+1}$ will be defined if $i + 1 \leq \gamma$, δ is Woodin in $\mathcal{P}'_i$ with respect to definable subsets of $\mathcal{P}'_i$ and the ultimate projectum of $\mathcal{P}'_i$ is not less than δ. In that case, $\mathcal{P}'_{i+1}$ is the result of constructing one step further (i.e., taking the rudimentary closure of $\mathcal{P}'_i$). The construction terminates at stage $i \leq \gamma$ if δ is not definably Woodin over $\mathcal{P}'_i$ or $\rho_\omega(\mathcal{P}'_i) < \delta$ or $i = \gamma$, and we then set $\bar{\gamma} = i$. We write $\mathcal{P}'(\mathcal{M}, \bar{\mathcal{P}}, \delta) = \mathcal{P}'_{\bar{\gamma}}$.

Now, for an iteration tree $\mathcal{U} \in M[G]$ on a premouse $\mathcal{M}^{\mathcal{U}}_0$, we define $\mathcal{P}^{M[G]}(\mathcal{U})$ by

[5] Here, $|\mathcal{N}|$ denotes the universe of the the model $\mathcal{N}$.

$$\mathcal{P}^{M[G]}(\mathcal{U}) := \begin{cases} \mathcal{P}(M[G], \mathcal{M}(\mathcal{U})+\omega, \delta(\mathcal{U})) & \text{if } \mathcal{U} \text{ has limit length} \\ & \text{and this is defined,} \\ \mathcal{P}(M[G], \mathcal{M}^{\mathcal{U}}_\infty || \delta^{\mathcal{M}^{\mathcal{U}}_\infty}+\omega, \delta^{\mathcal{M}^{\mathcal{U}}_\infty}) & \text{if } \mathcal{U} \text{ has successor length} \\ & \text{and this is defined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Let $\mathcal{T} \in L[E][G]$ be an iteration tree on some premouse $\mathcal{M}$ which lives strictly below $\delta^{\mathcal{M}}$. We say that $\mathcal{T}$ is *guided by $\mathcal{P}$–constructions in $L[E][G]$* provided the following holds true. For every limit ordinal $\lambda < \mathrm{lh}(\mathcal{T})$, there is an iteration tree $\mathcal{U}_\lambda \in L[E][G]$ on $\mathcal{M}(\mathcal{T} \upharpoonright \lambda)$ of successor length such that for every limit ordinal $\eta < \mathrm{lh}(\mathcal{U}_\lambda)$, $\mathcal{P}^{M[G]}(\mathcal{U}_\lambda \upharpoonright \eta)$ is well–defined and

$$\mathcal{P}^{M[G]}(\mathcal{U}_\lambda \upharpoonright \eta) \trianglelefteq \mathcal{M}^{\mathcal{U}_\lambda}_\eta \tag{2}$$

is a $\mathcal{Q}$–structure for $\mathcal{U}_\lambda \upharpoonright \eta$, and there is some $\mathcal{Q}$–structure $\mathcal{Q} \trianglelefteq \mathcal{M}^{\mathcal{T}}_\lambda$ together with some sufficiently elementary embedding $\sigma\colon \mathcal{Q} \to \mathcal{P}^{M[G]}(\mathcal{U}_\lambda)$ which is to exist inside $M^{\mathrm{Col}(\omega, \mathcal{P}^M(\mathcal{U}_\lambda))}$.

Definition 3.34. Let $\mathcal{M} \in L[E]$ be a premouse. We say that $\mathcal{M}$ is *fully iterable inside $M[G]$ as guided by $\mathcal{P}$–constructions* iff for every tree $\mathcal{T}$ on $\mathcal{M}$ of limit length which is guided by $\mathcal{P}$–constructions in $L[E][G]$ there is an iteration tree $\mathcal{T}^+$ on $\mathcal{M}$ of length $\mathrm{lh}(\mathcal{T})+1$, extending $\mathcal{T}$, which is also guided by $\mathcal{P}$–constructions in $L[E][G]$.

Definition 3.35. We recursively define a sequence $(\mathrm{Lp}'_i\colon 1 \leq i \leq \mathrm{On})$ as follows. $\mathrm{Lp}'_1 = J_\omega$, $\mathrm{Lp}'_\lambda = \bigcup_{i<\lambda} \mathrm{Lp}'_i$ for limit ordinals λ, and Lp'_{i+1} is the union of all sound premice $\mathcal{M} \triangleright \mathrm{Lp}'_i$ such that $\rho_\omega(\mathcal{M}) \leq \mathrm{Lp}'_i \cap \mathrm{OR}$ and $\mathcal{M}$ is fully iterable inside $M[G]$ as guided by $\mathcal{P}$–constructions. We write $(\mathrm{Lp}^{\mathcal{P}})^{M[G]}$ for $\mathrm{Lp}'_{\mathrm{On}}$ and call it the *maximal lower part model which is certified by $\mathcal{P}$–constructions.*

The definition of $(\mathrm{Lp}^{\mathcal{P}})^{M[G]}$ only needs

$$\{H^* \subset \mathrm{Col}(\omega,\theta)\colon \{n\colon (\bigcup H)(n) \neq (\bigcup H^*)(n)\} \text{ is finite }\}$$

(rather than H itself or G) as a parameter. This is because for any such H^*, $H^{M[G]}_{\theta^{+M}} = (H_{\theta^+})^{\bar{V}[H^*]}$, which is the crucial parameter in the definition of the maximal $\mathcal{P}$-constructions. We therefore have that $(\mathrm{Lp}^{\mathcal{P}})^{M[G]} \subseteq \bar{V}$.

It thus remains to verify the following.

Lemma 3.36. $(\mathrm{Lp}^{\mathcal{P}})^{M[G]} = \mathrm{Lp}^{\mathcal{P}}$.

Proof. Let us write $\mathrm{Lp}^G = (\mathrm{Lp}^{\mathcal{P}})^{M[G]}$ and $\mathrm{Lp} = \mathrm{Lp}^{\mathcal{P}}$. As both Lp^G and Lp are lower–part models, it is easy to see that if $\mathrm{Lp}^G \neq \mathrm{Lp}$, then there is some largest κ which is a cardinal of both Lp^G and Lp such that $\mathrm{Lp}^G|\kappa = \mathrm{Lp}|\kappa$.

Claim 1. Let $\mathrm{Lp}^G|\kappa \lhd \mathcal{M} \lhd \mathrm{Lp}^G$ be such that $\rho_\omega(\mathcal{M}) \leq \kappa$. Then $\mathcal{M} \lhd \mathrm{Lp}$.

Proof. Notice that $\mathcal{M}$ is OD in $M[G]$ and hence $\mathcal{M} \in \bar{V} \subseteq M$. We need to verify that $\mathcal{M}$ is fully iterable inside M as guided by $\mathcal{P}$–constructions. Otherwise there is a counterexample tree $\mathcal{T} \in M$, so that there is a tree $\mathcal{U}$ on $\mathcal{M}(\mathcal{T}) + \omega$ such that $\mathcal{P}^M(\mathcal{U})$ is class sized, $\delta(\mathcal{U})$ is Woodin in $\mathcal{P}^M(\mathcal{U})$, and for some k which is generic for the extender algebra, $\mathcal{P}^M(\mathcal{U})[k] = L[E]$. In particular, $\delta(\mathcal{U})$ is a regular cardinal in $L[E]$.

But by hypothesis, $L[E][G]$ can find the $\mathcal{Q}$–structure $\mathcal{Q}$ for $\mathcal{M}(\mathcal{U})$ which is in fact an initial segment of the limit model, call it $\mathcal{M}_\infty^{\mathcal{T}^\frown\mathcal{U}}$, of the iteration tree $\mathcal{T}^\frown\mathcal{U}$ on $\mathcal{M}$. We must have $\rho(\mathcal{M}_\infty^{\mathcal{T}^\frown\mathcal{U}}) < \delta(\mathcal{U})$ and hence $\mathrm{cf}(\delta(\mathcal{U})) < \delta(\mathcal{U})$ in $L[E][G]$. Contradiction!

Claim 2. Let $\mathrm{Lp}|\kappa \lhd \mathcal{M} \lhd \mathrm{Lp}$ be such that $\rho_\omega(\mathcal{M}) \leq \kappa$. Then $\mathcal{M} \lhd \mathrm{Lp}^G$.

Proof. This is similar to the proof of Lemma 3.12. Assume that $\mathcal{M}$ is not an initial segment of Lp^G. There is then a tree $\mathcal{T} \in L[E][G]$ on $\mathcal{M}$ such that we may not find a $\mathcal{Q}$–structure for $\mathcal{M}(\mathcal{T})$ by pulling back $\mathcal{Q}$–structures from $\mathcal{P}$–constructions over iterates of $\mathcal{M}(\mathcal{T})$.

Let $\mathcal{F}$ be the family of all such $\mathcal{T}$ of minimal length, and let $\mathcal{R}$ the result of pseudo–comparing all elements of $\mathcal{F}$. We can't find a $\mathcal{Q}$–structure for $\mathcal{R}$ by pulling back $\mathcal{Q}$–structures from $\mathcal{P}$–constructions over iterates of $\mathcal{R}$, as otherwise we might pull back such a $\mathcal{Q}$–structure to get a $\mathcal{Q}$–structure for $\mathcal{T}$.

Obviously, $\mathcal{R} \in L[E]$, as it has been defined in $L[E][G]$ just from the parameter $\mathcal{M} \in \bar{V} \subseteq M$ (and an ordinal). We may now pseudo–coiterate $\mathcal{M}$ with $\mathcal{R}$ inside $L[E]$. As $\mathcal{M} \lhd \mathrm{Lp}$, i.e., $\mathcal{M}$ is fully iterable inside M as guided by $\mathcal{P}$–constructions, this procedure will finally produce a $\mathcal{Q}$–structure for $\mathcal{R}$. Contradiction! $\square$

4 The Solid Core

4.1 Basics on solid sets and the Solid Core

Definition 4.1. A set x is *solid* if for every set a of ordinals, whenever there is a poset $\mathbb{P} \in L[a]$ and a filter G which is $\mathbb{P}$-generic over $L[a]$ such that $x \in L[a][G]$, then $x \in L[a]$.

So a set x is solid if it can't be added by forcing over an inner model of ZFC. The definition above is a first order version of this.

Lemma 4.2. *If x is solid in a forcing extension, then it is solid in the ground model. In fact, if $x \in W \subseteq \mathrm{V}$, where W is an inner model of* V *and x is solid in* V*, then x is solid in W.*

Proof. If x could be added by forcing over an inner model of W, then this inner model would also be an inner model of V, so that x wouldn't be solid in V. $\square$

Lemma 4.3. *Every solid set belongs to the mantle.*

Proof. If x is solid and W is a ground, then $x \in W$, by the definition of solidity. W was an arbitrary ground, so this shows that $x \in \mathbb{M}$. $\square$

Lemma 4.4. *If x is solid, then $x \in \mathsf{HOD}$.*

Proof. Since x is generic over HOD (by the Vopěnka algebra), it follows that $x \in \mathsf{HOD}$, because x is solid. $\square$

Question 4.5. Is the statement "x is solid" forcing absolute? I.e., in view of Lemma 4.2: Is "x is solid" necessary if true?

We will show in [FS] that the answer is no, in general, by showing that it is consistent that $M = K|\theta$ is solid, but in a set-forcing extension W, θ may be ω_1, and M may be added to an inner model of W by adding a Cohen real.

Observation 4.6. *If $\mathsf{MP}(\mathbb{R})$ holds, then "x is solid" is forcing absolute, for $x \in \mathbb{R}$.*[6]

Proof. Being solid passes down to grounds, as noted above. So suppose $x \in \mathbb{R}$ is solid, but not solid in some forcing extension. x will then fail to be solid in any further forcing extensions. By $\mathsf{MP}(\mathbb{R})$, x is not solid, a contradiction. $\square$

Definition 4.7. A set x is *generically solid* if every poset forces that $\check{x}$ is solid.

Lemma 4.8. *The statement that x is generically solid is forcing-absolute. In fact, generic solidity is downward absolute to arbitrary inner models.*

Proof. Generic solidity is clearly absolute to forcing extensions: If x is generically solid in V, and $\mathrm{V}[g]$ is a forcing extension of V, then x is solid in $\mathrm{V}[g]$, and since every forcing extension of $\mathrm{V}[g]$ is a forcing extension of V, x is solid in every forcing extension of $\mathrm{V}[g]$, which means that x is generically solid in $\mathrm{V}[g]$.

Vice versa: we show downward absoluteness to arbitrary inner models instead of to just ground models. So suppose x is generically solid in V, and suppose $W \subseteq \mathrm{V}$ is an inner model with $x \in W$. Suppose x was not generically solid in W.

[6] $\mathsf{MP}(\mathbb{R})$, the maximality principle with real parameters, is the scheme expressing that every statement about a real number that can be forced to be true in such a way that it stays true in any further forcing extension is already true. See [Ham03].

Let $h \subseteq \mathbb{Q} \in W$ be such that x is not solid in $W[h]$. Let $q \in \mathbb{Q}$ be a condition which forces over W that $\check{x}$ is not solid. Let $q \in h' \subseteq \mathbb{Q}$ be generic over V. In $W[h']$, there is an a and an $L[a]$-generic i such that $x \in L[a][i] \setminus L[a]$. So a and i also exist in $\mathrm{V}[h']$, showing that x is not solid in $\mathrm{V}[h']$. But $\mathrm{V}[h']$ is a set-forcing extension of V, where x is generically solid. This is a contradiction. □

Lemma 4.9. *Every generically solid set belongs to* gHOD *and* $\mathrm{g}\mathbb{M}$.

Proof. If x is generically solid, then x is solid in every generic extension $\mathrm{V}[G]$, so by what has been shown so far, x belongs to $\mathbb{M}^{\mathrm{V}[G]}$ and to $\mathsf{HOD}^{\mathrm{V}[G]}$. □

So the generically solid sets are canonically well-ordered. By listing all of those which are sets of ordinals, in that order, say by A, we get an inner model of ZFC, $L[A]$, which is forcing invariant, as gHOD and hence the ordering is. Moreover, $L[A] \subseteq \mathsf{gHOD}$, since $A \cap \alpha$ is OD in every forcing extension, for every α.

Let us fix a natural way to form a sum of two sets x and y of ordinals, $x \oplus y$, which is again a set of ordinals, from which both x and y can be defined in an absolute way, and which itself is also absolutely definable from x and y.

Definition 4.10. For sets of ordinals x and y, set

$$x \oplus y = \{\prec\alpha, 0\succ \mid \alpha \in x\} \cup \{\prec\beta, 1\succ \mid \beta \in y\}.$$

Observation 4.11. *If x and y are solid sets of ordinals, then so is $x \oplus y$.*

Proof. Let $x \oplus y \in L[a][G]$, where a and G are as in Definition 4.1. Since x is solid, it follows that $x \in L[a]$, as $x \in L[a][G]$, being definable from $x \oplus y$ there. For the same reason, $y \in L[a]$. But then $x \oplus y \in L[a]$ as well. □

Definition 4.12. The solid core is the class

$$\mathfrak{C} = \bigcup_{x \text{ solid}, x \subseteq \mathrm{On}} L[x].$$

The reason for restricting to solid sets of ordinals in the definition of the solid core is that we want to insure that the resulting model will satisfy the axiom of choice. Alternatively, one could have defined the solid core to be the union of all "self-well-ordered" solid sets (where a set x is self-well-ordered if $L(x)$ satisfies the axiom of choice). In this way, even though we do not know in general whether the solid core satisfies ZF, it automatically satisfies the axiom of choice. Note that it is not a good idea to consider the class of all solid sets in the hope of arriving at a canonical model, because the solid sets are not transitive. For example, $L_{\omega_1}[0^\#]$ is solid (and self-well-ordered), but contains real numbers that are Cohen-generic over L, and hence not solid.

Theorem 4.13. *$\mathfrak{C}$ is a definable, transitive class containing all the ordinals, closed under the Gödel operations.*

Proof. To see that $\mathfrak{C}$ is closed under the Gödel operations, let $x_0, x_1, x_2 \in \mathfrak{C}$. Then there are solid sets of ordinals y_0, y_1, y_2 such that $x_i \in L[y_i]$, for $i < 3$. By Observation 4.11, $z := (y_0 \oplus y_1) \oplus y_2$ is a solid set of ordinals, and $x_i \in L[z]$, for all $i < 3$. Since $L[z]$ is closed under the Gödel operations, the desired image of (x_0, x_1, x_2) is in $L[z] \subseteq \mathfrak{C}$. □

Question 4.14. Is $\mathfrak{C}$ almost universal? I.e., if $x \subseteq \mathfrak{C}$, then is there a $y \in \mathfrak{C}$ with $x \subseteq y$?

Definition 4.15. The solid sets of ordinals are *upward set-directed* if for every set Z of solid sets of ordinals, there is a solid set of ordinals z such that $Z \subseteq L[z]$.

Lemma 4.16. *If the solid sets of ordinals are upward set-directed then $\mathfrak{C}$ is a model of* ZFC.

Proof. To see that $\mathfrak{C}$ is a model of ZF, it suffices to show that given α, $v := V_\alpha \cap \mathfrak{C} \in \mathfrak{C}$, for this implies that $\mathfrak{C}$ is almost universal. For $a \in v$, pick a solid set of ordinals x_a such that $a \in V_\alpha \cap L[x_a]$. By upward set-directedness of the solid sets of ordinals, there is a solid set of ordinals y such that $L[x_a] \subseteq L[y]$, for all $a \in v$. But then $v = V_\alpha \cap L[y] = V_\alpha^{L[y]} \in L[y] \subseteq \mathfrak{C}$, which shows almost universality. That $\mathfrak{C}$ satisfies the axiom of choice is clear, since it is a union of models of choice. □

Lemma 4.17. *Assuming $\mathfrak{C}$ is a model of* ZFC, *it follows that $\mathfrak{C}^{\mathfrak{C}} = \mathfrak{C}$.*

Proof. Of course, the left hand side is contained in the right hand side. For the converse, suppose $a \in \mathfrak{C}$. Pick a solid x such that $a \in L[x]$. Clearly, $x \in \mathfrak{C}$. Solidity is downward absolute, so x is solid in $\mathfrak{C}$, and hence, $a \in \mathfrak{C}^{\mathfrak{C}}$. □

Lemma 4.18. *Assuming $\mathfrak{C}$ is a model of* ZFC, *$\mathfrak{C}$ satisfies the Ground Axiom.*

Proof. $\mathfrak{C} = \mathfrak{C}^{\mathfrak{C}} \subseteq \mathbb{M}^{\mathfrak{C}} \subseteq \mathfrak{C}$. So $\mathbb{M}^{\mathfrak{C}} = \mathfrak{C}$. □

Definition 4.19.

$$\mathrm{g}\mathfrak{C} = \bigcup_{x \subseteq \mathrm{On} \text{ generically solid}} L[x].$$

Lemma 4.20. *Assuming $\mathrm{g}\mathfrak{C}$ is a model of* ZFC, *it follows that*

$$\mathrm{g}\mathfrak{C}^{\mathrm{g}\mathfrak{C}} = \mathrm{g}\mathfrak{C}.$$

Proof. As before, using the fact that generic solidity is downward absolute to inner models. □

4.2 Computing the Solid Core

Theorem 4.21. *Suppose there is an inner model with a Woodin cardinal. Let W be a minimal such model, "minimal" in the sense of Definition 3.9. Then $\mathfrak{C} = \bigcap_\alpha W^\alpha$, using the notation introduced in Definition 3.16.*

Proof. For the inclusion from left to right, we can argue as in Theorem 3.17: Any solid set a (of ordinals) can be made generic over an iterate of W^α, the genericity iteration being above $\sup(a)$. By solidity of a then, a must belong to the iterate of W^α, and since the genericity iteration was above $\sup(a)$, it follows that $a \in W^\alpha$. So any solid set belongs to $\bigcap_\alpha W^\alpha$. As before, it is really a genericity pseudo iteration, where we don't have an embedding into the last model.

For the converse, we want to show that $W^\alpha||\kappa^{W^\alpha}$ is solid, for any α. To see this, fix α, and suppose M is an inner model in which there is a forcing notion $\mathbb{P}$, such that there is a $G \in \mathrm{V}$ which is M-generic for $\mathbb{P}$, and such that $W^\alpha||\kappa^{W^\alpha} \in M[G]$. We have to show that $W^\alpha||\kappa^{W^\alpha} \in M$.

Case 1: M has an inner model with a Woodin cardinal.

In this case, in M, let $\bar{W}$ be a minimal inner model with a Woodin cardinal. Coiterate $\bar{W}$ against $W^\alpha||\kappa^{W^\alpha}$ in $M[G]$. Get that $\bar{W}^\alpha||\kappa^{\bar{W}||\kappa_\alpha} = W^\alpha||\kappa^{W^\alpha}$, showing that $W^\alpha||\kappa^{W^\alpha} \in M$. (Cf. Lemmas 3.11 and 3.12.)

Case 2: M has no inner model with a Woodin cardinal.

Then K^M exists, and $K^{M[G]} = K^M$ is also the core model of $M[G]$. Inside $M[G]$, coiterate $K^M[G]$ against $W^\alpha||\kappa^{W^\alpha}$, producig iteration trees $\mathcal{T}$ on $K^M[G]$ and $\mathcal{U}$ on $W^\alpha||\kappa^{W^\alpha}$. Note that by absoluteness, $W^\alpha||\kappa^{W^\alpha}$ is iterable also in $M[G] \subseteq \mathrm{V}$. As $K^{M[G]}$ is universal, $K^{M[G]}$ wins the coiteration.

$W^\alpha||\kappa^{W^\alpha}$ is a lower–part model, i.e., it doesn't have any total extenders on its sequence, so that as $K^{M[G]}$ wins $\mathcal{U}$ must be trivial in the sense that $W^\alpha||\kappa^{W^\alpha}$ doesn't move in the comparison. But then $\mathcal{T}$ must be linear, and it has to look like this: For α_0 many steps, iterate E_{ξ_0} and its images. Then switch to a new index ξ_1 greater than the image of ξ_0, and iterate that and its images for α_1 many times. When switching to the "new" extender, there is a drop. Continue like this. Only finitely many switches can happen, because every time you switch, there is a drop. If $\mathcal{T}$ were to look different, then it would leave a total measure behind and $W^\alpha||\kappa^{W^\alpha}$ would end up with a total measure on its sequence.

The iteration $\mathcal{T}$ of $K^{M[G]}$ is thus determined by the ordinals $\vec{\alpha}$ and $\vec{\beta}$. So $\mathcal{T} \in M$, and $W^\alpha||\kappa^{W^\alpha}$ is an initial segment of the last model of the iteration. So in this case, too, $W^\alpha||\kappa^{W^\alpha} \in M$. $\square$

Let us now explore the solid core in the absence of an inner model with a Woodin cardinal. We will show in [FS] that in that case, $K \subseteq \mathfrak{C}$. However, the opposite inclusion does not hold in general in this situation, as the following theorem shows.

Theorem 4.22. *It is consistent that there is no inner model with a Woodin cardinal and $K \neq \mathfrak{C}$. In fact, this is true in R. David's class forcing extension $L[r]$ of L: there, $0^{\#}$ does not exist, $r \notin L$, and r is generically solid. So $L[r]$ is its own generic solid core (and solid core, and mantle, and generic mantle, and generic* HOD*, and* HOD*).*

Proof. R. David proved in [Dav82, Theorem 1] that there is a class forcing extension $L[r]$ of L, where $r \subseteq \omega$, such that $r \notin L$, $0^{\#} \notin L[r]$, and r is a set-forcing-absolute Π^1_2-singleton, in the strong sense that there is a Π^1_2-formula φ such that in every set-forcing extension of $L[r]$, r is unique with $\varphi(r)$. That is, any set-forcing extension of $L[r]$ satisfies $\forall s \quad (\varphi(s) \iff s = r)$. These properties, taken together, imply that r is generically solid in $L[r]$. To see this, let g be set-generic over $L[r]$, let $W \subseteq L[r][g]$ be an inner model, and let $h \in L[r][g]$ be generic for some forcing in W, and assume that $r \in W[h]$. We have to show that $r \in W$.

Note that $L[r] \subseteq W[h] \subseteq L[r][g]$, so there is an $W[h]$-generic filter i such that $W[h][i] = L[r][g]$, since $W[h]$ is intermediate in between a model of ZFC and its set-forcing extension. Let α be a cardinal at least as large as the cardinality of the forcings for which g and h are generic. Let G be $\mathrm{Col}(\omega, \alpha)$-generic over $L[r][g]$. By the absorption property of the collapse, there is a W-generic filter $H \subseteq \mathrm{Col}(\omega, \alpha)$ such that

$$L[r][g][G] = W[h][i][G] = W[H]$$

Since $W[H] = L[r][g][G]$ is a forcing extension of $L[r]$, r is a Π^1_2-singleton in $W[H]$. So by the homogeneity of the collapse, it follows that $r \in W$. □

References

[Dav82] René David. A very absolute Π^1_2 real singleton. *Annals of Mathematical Logic*, 23:101–120, 1982.

[FHR15] Gunter Fuchs, Joel David Hamkins, and Jonas Reitz. Set-theoretic geology. *Annals of Pure and Applied Logic*, 166(4):464–501, 2015.

[FS] Gunter Fuchs and Ralf Schindler. The solidity and nonsolidity of initial segments of the core model. *In preparation.*

[Fuc08] Gunter Fuchs. Closed maximality principles: implications, separations and combinations. *Journal of Symbolic Logic*, 73(1):276–308, 2008.

[Ham03] Joel David Hamkins. A simple maximality principle. *Journal of Symbolic Logic*, 68(2):527–550, June 2003.

[Lav07] Richard Laver. Certain very large cardinals are not created in small forcing extensions. *Annals of Pure and Applied Logic*, 149:1–6, 2007.

[MS94] William J. Mitchell and John R. Steel. *Fine Structure and Iteration Trees*. Lecture Notes in Logic **3**. Springer, 1994.

[MS04] William Mitchell and Ralf Schindler. A universal extender model without large cardinals in V. *Journal of Symbolic Logic*, 69(2):371–386, 2004.

[Sch07] Farmer Schlutzenberg. *Measures in mice*. PhD thesis, University of Berkeley, 2007.

[SS09] Ralf Schindler and John Steel. The self-iterability of $L[E]$. *Journal of Symbolic Logic*, 74(3):751–779, 2009.

[Ste96] John R. Steel. *The Core Model Iterability Problem*. Lecture Notes in Logic **8**. Springer, Berlin, 1996.

[Ste09] John R. Steel. An outline of inner model theory. In Matthew Foreman, Akihiro Kanamori, and Menachem Magidor, editors, *Handbook of Set Theory*. Springer, 2009.

[WDR12] W. H. Woodin, J. Davis, and D. Rodriguez. The HOD dichotomy. *Notes of the Apalachian Set Theory meeting 2012 at Cornell, unpublished, available at http://www.math.cmu.edu/ eschimme/Appalachian/WoodinDavisRodriguez.pdf*, pages 1–19, 2012.

[Woo04] W. Hugh Woodin. The continuum hypothesis, the generic-multiverse of sets, and the Ω conjecture. In *Proceedings of the Conference on the Continuum in Philosophy and Mathematics*, 2004.