Martin's Maximum^{++} implies Woodin's Axiom $(*)$

David Asperó[*] and Ralf Schindler[†]

March 12, 2020

Abstract

We show that Martin's Maximum^{++} implies Woodin's $\mathbb{P}_{\max}$ axiom $(*)$. This answers a question from the 1990's and amalgamates two prominent axioms of set theory which were both known to imply that there are $\aleph_2$ many real numbers.

1 Introduction.

Cantor's Continuum Problem, which later became Hilbert's first Problem (see [15]), asks how many real numbers there are. After having proved his celebrated theorem according to which $\mathbb{R}$ is uncountable, i.e., $2^{\aleph_0} > \aleph_0$, see [3], Cantor conjectured that every uncountable set of reals has the same size as $\mathbb{R}$, i.e., $2^{\aleph_0} = \aleph_1$. This statement is known as Cantor's Continuum Hypothesis (CH). Gödel [11] proved in the 1930's that CH is consistent with the standard axiom system for set theory, ZFC, by showing that CH holds in his constructible universe L, the minimal transitive model of ZFC containing all the ordinals. The axiom $V = L$, saying that the universe V of all sets is simply identical with L, has often been rejected, however, as an undesirable minimalistic assumption about V. For instance, L cannot have measurable cardinals by a result of Scott [32]. Gödel himself believed that CH would be shown not to follow from ZFC, and at least for part of his life he held the view that CH is indeed false and that actually

$$2^{\aleph_0} = \aleph_2. \tag{1}$$

[*]Funded by EPSRC Grant EP/N032160/1.

[†]Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germanys Excellence Strategy EXC 2044 390685587, Mathematics Münster: Dynamics - Geometry - Structure.

(see [14] and [13, pp. 173ff.]). Luzin [24] proposed a related hypothesis which also refutes CH, namely

$$2^{\aleph_0} = 2^{\aleph_1}. \tag{2}$$

That CH does not follow from ZFC was confirmed by Cohen in 1963 through the discovery of the method of *forcing*: Every model of ZFC can be generically extended to a model of ZFC in which CH fails, see [5]. In fact, using forcing one can show that it is relatively consistent with ZFC that the cardinality of the continuum is $\aleph_1$, $\aleph_2$, $\aleph_{155}$, $\aleph_{\omega^2+17}$, or $\aleph_\alpha$ for many other values of α, see [33].

Ever since Cohen's work, set-theorists have been searching for natural new axioms which extend ZFC and which settle the Continuum Problem (see e.g. [43], [44], [20], and the discussion in [8]). One family of such axioms is the hierarchy of large cardinal axioms. It was realized early on, though, that these axioms cannot settle the Continuum Problem: one can always force CH to hold or be false by small forcing notions, and all large cardinals which exist in V will retain their large cardinal properties in the respective extensions, see [23].

One axiom which does settle the Continuum Problem is CH itself; after all, CH looks natural in that it gives the least possible value to $2^{\aleph_0}$ consistent with Cantor's theorem, $2^{\aleph_0} > \aleph_0$. CH allows the "diagonal" construction of objects of size $\aleph_1$ with specific combinatorial properties, e.g. Luzin or Sierpiński sets. In 1985, Woodin proved his Σ^2_1 absoluteness result conditioned on CH. Namely, if CH holds true, there is a proper class of measurable Woodin cardinals, and σ is a statement of the form "There is a set of reals X such that $\varphi(X, r)$," where r is a real and $\varphi(X)$ is a formula of set theory all of whose quantifiers are restricted to reals, such that σ can be forced over V, then σ actually holds true in V, see e.g. [7, Theorem 4.1]. Over the last decade, Woodin has developed a sophisticated scenario for set theory according to which CH is true, see e.g. [45] and [46].

Nevertheless, and despite the appeal of Σ^2_1 absoluteness, CH is often regarded as a minimalistic assumption on a par with its parent, $V = L$. To give an illustrative example, under CH one can easily find sets X and Y of reals without endpoints which are both $\aleph_1$-dense—in the sense that every interval of points contains exactly $\aleph_1$ many points—but such that X and Y are not order-isomorphic. On the other hand, by a theorem of Baumgartner [2], given any such X and Y, there is a nicely behaved forcing notion which adds an order-isomorphism between X and Y. Thus, adopting CH precludes the existence of sufficiently generic filters for such forcing notions—which may consistently exist.

A dual approach to CH is then to formulate axioms stipulating the existence of objects which may possibly exist, i.e., to look for "maximality principles" expressing some form of saturation of the universe of all sets with respect to its generic extensions. Such principles are known as *forcing axioms*.

Shortly after the discovery of forcing, it was realized that it is possible to iterate the process of forming generic extensions $V \subset V[g_0] \subset V[g_1] \subset \ldots \subset V[g_\alpha]$ of V in any length α in such a way that the final model is itself a generic extension of V. By "closing off" one may then try to get to final models which are saturated with respect to the existence of certain (partial) generics.

The first forcing axiom shown to be consistent was Martin's Axiom at ω_1, MA_{ω_1}, see [34]. MA_{ω_1} says that for every partial order $\mathbb{P}$ with the countable chain condition (i.e., there is no uncountable family of pairwise incompatible conditions in $\mathbb{P}$) and for every collection $\mathcal{D}$ of size $\aleph_1$ consisting of dense subsets of $\mathbb{P}$ there is a filter $g \subset \mathbb{P}$ which is $\mathcal{D}$-generic (i.e., is such that $g \cap D \neq \emptyset$ for each $D \in \mathcal{D}$). Over the following years, generalizations of MA_{ω_1} got isolated. This line of research culminated in the proof by Foreman-Magidor-Shelah of the consistency of *Martin's Maximum*, MM, see [10].

Martin's Maximum is the statement that if $\mathbb{P}$ is a forcing notion such that forcing with $\mathbb{P}$ preserves stationary subsets of ω_1 in V and if $\mathcal{D}$ is a collection of size $\aleph_1$ consisting of dense subsets of $\mathbb{P}$, then there is a $\mathcal{D}$-generic filter $g \subset \mathbb{P}$. MM is provably maximal in the sense that the forcing axiom for any forcing notion destroying some stationary subset S of ω_1 necessarily fails. The conclusion of Baumgartner's result [2] holds true under (a weakening of) MM, and MM admits many other features of maximality.

At the same time, MM can be forced by means of a forcing iteration $\mathbb{P} \subset V_\kappa$, assuming that κ is a supercompact cardinal. The natural such forcing $\mathbb{P}$ actually produces a model of a strengthening of MM, called MM^{++}. This is the statement that if $\mathbb{P}$ is a forcing notion preserving stationary subsets of ω_1, $\mathcal{D}$ is a collection of size $\aleph_1$ consisting of dense subsets of $\mathbb{P}$, and $\{\tau_\alpha : \alpha < \omega_1\}$ is a collection of $\mathbb{P}$-names for stationary subsets of ω_1, then there is a filter $g \subset \mathbb{P}$ which is $\mathcal{D}$-generic and which, furthermore, interprets every τ_α, $\alpha < \omega_1$, as a truly stationary set in V (i.e., $\{\nu < \omega_1 : \exists p \in g \, p \Vdash_{\mathbb{P}} \check{\nu} \in \tau_\alpha\}$ is stationary for every $\alpha < \omega_1$).

Already MA_{ω_1} contradicts CH, and it even proves Luzin's hypothesis (2), i.e., $2^{\aleph_0} = 2^{\aleph_1}$. More interestingly, MM (in contrast to MA_{ω_1}) decides the cardinality of the continuum, and in fact it confirms Gödel's conjecture (1), $2^{\aleph_0} = \aleph_2$. This is shown by producing an affirmative answer to Friedman's Problem under MM, see [10, Theorem 10].[1] MM^{++} is—by its very definition and the fact that no strictly stronger forcing axiom can be consistent—a prototype maximality principle for V.

[1] It was later verified by Moore that already the much weaker forcing axiom BPFA implies (1), see [27].

Remarkably, the empirical evidence seems to suggest that MM^{++} provides a complete theory of the initial segment H_{ω_2} of the universe of sets, at least with respect to natural questions. Here, H_{ω_2} is the collection of all sets which are hereditarily of size $< \aleph_2$.

There is another maximality principle, though, which Magidor called a "competitor" of MM, see [25, p. 18], and which is denoted by $(*)$. In much the same way as MM, $(*)$ is inspired by and formulated in the language of forcing, and they both have "the same intuitive motivation: Namely, the universe of sets is rich" ([25, p. 18]). $(*)$, introduced by Woodin in [42, Definition 5.1], is the conjunction of the following two statements.

(i) AD, the Axiom of Determinacy,[2] holds in $L(\mathbb{R})$, and

(ii) there is some g which is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$ such that $\mathcal{P}(\omega_1) \cap V \subset L(\mathbb{R})[g]$.

Item (i), that AD holds in $L(\mathbb{R})$, follows from the existence of large cardinals, e.g. from the existence of infinitely many Woodin cardinals with a measurable cardinal above them all, see [26, p. 91]. Item (ii) is the part of $(*)$ which goes beyond assuming the existence of large cardinals. $\mathbb{P}_{\max}$ is a forcing notion which was isolated by Woodin, see [42, Defiition 4.33]. It arose out of earlier work by Steel-Van Wesep [37] and by himself [41] on the size of $\boldsymbol{\delta}^1_2$ and the question if NS_{ω_1}, the nonstationary ideal on ω_1, can be saturated.

$\mathbb{P}_{\max}$ consists of countable transitive structures, and membership in $\mathbb{P}_{\max}$ is uniformly Π^1_2 in the codes. $\mathbb{P}_{\max}$ is ω-closed and homogeneous, see [42, Lemma 4.43]. The homogeneity of $\mathbb{P}_{\max}$ yields that under $(*)$, the theory of $L(\mathcal{P}(\omega_1))$ becomes part of the theory of $L(\mathbb{R})$. If AD holds in $L(\mathbb{R})$, then there is no well-order of the reals in $L(\mathbb{R})$ (see e.g. [30, Lemma 12.2]), but if g is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$, then ZFC is true in $L(\mathbb{R})[g]$ (see [42, Theorem 4.54]), and moreover NS_{ω_1} is saturated in $L(\mathbb{R})[g]$ (see [42, Theorem 4.50]) and $L(\mathbb{R})[g]$ provides an effective failure of CH in that $\boldsymbol{\delta}^1_2 = \omega_2$ is true in $L(\mathbb{R})[g]$ (see [42, Theorem 4.53]).

Like the "classical" forcing axioms culminating with MM^{++}, $(*)$ is also a maximality principle, as it implies (and is in fact equivalent with) what is dubbed "Π_2 maximality." A sentence σ (in the language of set theory, possibly augmented with some additional predicates) is said to be Π_2 if it is of the form $\forall x \exists y \varphi(x, y)$, with $\varphi(x, y)$ being a formula with only restricted quantifiers. There is a whole family of interesting statements which are Π_2 in the language for the structure

$$(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}),$$

[2] See e.g. [30, Chapter 12].

see e.g. the discussion in [6]. The formulation of "Π_2 maximality" involves the concept of Ω-logic, see [42, Section 10.4]; for a sentence σ to be "Ω-consistent" is stronger than it just being consistent in that σ needs to be true in models which are closed under arbitrarily complicated universally Baire operations, see [42, Definition 10.144]. The Π_2 maximality theorem, see [42, Theorem 10.150], then runs as follows.

Theorem 1.1 (Woodin) *Suppose there is a proper class of Woodin cardinals. Then the following statements are equivalent.*

(1) $(*)$.

(2) Let σ *be a* Π_2 *sentence in the language for the structure*

$$(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A\colon A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})).$$

If σ *is* Ω*-consistent, then* σ *is true.*

One specific instance of σ in (2) of Theorem 1.1 is called ψ_{AC}, see [42, Definition 5.12]. It is in spirit a local version of an affirmative solution to Friedman's Problem. Woodin showed, see [42, Theorem 5.14, Lemmata 5.15 and 5.18], that ψ_{AC} follows from both MM and $(*)$ and that ψ_{AC} implies Gödel's conjecture (1), i.e. $2^{\aleph_0} = \aleph_2$.

The homogeneity of $\mathbb{P}_{\max}$ gives that in the presence of large cardinals, $(*)$ yields a complete theory for $L(\mathcal{P}(\omega_1))$ modulo set-forcing: all set-generic extensions of V in which $(*)$ holds agree on the theory of this inner model.

Despite its nice properties, in order for $(*)$ to be a convincing candidate for a natural axiom, it would have to be compatible with all consistent large cardinal axioms. While $L(\mathbb{R})[g]$ is trivially a model of $(*)$, provided that g is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$, Scott's result [32] carries over from L to $L(\mathbb{R})[g]$ and shows that $L(\mathbb{R})[g]$ cannot have measurable cardinals either. [42] and subsequent work had left open the problem if $(*)$ would be compatible with large cardinals beyond the level of Woodin cardinals.

Prior to the current paper, the relation between classical forcing axioms like MM, which could be forced by iterated forcing over models of ZFC with large cardinals, and the axiom $(*)$, whose models were obtained by forcing over models satisfying the Axiom of Determinacy, remained a complete mystery. It had been known by a result of P. Larson [21] that even $\mathsf{MM}^{+\omega}$, an axiom strictly between MM and MM^{++}, does not imply $(*)$.[3] One can build models of $\mathrm{MM}^{+\omega}$ with a well-order of H_{ω_2} which is

[3] $\mathrm{MM}^{+\omega}$ is the strenghening of MM obtained by replacing, in the formulation of MM^{++}, collections of $\aleph_1$-many names for stationary sets with collections of only countably many such names.

definable over $(H_{\omega_2}; \in)$ by a formula without parameters, and the existence of such a well-order is incompatible with $(*)$ by the homogeneity of $\mathbb{P}_{\max}$. It remained even unclear whether classical strong forcing axioms would be compatible at all with $(*)$, see [42, p. 846]. See also [42, Question (18) a) on p. 924], [25, Conjecture 6.8 on p. 19], and [28, Problem 14.7].

This paper resolves the tension between MM and $(*)$. We prove:

Theorem 1.2 *Assume* Martin's Maximum^{++}. *Then Woodin's* $\mathbb{P}_{\max}$*-axiom* $(*)$ *holds true.*

In particular, MM and $(*)$ are compatible with one another, and $(*)$ is compatible with all consistent large cardinal axioms: If κ is a supercompact cardinal and $\mathbb{P} \subset V_\kappa$ is the partial order from [10] to force MM^{++}, then by Theorem 1.2 the axiom $(*)$ holds in $V^{\mathbb{P}}$, but all the large cardinals of V above κ are preserved by $\mathbb{P}$.

It also follows from Theorem 1.2 that MM^{++} imples (2) of Theorem 1.1, and that $(*)$ can be characterized, in the presence of large cardinals, by a statement which on the face of its formulation is weaker than (2) of Theorem 1.1. We will prove the following theorem at the end of the next section.

Theorem 1.3 *Suppose there is a supercompact cardinal. Then the following statements are equivalent.*

(1) $(*)$.

(2) Let σ *be a* Π_2 *sentence in the language for the structure*

$$(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A \colon A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})).$$

If there is a stationary set preserving forcing $\mathbb{P}$ *such that* σ *holds in* $V^{\mathbb{P}}$*, then* σ *is true in* V*.*

This equivalence of $(*)$ is a variant of one which we are going to state below, see Theorem 2.10, and which characterizes $(*)$ as a strong version of a bounded forcing axiom.

In 1947, Gödel [12] had written:

> [...] one may on good reason suspect that the role of the continuum problem in set theory will be this, that it will finally lead to the discovery of new axioms which will make it possible to disprove Cantor's conjecture.

In light of our unifying result, Theorem 1.2, one could make the case that with MM^{++} and $(*)$, natural and strong such axioms have already been found.

2 Preliminaries.

Let us first restate MM^{++}. *Martin's Maximum*$^{++}$, abbreviated by MM^{++} and isolated by Foreman-Magidor-Shelah [10] (cf. also [42, Definition 2.45 (2)]). MM^{++} is the statement that if $\mathbb{P}$ is a forcing which preserves stationary subsets of ω_1, if $\{D_i \colon i < \omega_1\}$ is a collection of dense subsets of $\mathbb{P}$, and if $\{\tau_i \colon i < \omega_1\}$ is a collection of $\mathbb{P}$-names for stationary subsets of ω_1, then there is a filter $g \subset \mathbb{P}$ such that for every $i < \omega_1$,

(i) $g \cap D_i \neq \emptyset$ and

(ii) $(\tau_i)^g = \{\xi < \omega_1 \colon \exists p \in g\, p \Vdash_{\mathbb{P}} \check{\xi} \in \tau_i\}$ is stationary.

The forcing $\mathbb{P}_{\max}$ was designed by W. Hugh Woodin, see [42, Chapter 4]. The conditions in $\mathbb{P}_{\max}$ (see [42, Definition 4.33]) are countable transitive models of a sufficiently large fragment of ZFC plus MA_{ω_1} of the form $(M; \in, I, a)$, where

(i) $(M; I)$ is amenable[4] and $(M; I) \models$ "I is a normal uniform ideal on ω_1,"

(ii) $a \in \mathcal{P}(\omega_1^M) \cap M$ and $M \models$ "$\omega_1 = \omega_1^{L[a,x]}$ for some real x," and

(iii) $(M; \in, I)$ is generically iterable.

The last item, generic iterability, refers to generic iterations of $(M; \in, I)$. Given an ordinal $\gamma \leq \omega_1$, $\langle\langle(M_\alpha; \in, I_\alpha) \colon \alpha \leq \gamma\rangle, \langle\pi_{\alpha,\beta} \colon \alpha \leq \beta \leq \gamma\rangle, \langle g_\alpha \colon \alpha < \gamma\rangle\rangle$ is a generic iteration of $(M; \in, I)$ if the following hold true.

- $(M_0; \in, I_0) = (M; \in, I)$,
- for $\alpha < \gamma$, g_α is a $\mathcal{P}(\omega_1)^{M_\alpha} \setminus I_\alpha$-generic filter over M_α, $M_{\alpha+1}$ is the ultrapower of M_α by g_α, and $\pi_{\alpha,\alpha+1} \colon (M_\alpha; \in, I_\alpha) \to (M_{\alpha+1}; \in, I_{\alpha+1})$ is the corresponding generic elementary embedding,
- $\pi_{\alpha_0,\alpha_2} = \pi_{\alpha_1,\alpha_2} \circ \pi_{\alpha_0,\alpha_1}$ for all $\alpha_0 \leq \alpha_1 \leq \alpha_2$, and
- if β is a nonzero limit ordinal $\leq \gamma$, then $(M_\beta, (\pi_{\alpha,\beta} \colon \alpha < \beta))$ is the direct limit of $(M_\alpha, \pi_{\alpha,\alpha'} \colon \alpha \leq \alpha' < \beta)$.

$(M; \in, I)$ being generically iterable means that all models in any generic iteration iteration of $(M; \in, I)$ are well-founded, irrespective of the filters g_α chosen at any stage α, see [42, Definition 4.1].

[4] i.e., $x \cap I \in M$ for all $x \in M$

The current paper will only consider such generic iterations rather than iterations of mice as being studied in inner model theory.

$\mathbb{P}_{\max}$ becomes a partial order by declaring $(N;\in,J,b)<(M;\in,I,a)$, "$(N;\in,J,b)$ is stronger than $(M;\in,I,a)$," iff $(M;\in,I,a)\in N$ and inside N there is a generic iteration of $(M;\in,I,a)$ of length ω_1^N+1 with last model $(M^*;\in,I^*,a^*)$ such that $I^*=J\cap M^*$ and $a^*=b$.

Most of [42] studies the effect of forcing with $\mathbb{P}_{\max}$ or variants thereof over a model of the Axiom of Determinacy. Let us restate Woodin's $\mathbb{P}_{\max}$ axiom $(*)$, see [42, Definition 5.1]. $(*)$ says that

(i) AD holds in $L(\mathbb{R})$ and

(ii) there is some g which is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$ such that $\mathcal{P}(\omega_1)\cap V\subset L(\mathbb{R})[g]$.

Already PFA, the *Proper Forcing Axiom*, which is weaker than MM^{++}, implies $\mathsf{AD}^{L(\mathbb{R})}$ and much more, see [35], [17], and [29, Chapter 12].

The current paper produces a proof of Theorem 1.2. Our key new idea is $(\Sigma.8)$ on page 20 below. We try to give an overview of the proof of Theorem 1.2 at the end of this section.

Theorem 1.2 is optimal in that P. Larson [21] and [22] has shown that $\mathsf{MM}^{+\omega}$ is consistent with $\neg(*)$ relative to a supercompact limit of supercompact cardinals. Our proof is also optimal in that the forcing which we will use to verify Theorem 1.2 has size $2^{\aleph_2}$, while Woodin has shown that MM^{++} for forcings of size $2^{\aleph_0}=\aleph_2$ does not imply $(*)$, see [42, Theorem 10.90], and it is consistent with MM^{++} that $2^{\aleph_2}=\aleph_3$.

Throughout our entire paper, "ω_1" will *always* denote ω_1^V, the ω_1 of V.

Convention 2.1 *Let us fix throughout this paper some $A\subset\omega_1$ such that $\omega_1^{L[A]}=\omega_1$. Let us define g_A as the set of all $\mathbb{P}_{\max}$ conditions $p=(N;\in,I,a)$ such that there is a generic iteration*

$$(N_i,\sigma_{ij}\colon i\leq j\leq\omega_1)$$

of $p=N_0$ of length ω_1+1 such that if we write $N_{\omega_1}=(N_{\omega_1};\in,I^*,a^*)$,[5] then $I^*=(\mathsf{NS}_{\omega_1})^V\cap N_{\omega_1}$ and $a^*=A$.

Lemma 2.2 (Woodin) *Assume that NS_{ω_1} is saturated and that $\mathcal{P}(\omega_1)^{\#}$ exists.*

(1) g_A is a filter.

[5] Here and elsewhere we often confuse a model with its underlying universe.

(2) If g_A is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$, then $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[g]$.

PROOF. This routinely follows from the proof of [42, Lemma 3.12 and Corollary 3.13] and from [42, Lemma 3.10]. □

One may also use BMM (Bounded Martin's Maximum) plus "NS_{ω_1} is precipitous" to show that g_A is a filter, this is by the proof from [4].

Let $1 \leq k < \omega$, and let $D \in \mathcal{P}(\mathbb{R}^k)$. We say that *the trees T and U on ${}^k\omega \times \mathrm{OR}$ witness that D* is *universally Baire* iff $D = p[T]$ and for all posets $\mathbb{P}$,

$$\Vdash_{\mathbb{P}} p[U] = \mathbb{R}^k \setminus p[T]. \tag{3}$$

D is called *universally Baire* iff there are trees T and U witnessing that D is universally Baire. The concept of universally Baire set was isolated by Feng-Magidor-Woodin in [9, Section 2]; see also [30, Definition 8.6].

Let us denote by Γ^∞ the collection of all $D \in \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k)$ which are universally Baire. If $D \in \Gamma^\infty$, then there is an unambiguous version of D in any forcing extension $V[g]$ of V, which as usual we denote by D^* and which is equal to $p[T] \cap V[g]$ for some/all trees T and U which witness that D is universally Baire. See [30, p. 149f.].

Definition 2.3 *Let $\Gamma \subset \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k)$. We say that Γ is* productive *iff*

(a) $\Gamma \subset \Gamma^\infty$,

(b) for all $k < \omega$ and all $D \in \Gamma \cap \mathcal{P}(\mathbb{R}^{k+1})$, $\mathbb{R}^{k+1} \setminus D \in \Gamma$ and if $k > 0$, then $\exists^{\mathbb{R}} D = \{(x_0, \ldots, x_{k-1}) \colon \exists x_k (x_0, \ldots, x_{k-1}, x_k) \in D\} \in \Gamma$, and

(c) for all $k < \omega$ and all $D \in \Gamma \cap \mathcal{P}(\mathbb{R}^{k+2})$, if the trees T and U on ${}^{k+2}\omega \times \mathrm{OR}$ witness that D is universally Baire and if

$$\tilde{U} = \{(s \restriction (k+1), (s(k+1), t)) \colon (s,t) \in U\}, \tag{4}$$

then there is a tree $\tilde{T}$ on ${}^{k+1}\omega \times \mathrm{OR}$ such that for all posets $\mathbb{P}$,

$$\Vdash_{\mathbb{P}} p[\tilde{U}] = \mathbb{R}^{k+1} \setminus p[\tilde{T}]. \tag{5}$$

Hence a pointclass consisting of universally Baire sets is productive iff it is closed under complements and projections in the sense of (b) of Definition 2.3 and for all $k < \omega$ and $D \in \Gamma \cap \mathbb{R}^{k+2}$,

$$(\exists^{\mathbb{R}} D)^* = \{\vec{x} \in \mathbb{R}^{k+1} \colon \exists y \in \mathbb{R}\, (\vec{x}, y) \in D^*\} \tag{6}$$

will be true in every generic extension. On the other hand, at least on the face of its definition, Γ being productive is stronger than having that $\Gamma \subset \Gamma^\infty$ and Γ is closed under complements and projections (this is exacly [9, Question 3]).

If Γ is productive and if $D \in \Gamma$, then any projective statement about D is absolute between V and any forcing extension of V, i.e., if φ is projective, $x_1, \ldots, x_k \in \mathbb{R}$, and $\mathbb{P}$ is any poset, then

$$V \models \varphi(x_1, \ldots, x_k, D) \iff \Vdash_{\mathbb{P}} \varphi(\check{x}_1, \ldots, \check{x}_k, D^*).$$

This is shown by a trivial induction on the complexity of φ. [9, Question 3] is concerned with the question about the connection of, on the one hand, projective absoluteness with respect to forcing extensions and, on the other hand, having that every projective set is universally Baire (see [9, Questions 1 and 7]).

Theorem 2.4 (Woodin) *Assume that there is a proper class of Woodin cardinals. Then Γ^∞ is productive.*

Proof. This is true by a theorem of Woodin, see e.g. [36, Theorem 1.2], combined with the key result of Martin and Steel from [26]. □

For any set X, $M_\omega^\#(X)$ denotes the least active X-mouse which has infinitely many Woodin cardinals. See [38].

Theorem 2.5 (Steel) *Assume* PFA. *Then the universe is closed under the operation $X \mapsto M_\omega^\#(X)$. In particular, every set of reals in $L(\mathbb{R})$ is universally Baire, and $\bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\mathbb{R})$ is productive.*

Proof. The proof from [35] produces the result that under PFA, the universe is closed under the operation $X \mapsto M_\omega^\#(X)$. The rest is given by standard inner model theoretic arguments, see e.g. [31, Section 3, pp. 187f.]. □

By Lemma 2.2 and Theorem 2.5, Theorem 1.2 follows from the following more general statement.

Theorem 2.6 *Let $\Gamma \subset \bigcup_{1\leq k<\omega} \mathcal{P}(\mathbb{R}^k)$. Assume that*

(i) $\Gamma = \bigcup_{1\leq k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) Γ *is productive, and*

(iii) Martin's Maximum^{++} holds true.

Then g_A is $\mathbb{P}_{\max}$-generic over $L(\Gamma, \mathbb{R})$.

The abbreviation $(*)_\Gamma$ was introduced in [31, Definition 4.1] to denote a straightforward generalization of $(*)$ to larger pointclasses.

Corollary 2.7 *Assume that there is a proper class of Woodin cardinals. Let $\Gamma \subset \bigcup_{1 \le k < \omega} \mathcal{P}(\mathbb{R}^k) \cap \Gamma^\infty$. Suppose that (i)–(iii) from the statement of Theorem 2.6 are satisfied. Then $(*)_\Gamma$ holds true.*

Let $e \colon \mathbb{R} \to \mathrm{HC}$ be a fixed simple coding of hereditarily countable sets by reals, see e.g. [31, p. 179]. A set $D \subset \mathrm{HC}$ is then called *universally Baire in the codes* iff the code set $\{x \in \mathbb{R} \colon e(x) \in D\}$ of D is universally Baire. If this is the case, then every forcing extension of V will have its unique new version of D, which we denote by D^*. If the code set of D is a member of a productive pointclass, then for every forcing $\mathbb{P}$,

$$(\mathrm{HC}; \in, D) \prec (\mathrm{HC}^{V^{\mathbb{P}}}; \in D^*). \tag{7}$$

Here and in what follows, $(M; \in, \ldots) \prec (N; \in, \ldots)$ means that $(M; \in, \ldots)$ is an elementary substructure of $(N; \in, \ldots)$, i.e., $M \subset N$, the two models have the same first order language associated to it, and for all formulae φ of that common language and all $x_1, \ldots, x_k \in M$,

$$(M; \in, \ldots) \models \varphi(x_1, \ldots, x_k) \iff (N; \in, \ldots) \models \varphi(x_1, \ldots, x_k).$$

Theorem 2.6 readily follows from the following Lemma via a standard application of MM^{++}.

Lemma 2.8 *Let $\Gamma \subset \bigcup_{1 \le k < \omega} \mathcal{P}(\mathbb{R}^k)$. Assume that*

(i) $\Gamma = \bigcup_{1 \le k < \omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) Γ *is productive, and*

(iii) NS_{ω_1} *is saturated.*

Let $D \subset \mathbb{P}_{\max}$ be open dense, $D \in L(\Gamma, \mathbb{R})$. There is then a stationary set preserving forcing $\mathbb{P}$ of size $2^{\aleph_2}$ such that in $V^{\mathbb{P}}$ there is some $p = (N; \in, I, a) \in D^*$ and some generic iteration

$$(N_i, \sigma_{ij} \colon i \le j \le \omega_1)$$

of $p = N_0$ of length $\omega_1 + 1$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, then $I^* = (\mathsf{NS}_{\omega_1})^{V^{\mathbb{P}}} \cap N_{\omega_1}$ and $a^* = A$.

Proof of Theorem 2.6 from Lemma 2.8. MM implies that NS_{ω_1} is saturated, see [10, Theorem 12]. By Lemma 2.2, it remains to show that $D \cap g_A \neq \emptyset$ for every open dense $D \subset \mathbb{P}_{\max}$, $D \in \Gamma$.

Let us fix such D. The statement that there is a p as in the conclusion of Lemma 2.8, which is tantamount to saying that there is a $p \in D \cap g_A$, is easily seen to be Σ_1 expressible over the structure $(H_{\omega_2}; \in, \mathsf{NS}_{\omega_1}, A, D)$. By the conclusion of Lemma 2.8, the existence of such a p may be forced by a stationary set preserving forcing. Hence by MM^{++}, cf. [42, Theorem 10.124], there is such a p in V. $\square$

The attentive reader will notice that with virtually the same proof that is to come we could weaken the hypothesis (iii) of Lemma 2.8 to "NS_{ω_1} is precipitous" at the cost of increasing the size of $\mathbb{P}$ to become $(2^{2^{\aleph_1}})^+$. The same remark applies to hypothesis (iii) in Theorem 2.9. We decided to present our proof working under the stronger hypothesis, though, as doing so simplifies the notation, and in practice the weaker versions of Lemma 2.8 and Theorem 2.9 are as interesting as their stronger counterparts.

As the proof of Theorem 2.6 from Lemma 2.8 shows, we don't need the full power of MM^{++} in order to derive Theorem 2.6 from Lemma 2.8, the hypothesis that D-BMM^{++} holds true for all $D \in \Gamma^\infty$ would suffice, see [42, Definition 10.123]. $(*)$ is then actually *equivalent* to a version of BMM. Let us first state a more general fact, Theorem 2.9, which gives the characterization of $(*)$, i.e. Theorem 2.10, as a special case.

Theorem 2.9 *Assume that there is a proper class of Woodin cardinals. Let* $\Gamma \subset \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k)$*. Assume that*

(i) $\Gamma = \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) Γ *is productive.*

The following statements are then equivalent.

(1) D-BMM^{++} *holds true for all* $D \in \Gamma$.

(2) g_A *is* $\mathbb{P}_{\max}$*-generic over* $L(\Gamma, \mathbb{R})$.

Proof. (2) $\Longrightarrow$ (1): This is exactly by the proof of (A) $\Longrightarrow$ (B) of [1, Theorem 2.7].

(1) $\Longrightarrow$ (2): We may first force NS_{ω_1} to be saturated by a stationary set preserving forcing, see e.g. [42, Theorem 2.64]. The rest is then by the proof of Theorem 2.6 from Lemma 2.8 which was given above. $\square$

Theorem 2.10 *Assume that there is a proper class of Woodin cardinals. The following statements are then equivalent.*

(1) D-BMM^{++} *holds true for all* $D \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$.

(2) $(*)$.

Let us now give a *Proof* of Theorem 1.3. (1) $\Longrightarrow$ (2) is weaker than (1) $\Longrightarrow$ (2) of Theorem 1.1. Let us now assume (2) and show (1). Fix $D \subset \mathbb{P}_{\max}$, any open dense set in $L(\mathbb{R})$. As the statement of the theorem assumes a supercompact cardinal to exist, there is a semi-proper (and hence stationary set preserving) forcing $\mathbb{P}$ such that MM^{++} holds true in $V^{\mathbb{P}}$. Inside $V^{\mathbb{P}}$, we will have that MM^{++} yields via (the proof of) Lemma 2.8 that for all $A' \subset \omega_1$ with $\omega_1^{L[A']} = \omega_1$ there will be some $p = (N; \in, I^*, a^*) \in D^*$ and some generic iteration

$$(N_i, \sigma_{ij} \colon i \leq j \leq \omega_1)$$

of $p = N_0$ of length $\omega_1 + 1$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, then $I^* = (\mathsf{NS}_{\omega_1})^{V^{\mathbb{P}}} \cap N_{\omega_1}$ and $a^* = A'$. This is a statement which is Π_2 over the structure mentioned in (2). This statement will therefore be true in V, which readily implies that g_A is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$ and $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[g_A]$. $\square$

The forcing which we designed in order to produce Lemma 2.8 is a souped up version of the forcings from [4] and [6], which are in turn variants of the $\mathcal{L}$-forcing of Jensen as being developed e.g. in [16].[6] All these forcings may be construed as building uncountable models as term models of a given language, $\mathcal{L}$, with the forcing conditions being finite fragments of a consistent and complete $\mathcal{L}$-theory which will give those term models, augmented by "side conditions" which will guarantee that the forcing only collapses cardinals in a controlled way. Our forcing will change the cofinalities of ω_2 and ω_3 to ω and ω_1, respectively, and does not collapse any other cardinal outside of the (possibly empty) half-open interval $(\omega_3, 2^{\aleph_2}]$.

To prove Lemma 2.8, we aim to build a stationary set preserving forcing $\mathbb{P}$ which adds a generic iteration of some $\mathbb{P}_{\max}$-condition $(N; \in, I, a)$ coded by a real in the projection of a tree T projecting to the set of codes for conditions in our given dense set D. Moreover, we want this iteration to send the distinguished set a of $(N; \in, I, a)$ to A, and we want every I^*-positive set in the final model $(N^*; \in, I^*, A)$ to be a stationary subset of ω_1 in $V^{\mathbb{P}}$. Our approach is to think of all the relevant

[6]The referee informs us that J. Keisler in [18] and [19] developed forcings which work in a similar fashion.

objects – $(N;\in,I,a)$, a branch through T projecting to a real coding $(N;\in,I,a)$, and the generic iteration of $(N;\in,I,a)$ of length ω_1+1 – as being given by "term models" in a suitable language, $\mathcal{L}$, and add them via finite approximations. Thus, the working parts of our forcing will be finite sets p of sentences from $\mathcal{L}$ providing partial information about the above objects. We will require these finite bits of information p to be realized in some outer model. The existence of such an outer models will be absolute to any generic extensions of V via $\mathrm{Col}(\omega,\omega_2)$.

In $V^{\mathrm{Col}(\omega,\omega_2)}$,

$$(H^V_{\omega_2};\in,\mathsf{NS}^V_{\omega_1},A)$$

becomes a $\mathbb{P}_{\max}$-condition, and $p[T]=D^*$ is still dense, so that in $V^{\mathrm{Col}(\omega,\omega_2)}$ there is a $\mathbb{P}_{\max}$-condition $(N;\in,I,a)\in D^*$ which is stronger than $(H^V_{\omega_2};\in,\mathsf{NS}^V_{\omega_1},A)$. We may now iterate $(N;\in,I,a)$ in length $\omega_1^{V^{\mathrm{Col}(\omega,\omega_2)}}+1=\omega_3^V+1$ so as to produce

$$\sigma\colon (N;\in,I,a)\to(N^*;\in,I^*,a^*).$$

If $(M_i,\pi_{ij}\colon i\le j\le\omega_1^N)\in N$ is the generic iteration of $(H^V_{\omega_2};\in,\mathsf{NS}^V_{\omega_1},A)$ witnessing that $(N;\in,I,a)$ is stronger than $(H^V_{\omega_2};\in,\mathsf{NS}^V_{\omega_1},A)$, then $j((M_i,\pi_{ij}\colon i\le j\le\omega_1^N))=(M_i,\pi_{ij}\colon\ \le j\le\omega_3^V)$ is an extension of that iteration such that

$$\pi_{\omega_1^V,\omega_3^V}\colon M_{\omega_1^V}=(H^V_{\omega_2};\in,\mathsf{NS}^V_{\omega_1},A)\to M_{\omega_3^V}$$

may be lifted to a generic iteration

$$\tilde\pi\colon V\to M$$

of V in a way that M is transitive. One can now verify that $V^{\mathrm{Col}(\omega,\omega_2)}$ contains the objects we intend to approximate, albeit not defined relative to the parameters T and A, but relative to $\tilde\pi(T)$ and $\tilde\pi(A)$. By absoluteness, such objects will also exist in $M^{\mathrm{Col}(\omega,\tilde\pi(\omega_2))}$, so that by pulling back via $\tilde\pi$, the objects we intend to approximate, this time with the right parameters T and A, will exist in $V^{\mathrm{Col}(\omega,\omega_2)}$.

For technical reasons, not only the objects we are ultimately interested in obtaining, but also the iteration $(M_i,\pi_{ij}\colon i\le j\le\omega_1^N)\in N$ will be among the objects our forcing $\mathbb{P}$ will need to approximate. We will think of the objects themselves, which exist in $V^{\mathrm{Col}(\omega,\omega_2)}$, as "certificates" of some finite piece of information about them. The idea is then to have our forcing consist of finite sets of $\mathcal{L}$-sentences for which there is a "certificate" in $V^{\mathrm{Col}(\omega,\omega_2)}$.

The problem with the above strategy is that, although a forcing $\mathbb{P}$ like the one we have described would in fact add the desired objects, one would still need to show that it preserves stationary subsets of ω_1 and that every positive set in the final model of the iteration being added by $\mathbb{P}$ is in fact stationary in that extension. Our forcing $\mathbb{P}$ will be a subset of H_{ω_3}, and one tool for taking care of these issues is the use of a diamond sequence $\langle (Q_\lambda, A_\lambda) : \lambda < \omega_3 \rangle$ consisting of transitive structures in H_{ω_3} in order to guess $(H_{\omega_3}, \dot{C})$, where $\dot{C}$ is a $\mathbb{P}$-name for a club in ω_1, $\dot{C} \subset H_{\omega_3}$. That $(H_{\omega_3}, \dot{C})$ be guessed means that there are stationarily many $\lambda < \omega_3$ such that (Q_λ, A_λ) is an elementary substructure of $(H_{\omega_3}, \dot{C})$.

Imagine that $\mathbb{P}$ is a forcing which adds the desired objects, but which also preserves stationary subsets of ω_1. Let $\dot{C}$ be a $\mathbb{P}$-name for a club in ω_1, $\dot{C} \subset H_{\omega_3}$, and let $S \subset \omega_1$ be stationary in V. Let g be $\mathbb{P}$-generic over V. There will be some $\lambda < \omega_3$ such that $(H_{\omega_3}, \dot{C})$ is guessed by (Q_λ, A_λ) and in $V[g]$ there will be some countable elementary substructure X of (Q_λ, A_λ) such that

(a) $X \cap \omega_1 \in S$, and

(b) $X[g] \cap Q_\lambda = X \cap Q_\lambda$.

Here, $X[g] = \{\tau^g : \tau \in V^{\mathbb{P}} \cap X\}$. That (Q_λ, A_λ) is an elementary substructure of $(H_{\omega_3}, \dot{C})$ will then mean in practive that $X \cap \omega_1 \in \dot{C}^g$, so that $X \cap \omega_1$ witnesses that S is still stationary in $V[g]$. Calling some g with (b) "$(\mathbb{P}, X)$-generic" there is, however, no reason to expect an X such that g is $(\mathbb{P}, X)$-generic to exist in V (in fact, V won't have such X).

When defining $\mathbb{P}$, we will turn this around and have our conditions also approximate finite bits of information about such elementary substructures X.

Our key tool for taking care of the above issue is then to define $\mathbb{P}$ as the last forcing from a recursively defined $\subset$-increasing sequence $\vec{\mathbb{P}} = (\mathbb{P}_\lambda : \lambda \leq \omega_3)$. Each $\mathbb{P}_\lambda$ will be a subset of Q_λ. Hence, when defining $\mathbb{P}_\lambda$, $\lambda \leq \omega_3$, if $\delta < \lambda$, then we already know what it means for some g to be (partially) $\mathbb{P}_\delta$-generic over (Q_δ, A_δ), and if X is a countable elementary substructure of (Q_δ, A_δ), then $X[g]$ may be assigned a meaningful interpretation as $X[g] = \{\tau^g : \tau \in V^{\mathbb{P}_\delta} \cap X\}$. We will maintain that at each stage λ in the construction of $\vec{\mathbb{P}}$ we define $\mathbb{P}_\lambda$ by saying that a finite set p of $\mathcal{L} \cap Q_\lambda$-sentences is in $\mathbb{P}_\lambda$ if and only if there is a certificate for p which, when intersected with each of the side conditions X_δ (also given by the certificate), for $\delta < \lambda$, is generic over X_δ for the already defined forcing $\mathbb{P}_\delta$.

Our next section is entirely devoted to a proof of Lemma 2.8.

The authors thank Andreas Lietz and Matteo Viale for their comments on earlier versions of this paper. They would also like to thank the anonymous referee for their report which has been of extraordinary quality.

3 The forcing.

Recall our Convention 2.1. Let us assume throughout the hypotheses of Lemma 2.8. We aim to verify its conclusion.

Let us fix $D \subset \mathbb{P}_{\max}$, an open dense set in $L(\Gamma, \mathbb{R})$. By hypotheses (ii) in the statement of Lemma 2.8 (i.e., $\Gamma \subset \Gamma^\infty$ is productive) we will have that

(D.1) $V^{\mathrm{Col}(\omega,\omega_2)} \models$ "D^* is an open dense subset of $\mathbb{P}_{\max}$."

Let us identify D with a canonical set of reals coding the elements of D,[7] and let $\tilde{T} \in V$ be a tree on $\omega \times 2^{\aleph_2}$ such that

(D.2) $V^{\mathrm{Col}(\omega,\omega_2)} \models D^* = p[\tilde{T}]$.

Let h be $\mathrm{Col}(\omega, \omega_2)$-generic over V. Inside $V[h]$,

$$((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A)$$

is a $\mathbb{P}_{\max}$ condition, call it p. Let $q^* \in (\mathbb{P}_{\max})^{V[h]}$, $q^* < p$, $q^* \in D^*$, cf. (D.1). Let $q^* = (N^*; \in, I^*, a^*)$. Identifying q^* with some real coding it, we have that $q^* \in p[\tilde{T}]$, cf. (D.2). Let $T \in V$ be a tree on $\omega \times \omega_2$ such that

$$q^* \in p[T] \subset p[\tilde{T}]. \tag{8}$$

Let us write

$$\kappa = \aleph_3, \tag{9}$$

so that $T \in H_\kappa$. Let d be $\mathrm{Col}(\kappa, \kappa)$-generic over V. In $V[d]$, let $(\bar{A}_\lambda \colon \lambda < \kappa)$ be a $\diamondsuit_\kappa$-sequence, i.e., for all $\bar{A} \subset \kappa$, $\{\lambda < \kappa \colon \bar{A} \cap \lambda = \bar{A}_\lambda\}$ is stationary. Also, let $c \colon \kappa \to H_\kappa^V = H_\kappa^{V[d]}$, $c \in V[d]$, be bijective. For $\lambda < \kappa$, let

$$Q_\lambda = c"\lambda \text{ and } A_\lambda = c"\bar{A}_\lambda. \tag{10}$$

An easy closure argument will give us some club $C \subset \kappa$ such that for all $\lambda \in C$,

(i) Q_λ is transitive,

(ii) $\{T, ((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A)\} \cup (\omega_2 + 1) \subset Q_\lambda$,

[7] We will later have to spell out a bit more precisely in which way we aim to have the elements of $p[T]$ code the elements of D, see (C.2) and (Σ.5) below.

(iii) $Q_\lambda \cap \mathrm{OR} = \lambda$ (so that $c \restriction \lambda \colon \lambda \to Q_\lambda$ is bijective), and

(iv) $(Q_\lambda; \in) \prec (H_\kappa; \in)$.

We will fix a C with the above properties from now on.

In $V[d]$, for all P, $B \subset H_\kappa$, the set of all $\lambda \in C$ such that

$$(Q_\lambda; \in, P \cap Q_\lambda, B \cap Q_\lambda) \prec (H_\kappa; \in, P, B)$$

is club, and the set of all $\lambda \in C$ such that $B \cap Q_\lambda = A_\lambda$ is stationary, so that

($\diamond$) For all P, $B \subset H_\kappa$ the set

$$\{\lambda \in C \colon (Q_\lambda; \in, P \cap Q_\lambda, A_\lambda) \prec (H_\kappa; \in, P, B)\}$$

is stationary.

We shall sometimes also write $Q_\kappa = H_\kappa$. It is easy to see that the principle which we refer to as ($\diamond$) is actually equivalent to $\diamond_\kappa$. We shall use ($\diamond$) to guess information about names for club subsets of ω_1; this will play a crucial role in the verification that our forcing preserves stationary subsets of ω_1.

We shall now go ahead and produce a stationary set preserving forcing $\mathbb{P} \in V[d]$ of size κ which adds some $p \in D^*$ and some generic iteration

$$(N_i, \sigma_{ij} \colon i \leq j \leq \omega_1)$$

of $p = N_0$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, then $I^* = (\mathsf{NS}_{\omega_1})^{V[d]^{\mathbb{P}}} \cap N_{\omega_1}$ and $a^* = A$. As the forcing $\mathrm{Col}(\kappa, \kappa)$ which added d is certainly stationary set preserving, this will verify Lemma 2.8.

NS_{ω_1} is still saturated in $V[d]$. This is true simply because forcing with $\mathrm{Col}(\kappa, \kappa)$ doesn't add any sequences of elements of V of length $\aleph_2$. Moreover, (D.1) and (D.2) are true with V being replaced by $V[d]$, so that in order to simplify our notation, we shall in what follows confuse $V[d]$ with V, i.e., pretend that, in addition to "NS_{ω_1} is saturated" plus (D.1) and (D.2), ($\diamond$) is also true in V.

Working under these hypotheses, we shall now recursively define a $\subset$-increasing and continuous chain of forcings $\mathbb{P}_\lambda$ for all $\lambda \in C \cup \{\kappa\}$. The forcing $\mathbb{P}$ will be $\mathbb{P}_\kappa$. Each $\mathbb{P}_\lambda$ will consist of a finite set of formulae of an associated first order language, $\mathcal{L}^\lambda$, which will be defined below. The order of each $\mathbb{P}_\lambda$ will be just reverse inclusion, i.e., $q \leq_{\mathbb{P}_\lambda} p$ iff $q \supset p$ for p, $q \in \mathbb{P}_\lambda$.

Assume that $\lambda \in C \cup \{\kappa\}$ and $\mathbb{P}_\mu$ has already been defined in such a way that $\mathbb{P}_\mu \subset Q_\mu$ for all $\mu \in C \cap \lambda$.

We shall be interested in objects $\mathfrak{C}$ which exist in some outer model[8] and which have the following properties.

$$\mathfrak{C} = \langle\langle M_i, \pi_{ij}, N_i, \sigma_{ij} \colon i \leq j \leq \omega_1\rangle, \langle(k_n, \alpha_n) \colon n < \omega\rangle, \langle\lambda_\delta, X_\delta \colon \delta \in K\rangle\rangle, \tag{11}$$

where

(C.1) M_0, $N_0 \in \mathbb{P}_{\max}$,

(C.2) $x = \langle k_n \colon n < \omega\rangle$ is a real code for $N_0 = (N_0; \in, I, a)$, by which we mean that there is some surjection $f \colon \omega \to N_0$ such that x is the monotone enumeration of the Gödel numbers of all expressions of the form $\ulcorner \dot{N}_0 \models \varphi(\dot{n}_1, \ldots, \dot{n}_\ell, \dot{a}, \dot{I})\urcorner$ with φ being a first order formula of the language associated to $(N_0; \in, I, a)$, $N_0 \models \varphi(f(n_1), \ldots, f(n_\ell), a, I)$, and $\langle(k_n, \alpha_n) \colon n < \omega\rangle \in [T]$,

(C.3) $\langle M_i, \pi_{ij} \colon i \leq j \leq \omega_1^{N_0}\rangle \in N_0$ is a generic iteration of M_0 which witnesses that $N_0 < M_0$ in $\mathbb{P}_{\max}$,

(C.4) $\langle N_i, \sigma_{ij} \colon i \leq j \leq \omega_1\rangle$ is a generic iteration of N_0 such that if

$$N_{\omega_1} = (N_{\omega_1}; \in, I^*, A^*),$$

then $A^* = A$,[9]

(C.5) $\langle M_i, \pi_{ij} \colon i \leq j \leq \omega_1\rangle = \sigma_{0\omega_1}(\langle M_i, \pi_{ij} \colon i \leq j \leq \omega_1^{N_0}\rangle)$ and

$$M_{\omega_1} = ((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A), \tag{12}$$

(C.6) $K \subset \omega_1$,

and for all $\delta \in K$,

(C.7) $\lambda_\delta \in \lambda \cap C$, and if $\gamma < \delta$ is in K, then $\lambda_\gamma < \lambda_\delta$ and $X_\gamma \cup \{\lambda_\gamma\} \subset X_\delta$, and

(C.8) $X_\delta \prec (Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta})$ and $X_\delta \cap \omega_1 = \delta$.

[8] W is an outer model iff W is a transitive model of ZFC with $W \supset V$ and which has the same ordinals as V; in other words, W is an outer model iff V is an inner model of W.

[9] There is no requirement on I^* matching the non-stationary ideal of some model in which $\mathfrak{C}$ exists.

For future purposes, let us refer to any object $\mathfrak{C}$ as in (11) which satisfies the above properties (C.1) through (C.8) as a *potential certificate*.

We need to define a first order language $\mathcal{L}$ (independently from λ) whose formulae will be able to describe $\mathfrak{C}$ with the above properties by producing the models M_i and N_i, $i < \omega_1$, as term models out of equivalence classes of terms of the form $\dot{n}$, $n < \omega$. The language $\mathcal{L}$ will have the the following constants.

$\dot{T}$	intended to denote T
x for every $x \in H_\kappa$	intended to denote x itself
$\dot{n}$ for every $n < \omega$	as terms for elements of M_i and $N_i, i < \omega_1$
$\dot{M}_i$ for $i < \omega_1$	intended to denote M_i
$\dot{\pi}_{ij}$ for $i \le j \le \omega_1$	intended to denote π_{ij}
$\dot{\vec{M}}$	intended to denote $(M_j, \pi_{jj'} \colon j \le j' \le \omega_1^{N_i})$ for $i < \omega_1$
$\dot{N}_i$ for $i < \omega_1$	intended to denote N_i
$\dot{\sigma}_{ij}$ for $i \le j < \omega_1$	intended to denote σ_{ij}
$\dot{a}$	intended to denote the distinguished a-predicate of $M_i, N_i, i < \omega_1$
$\dot{I}$	intended to denote the distinguished ideal of $N_i, i < \omega_1$
$\dot{X}_\delta$ for $\delta < \omega_1$	intended to denote X_δ.

The formulae of $\mathcal{L}$ will be exactly the expressions of the following form.[10]

$$\ulcorner \dot{N}_i \vDash \varphi(\xi_1, \ldots, \xi_k, \dot{n}_1, \ldots, \dot{n}_\ell, \dot{a}, \dot{I}, \dot{M}_{j_1}, \ldots, \dot{M}_{j_m}, \dot{\pi}_{q_1 r_1}, \ldots, \dot{\pi}_{q_s r_s}, \dot{\vec{M}}) \urcorner$$
for $i < \omega_1, \xi_1, \ldots, \xi_k < \omega_1, n_1, \ldots, n_\ell < \omega, j_1, \ldots, j_m < \omega_1, q_1 \le r_1 < \omega_1, \ldots, q_s \le r_s < \omega_1$
and φ being a first order formula of the language of set theory
associated with $\mathbb{P}_{\max}$-structures

$$\ulcorner \dot{\pi}_{i\omega_1}(\dot{n}) = x \urcorner \quad \text{for } i < \omega_1 \text{ and } x \in H_{\omega_2}$$
$$\ulcorner \dot{\pi}_{\omega_1\omega_1}(x) = x \urcorner \quad \text{for } x \in H_{\omega_2}$$
$$\ulcorner \dot{\sigma}_{ij}(\dot{n}) = \dot{m} \urcorner \quad \text{for } i \le j < \omega_1, n, m < \omega$$
$$\ulcorner (\vec{u}, \vec{\alpha}) \in \dot{T} \urcorner \quad \text{for } \vec{u} \in {}^{<\omega}\omega \text{ and } \vec{\alpha} \in {}^{<\omega}\omega_2$$
$$\ulcorner \delta \mapsto \bar{\lambda} \urcorner \quad \text{for } \delta < \omega_1, \bar{\lambda} < \kappa$$
$$\ulcorner x \in \dot{X}_\delta \urcorner \quad \text{for } \delta < \omega_1, x \in H_\kappa$$

[10] There are no dots on top of ξ_1, …, ξ_k in the first formula, as those are constants of the type which intend to denote themselves. The term $\dot{n}$ in the second formula is of the type denoting an element of M_i, etc.

Let us write $\mathcal{L}^\lambda$ for the collection of all $\mathcal{L}$-formulae except for the formulae which mention elements outside of Q_λ, i.e., except for the formulae of the form $\ulcorner\delta \mapsto \bar\lambda\urcorner$ for $\delta < \omega_1$ and $\lambda \leq \bar\lambda < \kappa$ as well as $\ulcorner x \in \dot X_\delta\urcorner$ for $\delta < \omega_1$ and $x \in H_\kappa \setminus Q_\lambda$. We may and shall assume that $\mathcal{L}$ is built in a canonical way so that $\mathcal{L}^\lambda \subset Q_\lambda$ and in fact $\mathcal{L}^\lambda = \mathcal{L} \cap Q_\lambda$.

We say that a potential certificate $\mathfrak{C}$ as in (11) is *pre-certified* by a collection Σ of $\mathcal{L}^\lambda$-formulae if and only if (C.1) through (C.8) are satisfied by $\mathfrak{C}$ and there are surjections $e_i \colon \omega \to N_i$ for $i < \omega_1$ such that the following hold true.

(Σ.1) $\ulcorner \dot N_i \models \varphi(\xi_1, \ldots, \xi_k, \dot n_1, \ldots, \dot n_\ell, \dot a, \dot I, \dot M_{j_1}, \ldots, \dot M_{j_m}, \dot\pi_{q_1r_1}, \ldots, \dot\pi_{q_sr_s}, \dot{\vec{M}})\urcorner \in \Sigma$ iff $i < \omega_1$, $\xi_1, \ldots, \xi_k \leq \omega_1^{N_i}$, $n_1, \ldots, n_\ell < \omega$, $j_1, \ldots, j_m \leq \omega_1^{N_i}$, $q_1 \leq r_1 \leq \omega_1^{N_i}, \ldots,$ $q_s \leq r_s \leq \omega_1^{N_i}$ and
$N_i \models \varphi(\xi_1, \ldots, \xi_k, e_i(n_1), \ldots, e_i(n_\ell), A \cap \omega_1^{N_i}, I^{N_i}, M_{j_1}, \ldots, M_{j_m}, \pi_{q_1r_1}, \ldots, \pi_{q_sr_s}, \vec M)$, where I^{N_i} is the distinguished ideal of N_i and $\vec M = \langle M_j, \pi_{jj'} \colon j \leq j' \leq \omega_1^{N_i}\rangle$,

(Σ.2) $\ulcorner \dot\pi_{i\omega_1}(\dot n) = x\urcorner \in \Sigma$ iff $i < \omega_1$, $n < \omega$, and $\pi_{i\omega_1}(e_i(n)) = x$,

(Σ.3) $\ulcorner \dot\pi_{\omega_1\omega_1}(x) = x\urcorner \in \Sigma$ iff $x \in H_{\omega_2}$,

(Σ.4) $\ulcorner \dot\sigma_{ij}(\dot n) = \dot m\urcorner \in \Sigma$ iff $i \leq j < \omega_1$, n, $m < \omega$, and $\sigma_{ij}(e_i(n)) = e_j(m)$,

(Σ.5) letting F with $\mathrm{dom}(F) = \omega$ be the monotone enumeration of the Gödel numbers of all formulae of the form $\ulcorner \dot N_0 \models \varphi(\dot n_1, \ldots, \dot n_\ell, \dot a, \dot I)\urcorner$ with $\ulcorner \dot N_0 \models \varphi(\dot n_1, \ldots, \dot n_\ell, \dot a, \dot I)\urcorner \in \Sigma$, we have that $\ulcorner(\vec u, \vec\alpha) \in \dot T\urcorner \in \Sigma$ iff there is some $n < \omega$ such that $\langle \vec u, \vec \alpha\rangle = \langle (F(m), \alpha_m) \colon m < n\rangle$ and $F(m) = k_m$ for all $m < n$,

(Σ.6) $\ulcorner\delta \mapsto \bar\lambda\urcorner \in \Sigma$ iff $\delta \in K$ and $\bar\lambda = \lambda_\delta$, and

(Σ.7) $\ulcorner x \in \dot X_\delta\urcorner \in \Sigma$ iff $\delta \in K$ and $x \in X_\delta$.

We say that a potential certificate $\mathfrak{C}$ as in (11) is *certified* by a collection Σ of formulae if and only if $\mathfrak{C}$ is pre-certified by Σ and, in addition,

(Σ.8) if $\delta \in K$, then $[\Sigma]^{<\omega} \cap X_\delta \cap E \neq \emptyset$ for every $E \subset \mathbb{P}_{\lambda_\delta}$ which is dense in $\mathbb{P}_{\lambda_\delta}$ and definable over the structure

$$(Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta})$$

from parameters in X_δ.[11]

[11] Equivalently, $[\Sigma]^{<\omega} \cap E \neq \emptyset$ for every $E \subset \mathbb{P}_{\lambda_\delta} \cap X_\delta$ which is dense in $\mathbb{P}_{\lambda_\delta} \cap X_\delta$ and definable over the structure

$$(X_\delta; \in, \mathbb{P}_{\lambda_\delta} \cap X_\delta, A_{\lambda_\delta} \cap X_\delta)$$

from parameters in X_δ.

By way of definition, we call a potential certificate $\mathfrak{C}$ as in (11) a *semantic certificate* iff there is a collection Σ of formulae such that $\mathfrak{C}$ is certified by Σ. We call Σ a *syntactic certificate* iff there is a semantic certificate $\mathfrak{C}$ such that $\mathfrak{C}$ is certified by Σ. Given a syntactic certificate Σ, there is a *unique* semantic certificate $\mathfrak{C}$ such that $\mathfrak{C}$ is certified by Σ. Even though it is obvious how to construct $\mathfrak{C}$ from Σ, in the proof of Lemma 3.3 below we will provide details on how to derive a semantic certificate from a given Σ.

It is worth stressing that not every collection of $\mathcal{L}^\lambda$-formulae which is merely consistent is already a syntactic certificate. The requirement that the constant $x \in H_\kappa$ is to be interpreted by itself (cf. (Σ.2), (Σ.3), and (Σ.7)) may be restated as saying that for a consistent $\mathcal{L}^\lambda$-theory to be a syntactic certificate it is to be true that certain types are omitted.

Let $\Sigma \cup p$ be a set of formulae, where p is finite. We say that p is *certified by* Σ if and only if there is some (unique) $\mathfrak{C}$ as in (11) such that $\mathfrak{C}$ is certified by Σ and

(Σ.9) $p \in [\Sigma]^{<\omega}$.

We may also say that p is *certified by* $\mathfrak{C}$ as in (11) iff there is some Σ such that $\mathfrak{C}$ and p are both certified by Σ—and we will then also refer to Σ as a syntactical certificate for p and to $\mathfrak{C}$ as the associated semantic certificate.

We are then ready to define the forcing $\mathbb{P}_\lambda$. We say that $p \in \mathbb{P}_\lambda$ if and only if

$$V^{\mathrm{Col}(\omega,\lambda)} \models \text{"There is a set } \Sigma \text{ of } \mathcal{L}^\lambda\text{-formulae such that } p \text{ is certified by } \Sigma.\text{"} \tag{13}$$

Let p be a finite set of formulae of $\mathcal{L}^\lambda$. By the homogeneity of $\mathrm{Col}(\omega,\lambda)$, if there is some h which is $\mathrm{Col}(\omega,\lambda)$-generic over V and there is some $\Sigma \in V[h]$ such that p is certified by Σ, then for all h which are $\mathrm{Col}(\omega,\lambda)$-generic over V there is some $\Sigma \in V[h]$ such that p is certified by Σ. It is then easy to see that $\langle \mathbb{P}_\lambda \colon \lambda \in C \cup \{\kappa\}\rangle$ is definable over V from $\langle A_\lambda \colon \lambda < \kappa\rangle$ and C, and is hence an element of V.[12]

Again let p be a finite set of formulae of $\mathcal{L}^\lambda$. By Σ^1_1 absoluteness (see [30, Corollary 7.21]), if there is any outer model in which there is some Σ which certifies p, then there is some $\Sigma \in V^{\mathrm{Col}(\omega,\lambda)}$ which certifies p.[13] This simple observation is important in the verification that $\mathbb{P}_\lambda$ is actually non-empty, cf. Lemma 3.2, and in the proof of Lemma 3.8.

It is easy to see that

(i) $\mathbb{P} = \mathbb{P}_\kappa \subset H_\kappa$,

[12] To remind the reader, C is the club from p. 16.

[13] In fact, if P is a transitive model of KP plus the axiom *Beta* with $(Q_\lambda; \langle A_{\bar\lambda} \colon \bar\lambda < \lambda\rangle) \in P$ and if $p \in \mathbb{P}_\lambda$, then there is some $\Sigma \in P^{\mathrm{Col}(\omega,\lambda)}$ which certifies p.

(ii) if $\bar{\lambda} < \lambda$ are both in $C \cup \{\kappa\}$, then $\mathbb{P}_{\bar{\lambda}} \subset \mathbb{P}_\lambda$, and

(iii) if $\lambda \in C \cup \{\kappa\}$ is a limit point of $C \cup \{\kappa\}$, then $\mathbb{P}_\lambda = \bigcup_{\bar{\lambda} \in C \cap \lambda} \mathbb{P}_{\bar{\lambda}}$,

so that there is some club $D \subset C$ such that for all $\lambda \in D$,

$$\mathbb{P}_\lambda = \mathbb{P} \cap Q_\lambda.$$

Hence $(\diamondsuit)$ gives us the following.

$(\diamondsuit(\mathbb{P}))$ For all $B \subset H_\kappa$ the set

$$\{\lambda \in C \colon (Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\kappa; \in, \mathbb{P}, B)\}$$

is stationary.

The first one of the following lemmas is entirely trivial.

Lemma 3.1 *Let Σ be a syntactic certificate, and let p, $q \in [\Sigma]^{<\omega}$. Then p and q are compatible conditions in $\mathbb{P}$.*

Lemma 3.2 *Let $\lambda \in C \cup \{\kappa\}$. Then $\emptyset \in \mathbb{P}_\lambda$.*

Proof. This is a simple variant of the proofs of [1, Theorem 2.8] and of of [31, Theorem 4.2]. What needs to be done is to construct a semantic/syntactic certificate (for $\emptyset$) in some outer model. Notice that for all $\lambda \in C \cup \{\kappa\}$, $\emptyset \in \mathbb{P}_\lambda$ iff $\emptyset \in \mathbb{P}$.

Let h be $\mathrm{Col}(\omega, \omega_2)$-generic over V. Let $q^* = (N^*; \in, I^*, a^*) \in (\mathbb{P}_{\max})^{V[h]}$ be as in the paragraph preceeding (8) and such that (8) is true. Let $(M_i, \pi_{ij} \colon i \leq j \leq \omega_1^{N_0}) \in N^*$ be the unique generic iteration of p which witnesses $q^* < p = M_0 = (H^V_{\omega_2}; \in, \mathsf{NS}^V_{\omega_1}, A)$.

Let $(N_i, \sigma_{ij} \colon i \leq j \leq \kappa) \in V[h]$ be a generic iteration of $N_0 = N^*$ such that $\kappa = \omega_1^{N_\kappa}$.[14] Let

$$(M_i, \pi_{ij} \colon i \leq j \leq \kappa) = \sigma_{0\kappa}((M_i, \pi_{ij} \colon i \leq j \leq \omega_1^{N_0})) \tag{14}$$

Since $M_0 = ((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A)$ and $(\mathsf{NS}_{\omega_1})^V$ is assumed to be saturated in V, every maximal antichain in V consisting of stationary subsets of ω_1 is an element of M_0. We may hence lift the generic ultrapower map $\pi_{01} \colon M_0 \to M_1$ to act on all of V, and inductively we may lift the entire generic iteration (14) to a generic iteration

$$(M_i^+, \pi_{ij}^+ \colon i \leq j \leq \kappa) \tag{15}$$

[14]If we wished, then we could even arrange that writing $N_\kappa = (N_\kappa; \in, I', a')$, we have that $I' = (\mathsf{NS}_\kappa)^{V[h]} \cap N_\kappa$, but this is not relevant here; cf. footnote 9.

of V in such a way that all M_i^+, $i \leq \kappa$, are transitive. Cf. [42, Lemma 3.8]. Let us write $M = M_\kappa^+$ and $\pi = \pi_{0\kappa}^+$.

Let $\langle k_n, \alpha_n \colon n < \omega\rangle$ be such that $x = \langle k_n \colon n < \omega\rangle$ is a real code for N_0 à la (C.2) and $\langle (k_n, \alpha_n) \colon n < \omega\rangle \in [T]$. We then clearly have that $\langle (k_n, \pi(\alpha_n)) \colon n < \omega\rangle \in [\pi(T)]$.

It is now easy to see that

$$\mathfrak{C} = \langle\langle M_i, \pi_{ij}, N_i, \sigma_{ij} \colon i \leq j \leq \kappa\rangle, \langle (k_n, \pi(\alpha_n)) \colon n < \omega\rangle, \langle\rangle\rangle \tag{16}$$

certifies $\emptyset$, construed as the empty set of $\pi(\mathcal{L}^\kappa)$ formulae: as the third component $\langle\rangle$ of $\mathfrak{C}$ in (16) is empty, *any* set of surjections $e_i \colon \omega \to N_i$, $i < \omega_1$, will induce a syntactic certificate for $\emptyset$ whose associated sematic certificate is $\mathfrak{C}$. By Σ_1^1 absoluteness, there is then some $\mathfrak{C} \in M^{\mathrm{Col}(\omega, \pi(\omega_2))}$ as in (16) which certifies $\emptyset$, so that $\emptyset \in \pi(\mathbb{P})$. By the elementarity of π, then, $\emptyset \in \mathbb{P}$. $\square$

Lemma 3.3 *Let $\lambda \in C \cup \{\kappa\}$. Let $g \subset \mathbb{P}_\lambda$ be a filter such that $g \cap E \neq \emptyset$ for all dense $E \subset \mathbb{P}_\lambda$ which are definable over $(Q_\lambda; \in, \mathbb{P}_\lambda)$ from elements of Q_λ. Then $\bigcup g$ is a syntactic certificate.*

Proof. Let us first describe how to read off from $\bigcup g$ a candidate

$$\mathfrak{C} = \langle\langle M_i, \pi_{ij}, N_i, \sigma_{ij} \colon i \leq j \leq \omega_1\rangle, \langle (k_n, \alpha_n) \colon n < \omega\rangle, \langle \lambda_\delta, X_\delta \colon \delta \in K\rangle\rangle$$

for a semantic certificate for $\bigcup g$. A variant of what is to come shows how to derive $\mathfrak{C}$ from a given syntactic certificate Σ, where $\mathfrak{C}$ is unique such that Σ certifies $\mathfrak{C}$, cf. the remark on p. 21.

For i, $j < \omega_1$ and τ, $\sigma \in \{\dot{n} \colon n < \omega\} \cup \omega_1$ define

$$\begin{aligned}
\tau \sim_i \sigma \quad & \text{iff} \quad \ulcorner \dot{N}_i \models \tau = \sigma \urcorner \in \bigcup g \\
(i, \tau) \sim_{\omega_1} (j, \sigma) \quad & \text{iff } i \leq j \wedge \exists \rho \, \{\ulcorner \dot{\sigma}_{ij}(\tau) = \rho \urcorner, \ulcorner \dot{N}_j \models \rho = \sigma \urcorner\} \subset \bigcup g \\
& \quad \text{or } j \leq i \wedge \exists \rho \, \{\ulcorner \dot{\sigma}_{ji}(\sigma) = \rho \urcorner, \ulcorner \dot{N}_i \models \rho = \tau \urcorner\} \subset \bigcup g \\
[\tau]_i &= \{\sigma \colon \tau \sim_i \sigma\} \\
[(i, \tau)] &= \{(j, \sigma) \colon (i, \tau) \sim_{\omega_1} (j, \sigma)\} \\
M_i &= \{[\tau]_i \colon \tau \in \{\dot{n} \colon n < \omega\} \cup \omega_1 \wedge \ulcorner \dot{N}_i \models \tau \in \dot{M}_i \urcorner \in \bigcup g\} \\
M_{\omega_1} &= (H_{\omega_2})^V \\
N_i &= \{[\tau]_i \colon \tau \in \{\dot{n} \colon n < \omega\} \cup \omega_1\}
\end{aligned}$$

$$
\begin{aligned}
N_{\omega_1} = \{[i,\tau]\colon i<\omega_1 \wedge \ulcorner \dot N_i \vDash \tau=\tau\urcorner \in \textstyle\bigcup g\}& \\
[\tau]_i \,\tilde\in_i\, [\sigma]_i &\quad \text{iff} \quad \ulcorner \dot N_i \vDash \tau \in \sigma\urcorner \in \textstyle\bigcup g \\
[i,\tau]\tilde\in_{\omega_1}[j,\sigma] &\quad \text{iff } i\le j \wedge \exists\rho\,\{\ulcorner \dot\sigma_{ij}(\tau)=\rho\urcorner, \ulcorner \dot N_j \vDash \rho\in\sigma\urcorner\} \subset \textstyle\bigcup g \\
&\qquad \text{or } j\le i \wedge \exists\rho\,\{\ulcorner \dot\sigma_{ji}(\sigma)=\rho\urcorner, \ulcorner \dot N_i \vDash \tau\in\rho\urcorner\} \subset \textstyle\bigcup g \\
[\tau]_i \in I^{N_i} &\quad \text{iff} \quad \ulcorner \dot N_i \vDash \tau \in \dot I\urcorner \in \textstyle\bigcup g \\
[i,\tau] \in I^{N_{\omega_1}} &\quad \text{iff} \quad [\tau]_i \in I^{N_i} \\
[\tau]_i \in a^{N_i} &\quad \text{iff} \quad \ulcorner \dot N_i \vDash \tau \in \dot a\urcorner \in \textstyle\bigcup g \\
[i,\tau] \in a^{N_{\omega_1}} &\quad \text{iff} \quad [\tau]_i \in I^{N_i} \\
\pi_{ij}([\tau]_i) = [\sigma]_j &\quad \text{iff} \quad \ulcorner \dot N_j \vDash \dot\pi_{ij}(\tau)=\sigma\urcorner \in \textstyle\bigcup g \\
\pi_{i\omega_1}([\tau]_i) = x &\quad \text{iff} \quad \ulcorner \dot\pi_{i\omega_1}(\tau)=x\urcorner \in \textstyle\bigcup g \\
\pi_{\omega_1\omega_1}(x) = x &\quad \text{iff} \quad x \in (H_{\omega_2})^V \\
\sigma_{ij}([\tau]_i) = [\sigma]_j &\quad \text{iff} \quad \ulcorner \dot\sigma_{ij}(\tau)=\sigma\urcorner \in \textstyle\bigcup g \\
\sigma_{i\omega_1}([\tau]_i) = [i,\tau] &\\
(k,\alpha) = (k_n,\alpha_n) &\quad \text{iff } \exists\vec u\exists\vec\alpha(\ulcorner(\vec u,\vec\alpha)\in\dot T\urcorner \in \textstyle\bigcup g \wedge k=\vec u(n)\wedge \alpha=\vec\alpha(n)) \\
\delta \in K &\quad \text{iff } \exists\bar\lambda\,\ulcorner\delta\mapsto\bar\lambda\urcorner \in \textstyle\bigcup g \\
\bar\lambda = \lambda_\delta &\quad \text{iff } \delta\in K \wedge \ulcorner\delta\mapsto\bar\lambda\urcorner \in \textstyle\bigcup g \\
x \in X_\delta &\quad \text{iff } \delta\in K \wedge \ulcorner x\in \dot X_\delta\urcorner \in \textstyle\bigcup g
\end{aligned}
$$

We will first have that $\tilde\in_0$ is wellfounded and that in fact (the transitive collapse of) the structure $N_0 = (N_0; \tilde\in_0, a^{N_0}, I^{N_0})$ is an iterable $\mathbb{P}_{\max}$ condition. This is true because straightforward density arguments give (C.2), i.e., that $\langle (k_n,\alpha_n)\colon n<\omega\rangle \in [T]$ and $\langle k_n\colon n<\omega\rangle$ will code the theory of N_0 à la (C.2).

Another set of easy density arguments will give that $(N_i, \sigma_{ij}\colon i\le j\le\omega_1)$ is a generic iteration of N_0, were we identify N_i with the structure $(N_i; \tilde\in_i, a^{N_i}, I^{N_i})$. To verify this, let us first show:

Claim 3.4 *For each $i<\omega_1$ and for each $\xi\le\omega_1^{N_i}$, $[\xi]_i$ represents ξ in (the transitive collapse of the well-founded part of) the term model for N_i; moreover, $a^{N_i} = A\cap\omega_1^{N_i}$. Hence $a^{N_{\omega_1}} = A$.*

Proof of Claim 3.4. A straightforward induction on $\xi\le\omega_1^{N_i}$ using (Σ.1) shows that $[\xi]_i$ must always represent ξ in N_i as given by any certificate. Claim 3.4 then follows by straightforward density arguments. $\square$ (Claim 3.4)

Similarly:

Claim 3.5 *Let $i < \omega_1$. N_{i+1} is generated from* $\mathrm{ran}(\sigma_{ii+1}) \cup \{\omega_1^{N_i}\}$ *in the sense that for every $x \in N_{i+1}$ there is some function $f \in {}^{\omega_1^{N_i}}(N_i) \cap N_i$ such that $x = \sigma_{ii+1}(f)(\omega_1^{N_i})$.*

Claim 3.6 *Let $i < \omega_1$. $\{X \in \mathcal{P}(\omega_1^{N_i}) \cap N_i \colon \omega_1^{N_i} \in \sigma_{ii+1}(X)\}$ is generic over N_i for the forcing given by the I^{N_i}-positive sets.*

Claim 3.7 *Let $i \leq \omega_1$ be a limit ordinal. For every $x \in N_i$ there is some $j < i$ and some $\bar{x} \in N_j$ such that $x = \sigma_{ji}(\bar{x})$.*

$(N_i, \sigma_{ij} \colon i \leq j \leq \omega_1)$ is then indeed a generic iteration of N_0. As N_0 is iterable, we may and shall identify N_i with its transitive collapse, so that (C.4) holds true.

Another round of density arguments will show that $\mathfrak{C}$ satisfies (C.1), (C.3), (C.5), (C.6), and (C.7), where we identify M_i with the structure $(M_i; \in, (\mathsf{NS}_{\omega_1^{M_i}})^{M_i}, A \cap \omega_1^{M_i})$. Let us now verify (C.8) and (Σ.8).

As for (C.8), its second part, $X_\delta \cap \omega_1 = \delta$ for $\delta \in K$, is easy. We will now use the Tarski-Vaught test to verify the first part of (C.8). Let φ be any formula, and let $x_1, \ldots, x_k \in X_\delta$, $\delta \in K$. Suppose that

$$(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \exists v\, \varphi(v, x_1, \ldots, x_k). \tag{17}$$

Let $p \in g$ be such that $\{\ulcorner x_1 \in \dot{X}_\delta \urcorner, \ldots, \ulcorner x_k \in \dot{X}_\delta \urcorner, \ulcorner \delta \mapsto \lambda_\delta \urcorner\} \subset p$. Let $q \leq p$, and let Σ be a syntactical certificate for q whose associated semantic certificate is

$$\mathfrak{C}' = \langle \langle M_i', \pi_{ij}', N_i', \sigma_{ij}' \colon i \leq j \leq \omega_1 \rangle, \langle (k_n', \alpha_n') \colon n < \omega \rangle, \langle \lambda_\delta', X_\delta' \colon \delta \in K' \rangle \rangle.$$

Then $\delta \in K'$ and

$$\{x_1, \ldots, x_k\} \subset X_\delta' \prec (Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}),$$

so that by (17) we may choose some $x \in X_\delta'$ with

$$(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \varphi(x, x_1, \ldots, x_k).$$

Let $r = q \cup \{\ulcorner x \in \dot{X}_\delta \urcorner\}$.

By density, there is then some $y \in X_\delta$ such that

$$(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \varphi(y, x_1, \ldots, x_k).$$

The proof of (Σ.8) is similar. Let again $\delta \in K$. Let $E \subset \mathbb{P}_{\lambda_\delta} \cap X^g_\delta$ be dense in $\mathbb{P}_{\lambda_\delta} \cap X_\delta$, and $r \in E$ iff $r \in \mathbb{P}_{\lambda_\delta} \cap X_\delta$ and

$$(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \varphi(r, x_1, \ldots, x_k). \tag{18}$$

Let $p \in g$ be such that $\{\ulcorner x_1 \in \dot{X}_\delta \urcorner, \ldots, \ulcorner x_k \in \dot{X}_\delta \urcorner, \ulcorner \delta \mapsto \lambda_\delta \urcorner\} \subset p$. Let $q \leq p$, and again let Σ be a syntactical certificate for q whose associated semantic certificate is

$$\mathfrak{C}' = \langle \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) \colon n < \omega \rangle, \langle \lambda'_\delta, X'_\delta \colon \delta \in K' \rangle \rangle.$$

Then $[\Sigma]^{<\omega} \cap X'_\delta$ has an element, say r, such that (18) holds true. Let $s = q \cup r \cup \{\ulcorner r \in \dot{X}_\delta \urcorner\}$.

By density, then, $g \cap X_\delta \cap E \neq \emptyset$. $\square$

Lemma 3.8 *Let g be $\mathbb{P}$-generic over V. Let*

$$\mathfrak{C} = \langle \langle M_i, \pi_{ij}, N_i, \sigma_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) \colon n < \omega \rangle, \langle \lambda_\delta, X_\delta \colon \delta \in K \rangle \rangle$$

be the semantic certificate associated with the syntactic certificate $\bigcup g$. Let

$$N_{\omega_1} = (N_{\omega_1}; \in, A, I^*).$$

Then every element of $(\mathcal{P}(\omega_1) \cap N_{\omega_1}) \setminus I^*$ is stationary in $V[g]$.

Corollary 3.9 *$\mathbb{P}$ preserves stationary subsets of ω_1.*

Proof of Corollary 3.9 from Lemma 3.8. If $\mathfrak{C}$ and I^* are as in the statement of Lemma 3.8, then by Lemma 3.3 and (C.3) and (C.4) we will have that $(\mathsf{NS}_{\omega_1})^V = I^* \cap V$, so that the conclusion of Lemma 3.8 also gives that $\mathbb{P}$ preserves stationary subsets of ω_1. $\square$

Proof of Lemma 3.8. Let $\dot{N}_{\omega_1} \in V^{\mathbb{P}}$ be a canonical name for N_{ω_1}, and let $\dot{I}^* \in V^{\mathbb{P}}$ be a canonical name for I^*. Let $\bar{p} \in g$, $\dot{C}$, $\dot{S} \in V^{\mathbb{P}}$, and $i < \omega_1$ and $n < \omega$ be such that

(i) $\bar{p} \Vdash$ "$\dot{C} \subset \omega_1$ is club,"

(ii) $\bar{p} \Vdash$ "$\dot{S} \in (\mathcal{P}(\omega_1) \cap \dot{N}_{\omega_1}) \setminus \dot{I}^*$," and

(iii) $\bar{p} \Vdash$ "$\dot{S}$ is represented by $[i, \dot{n}]$ in the term model producing $\dot{N}_{\omega_1}$."

We may and shall also assume that

$$\ulcorner \dot{N}_i \models \dot{n} \text{ is a subset of the first uncountable cardinal, yet } \dot{n} \notin \dot{I}\urcorner \in \bar{p}, \tag{19}$$

because the $\mathcal{L}$-formula in (19) must belong to every syntactic certificate for $\bar{p}$, as $\bar{p}$ satisfies (ii) and (iii).

Let $p \le \bar{p}$ be arbitrary, $p \in \mathbb{P}$. We aim to produce some $q \le p$ and some $\delta < \omega_1$ such that $q \Vdash \check{\delta} \in \dot{C} \cap \dot{S}$, see Claim 3.11 below.

For $\xi < \omega_1$, let

$$D_\xi = \{q \le p \colon \exists \eta \ge \xi\, (\eta < \omega_1 \wedge q \Vdash \check{\eta} \in \dot{C})\},$$

so that D_ξ is open dense below p. Let

$$E = \{(q, \eta) \in \mathbb{P} \times \omega_1 \colon q \Vdash \check{\eta} \in \dot{C}\}.$$

Let us write

$$\tau = ((D_\xi \colon \xi < \omega_1), E).$$

We may and shall identify τ with some subset of H_κ which codes τ.

By $(\diamondsuit(\mathbb{P}))$, we may pick some $\lambda \in C$ such that $p \in \mathbb{P}_\lambda$ and

$$(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\kappa; \in, \mathbb{P}, \tau). \tag{20}$$

Let h be $\mathrm{Col}(\omega, \omega_2)$-generic over V, and let $g' \in V[h]$ be a filter on $\mathbb{P}_\lambda$ such that $p \in g'$ and g' meets every dense set which is definable over $(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$ from parameters in Q_λ. By Lemma 3.3, $\bigcup g'$ is a syntactic certificate for p, and we may let

$$\langle\langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \le j \le \omega_1\rangle, \langle(k'_n, \alpha'_n) \colon n < \omega\rangle, \langle\lambda'_\delta, X'_\delta \colon \delta \in K'\rangle\rangle$$

be the associated semantic certificate. In particular, $K' \subset \lambda$.

Let S denote the subset of ω_1 which is represented by $[i, \dot{n}]$ in the term model giving N'_{ω_1}, so that if $N'_{\omega_1} = (N'_{\omega_1}, \in, A, I')$, then by (19),

$$S \in (\mathcal{P}(\omega_1) \cap N'_{\omega_1}) \setminus I'. \tag{21}$$

Let us also write $\rho = \omega_1^{V[h]} = \omega_4^V$. Inside $V[h]$, we may extend $\langle N'_i, \sigma'_{ij} \colon i \le j \le \omega_1\rangle$ to a generic iteration

$$\langle N'_i, \sigma'_{ij} \colon i \le j \le \rho\rangle$$

such that

$$\omega_1 \in \sigma'_{\omega_1, \omega_1+1}(S). \tag{22}$$

This is possible as $\omega_1^{N'_{\omega_1}} = \sup\{\omega_1^{N_j} \colon j < \omega_1\} = \omega_1$ and by (21). Let

$$\langle M'_i, \pi'_{ij} \colon i \leq j \leq \rho\rangle = \sigma_{0,\rho}(\langle M'_i, \pi'_{ij} \colon i \leq j \leq \omega_1^{N'_0}\rangle),$$

so that $\langle M'_i, \pi'_{ij} \colon i \leq j \leq \rho\rangle$ is an extension of $\langle M'_i, \pi'_{ij} \colon i \leq j \leq \omega_1\rangle$.

Since $M'_{\omega_1} = ((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A)$, cf. (12), and $(\mathsf{NS}_{\omega_1})^V$ is assumed to be saturated in V, every maximal antichain in V consisting of stationary subsets of ω_1 is an element of M'_{ω_1}. We may hence lift the generic ultrapower map $\pi'_{\omega_1\omega_1+1} \colon M'_{\omega_1} \to M'_{\omega_1+1}$ to act on all of V, and inductively we may lift the entire generic iteration $\langle M'_i, \pi'_{ij} \colon \omega_1 \leq i \leq j \leq \rho\rangle$ to a generic iteration

$$\langle M^+_i, \pi^+_{ij} \colon \omega_1 \leq i \leq j \leq \rho\rangle$$

of V with all M^+_i, $\omega_1 \leq i \leq \rho$, being transitive. Cf. [42, Lemma 3.8]. Let us write $M = M^+_\rho$ and $\pi = \pi^+_{\omega_1,\rho}$.

The key point is now that $\langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \leq j \leq \rho\rangle$ may be used to extend $\pi\text{"}\bigcup g'$ to a syntactic certificate

$$\Sigma \supset \pi\text{"}\bigcup g' \tag{23}$$

for $\pi(p)$ in the following manner. Let $K^* = K' \cup \{\omega_1\}$. For $\delta \in K'$, let $\lambda^*_\delta = \pi(\lambda'_\delta)$ and $X^*_\delta = \pi\text{"}X'_\delta$. Also, write $\lambda^*_{\omega_1} = \pi(\lambda)$ and $X^*_{\omega_1} = \pi\text{"}Q_\lambda$. Notice that $\omega_1 \in \pi(C)$, so that $K^* \subset \pi(C)$. Let

$$\mathfrak{C}^* = \langle \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \leq j \leq \rho\rangle, \langle (k'_n, \pi(\alpha'_n)) \colon n < \omega\rangle, \langle \lambda^*_\delta, X^*_\delta \colon \delta \in K^*\rangle\rangle.$$

We claim that $\mathfrak{C}^*$ is a semantic certificate for $\pi(p)$ as an element of $\pi(\mathbb{P})$. First notice that $\langle (k'_n, \pi(\alpha'_n)) \colon n < \omega\rangle \in [\pi(T)]$. Next, if $\delta \in K'$, then $X^*_\delta = \pi\text{"}X'_\delta \prec (\pi(Q_{\lambda'_\delta}); \in, \pi(\mathbb{P}_{\lambda'_\delta}), \pi(A_{\lambda'_\delta}))$, and $\pi\text{"}g' \cap X^*_\delta = \pi\text{"}(g' \cap X'_\delta)$; as $\bigcup g'$ is a syntactic certificate for p, we thus have that $\pi\text{"}g' \cap X^*_\delta \cap E \neq \emptyset$ for every $E \subset \pi(\mathbb{P}_{\lambda'_\delta})$ which is dense in $\pi(\mathbb{P}_{\lambda'_\delta})$ and definable over the structure $(\pi(Q_{\lambda'_\delta}); \in, \pi(\mathbb{P}_{\lambda'_\delta}), \pi(A_{\lambda'_\delta}))$ from parameters in X^*_δ. Finally, $X^*_{\omega_1} = \pi\text{"}Q_\lambda$ and the choice of g' imply that $\pi\text{"}g' \cap X^*_{\omega_1} \cap E \neq \emptyset$ for every $E \subset \pi(\mathbb{P}_\lambda)$ which is dense in $\pi(\mathbb{P}_\lambda)$ and definable over the structure $(\pi(Q_\lambda); \in, \pi(\mathbb{P}_\lambda), \pi(A_\lambda))$ from parameters in $X^*_{\omega_1}$. This buys us that $\mathfrak{C}^*$ is indeed a semantic certificate for $\pi(p)$ as an element of $\pi(\mathbb{P})$, and that moreover there is some syntactic certificate Σ as in (23) such that $\mathfrak{C}^*$ is certified by Σ.

Now let $[\dot{m}]_{\omega_1+1}$ represent $\sigma'_{\omega_1\omega_1+1}(S)$ in the term model for N'_{ω_1+1} provided by Σ, so that[15]

$$\{\ulcorner \dot{\sigma}_{i\omega_1+1}(\dot{n}) = \dot{m} \urcorner, \ulcorner \dot{N}_{\omega_1+1} \models \omega_1 \in \dot{m} \urcorner\} \subset \Sigma,$$

[15]Here, $\dot{\sigma}_{i\omega_1+1}$ and $\dot{N}_{\omega_1+1}$ are terms of the language associated with $\pi(\mathbb{P}_\lambda)$, and $\ulcorner \dot{\sigma}_{i\omega_1+1}(\dot{n}) = \dot{m} \urcorner$ and $\ulcorner \dot{N}_{\omega_1+1} \models \omega_1 \in \dot{m} \urcorner$ are formulae of that language.

in other words,

$$\pi(p) \cup \{\ulcorner \dot\sigma_{i\omega_1+1}(\dot n) = \dot m \urcorner, \ulcorner \dot N_{\omega_1+1} \models \omega_1 \in \dot m \urcorner\} \text{ is certified by } \Sigma. \tag{24}$$

Let us now define

$$q^* = \pi(p) \cup \{\ulcorner \dot\sigma_{i\omega_1+1}(\dot n) = \dot m \urcorner, \ulcorner \dot N_{\omega_1+1} \models \omega_1 \in \dot m \urcorner, \ulcorner \omega_1 \mapsto \pi(\lambda) \urcorner\}. \tag{25}$$

We thus established the following.

Claim 3.10 *$q^* \in \pi(\mathbb{P})$, as being certified by Σ.*

The elementarity of $\pi\colon V \to M$ then gives some $\delta < \omega_1$ and some $\mu < \kappa$ such that

$$q = p \cup \{\ulcorner \dot\sigma_{i\delta+1}(\dot n) = \dot m \urcorner, \ulcorner \dot N_{\delta+1} \models \delta \in \dot m \urcorner, \ulcorner \delta \mapsto \lambda \urcorner\} \in \mathbb{P}. \tag{26}$$

Claim 3.11 *$q \Vdash \check\delta \in \dot C \cap \dot S$.*

Proof of Claim 3.11. $q \Vdash \check\delta \in \dot S$ readily follows from $\{\ulcorner \dot\sigma_{i\delta+1}(\dot n) = \dot m \urcorner, \ulcorner \dot N_{\delta+1} \models \delta \in \dot m \urcorner\} \subset q$, the fact that $\bar p \geq p$ forces that $\dot S$ is represented by $[i, \dot n]$ in the term model giving $\dot N_{\omega_1}$, and the fact that by Claim 3.4, $[\delta]_{\delta+1}$ represents δ in the model $N_{\delta+1}$ of any semantic certificate for q.

Let us now show that $q \Vdash \check\delta \in \dot C$. We will in fact show that q forces that $\check\delta$ is a limit point of $\dot C$. Otherwise there is some $r \leq q$ and some $\eta < \delta$ such that

$$r \Vdash \dot C \cap \check\delta \subset \check\eta. \tag{27}$$

Let

$$\langle\langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \leq j \leq \omega_1\rangle, \langle (k'_n, \alpha'_n) \colon n < \omega\rangle, \langle \lambda'_{\bar\delta}, X'_{\bar\delta} \colon \bar\delta \in K'\rangle\rangle \tag{28}$$

certify r. We must have that

(a) $\delta \in K'$,

(b) $X'_\delta \prec (Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$,

(c) $X'_\delta \cap \omega_1 = \delta$, and

(d) for some Σ such that the objects from (28) are certified by Σ, $[\Sigma]^{<\omega} \cap X'_\delta \cap E \neq \emptyset$ for every $E \subset \mathbb{P}_\lambda$ which is dense in $\mathbb{P}_\lambda \cap X'_\delta$ and definable over the structure

$$(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$$

from parameters in X'_δ.

We have that $A_\lambda = \tau \cap Q_\lambda$, and hence A_λ may be identified with $((D_\xi \cap Q_\lambda \colon \xi < \omega_1), E \cap Q_\lambda)$. As $\eta < \delta \subset X'_\delta$, D_η is definable over the structure

$$(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$$

from a parameter in X'_δ. By (20), $D_\eta \cap Q_\lambda$ is dense in $\mathbb{P}_\lambda$. By (d) above, there is then some $s \in [\Sigma]^{<\omega} \cap X'_\delta \cap D_\eta$.

By (20) again, the unique smallest $\eta' \geq \eta$ with $s \Vdash \check{\eta}' \in \dot{C}$ must be in X'_δ, hence $\eta' < \delta$ by (c) above. By Lemma 3.1, s is compatible with r. We have reached a contradiction with (27). □

4 Open questions.

Woodin [42] also introduced the axiom $(*)^+$ as a strengthening of $(*)$. $(*)^+$ says that there is some pointclass $\Gamma \subset \mathcal{P}(\mathbb{R})$ and some filter $g \subset \mathbb{P}_{\max}$ such that

(1) $L(\Gamma, \mathbb{R}) \models \mathsf{AD}^+$,

(2) g is $\mathbb{P}_{\max}$-generic over $L(\Gamma, \mathbb{R})$, and

(3) $\mathcal{P}(\mathbb{R}) \subset L(\Gamma, \mathbb{R})[g]$.

See [42, p. 908]. While the main result of the current paper gives a new twist to the question if MM is compatible with $(*)^+$, see [42, p. 923, Question (15) a)], it also leaves this question wide open.

There is a strengthening of MM^{++}, isolated by Viale [40], which has strong completeness properties modulo forcing similar to those of $(*)$. This is the axiom MM^{+++}. It says that a class $\mathbb{T}$ of towers of ideals with certain nice structural properties is dense in the category of stationary set preserving forcings; in other words, for every stationary set preserving forcing $\mathbb{P}$ there is a tower $\mathcal{T}$ in $\mathbb{T}$ such that $\mathbb{P}$ completely embeds into $\mathcal{T}$ in such a way that the quotient forcing preserves stationary sets in $V^{\mathbb{P}}$. MM^{+++} implies MM^{++}, if κ is an almost super-huge cardinal, then there is a partial order $\mathbb{P} \subset V_\kappa$ which forces MM^{+++}, and if there is a proper class of almost super-huge cardinals, then MM^{+++} is complete for the theory of the ω_1-Chang model[16] with respect to stationary set preserving partial orders forcing MM^{+++}.

[16]The ω_1-Chang model is the $\subseteq$-minimal transitive model of ZF containing all ordinals and closed under ω_1-sequences. It can be construed as $\bigcup_{\alpha \in \mathrm{Ord}} L([\alpha]^{\aleph_1})$ and it includes $L(\mathcal{P}(\omega_1))$ as a definable submodel.

Schindler [31, Definition 2.10] introduces $\mathsf{MM}^{*,++}$ as a strengthening of MM^{++} by relaxing "forceable by a stationary set preserving forcing" to "honestly consistent" in an appropriate formulation of MM^{++}, see [31].

It remains open if either of MM^{+++} or $\mathsf{MM}^{*,++}$ is really stronger than MM^{++}, and it even remains open if $\mathsf{MM}^{*,++}$ is consistent at all.

References

[1] D. Asperó and R. Schindler, *Bounded Martin's Maximum with an asterisk*, Notre Dame J. Formal Logic 55 (2014), pp. 333-348.

[2] J. Baumgartner, *All* $\aleph_1$*-dense sets of reals can be isomorphic*, Fund. Math. **79** (1973), pp. 101-106.

[3] G. Cantor, *Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*, Crelles Journal f. Mathematik **77** (1874), pp. 258-262.

[4] B. Claverie and R. Schindler, *Increasing* u_2 *by a stationary set preserving forcing*, Journal of Symb. Logic 74 (2009), pp. 187-200.

[5] P. Cohen, *Set theory and the continuum hypothesis*, Benjamin, New York, 1966.

[6] P. Doebler and R. Schindler, Π_2 *consequences of BMM plus NS is precipitous and the semiproperness of all stationary set preserving forcings*, Math. Res. Letters **16** (2009), pp. 797-815.

[7] P. Doebler and R. Schindler *The extender algebra and vagaries of* Σ^2_1 *absoluteness*, Münster Journal of Mathematics **6** (2013), pp. 117-166.

[8] S. Feferman, H. Friedman, P. Maddy, and J. Steel, *Does mathematics need new axioms?*, Bulletin of Symbolic Logic **6** (2000), pp. 401-446.

[9] Q. Feng, M. Magidor, and W.H. Woodin H., *Universally Baire Sets of Reals*, in: Judah H., Just W., Woodin H. (eds) Set Theory of the Continuum. Mathematical Sciences Research Institute Publications, vol 26, Springer, New York, NY 1992.

[10] M. Foreman, M. Magidor, and S. Shelah, *Martin's Maximum, saturated ideals and non-regular ultrafilters I*, Ann. of Mathematics 127 (1988), pp. 1 - 47.

[11] K. Gödel, *The consistency of the axiom of choice and the geeralized continuumhypothesis*, Proc. Nat. Acad. U.S.A. **24** (1938), pp. 556-557.

[12] K. Gödel, *What is Cantor's continuum problem?*, Amer. Math. Monthly **54** (1947), pp. 515-525.

[13] K. Gödel, *Collected works*, Vol. II, Publications 1938-1974, ed. by S. Feferman et al., Oxford University Press 1990.

[14] K. Gödel, *Some considerations leading to the probable conclusion that the true power of the continuum is* $\aleph_2$, in: "Collected works," Vol. II, Publications 1938-1974, ed. by S. Feferman et al., Oxford University Press 1995, pp. 420-422.

[15] D. Hilbert, *Mathematische Probleme*, Nachr. K. Ges. Wiss. Göttingen, Math.-Phys. Klasse (Göttinger Nachrichten), **3** (1900), pp. 253-297.

[16] R. Jensen, $\mathcal{L}$*-forcing*, handwritten notes, available at `https://www.mathematik.hu-berlin.de/~raesch/org/jensen.html`.

[17] R. Jensen, E. Schimmerling, R. Schindler, and J. Steel, *Stacking mice*, Journal of Symb. Logic 74 (2009), pp. 315-335.

[18] H.J. Keisler, *Model theory for infinitary logic. Logic with countable conjuctions and finite quantifiers*, Studies in Logic and the Foundations of Mathematics **62**, North Holland Publishing Co., Amsterdam 1971.

[19] H.J. Keisler, *Forcing and the omitting types theorem*, in: "Studies in model theory," (M. Morley, ed.), Studies in Mathematics **8**, Math. Assoc. Amer., 1973, pp. 96-133.

[20] P. Koellner, *The Continuum Hypothesis*, The Stanford Encyclopedia of Philosophy (Spring 2019 Edition), E. N. Zalta (ed.), `https://plato.stanford.edu/archives/spr2019/entries/continuum-hypothesis/`

[21] P. Larson, *Martin's Maximum and the Pmax axiom* $(*)$, Annals of Pure and Applied Logic 106 (2000), pp. 135-149.

[22] P. Larson, *Martin's Maximum and definability in* $H(\omega_2)$, Annals of Pure and Applied Logic 156 (2008), pp. 110-122.

[23] A. Lévy and R. Solovay, *Measurable cardinals and the continuum hypothesis*, Israel J. Math. **5** (1967), pp. 234-248.

[24] N. Luzin, *On some new results in descriptive function theory* (in Russian), Moscow-Leningrad 1935.

[25] M. Magidor, *Some set theories are more equal*, `http://logic.harvard.edu/EFI_Magidor.pdf`

[26] D.A. Martin and J. Steel, *A Proof of Projective Determinacy*, Journal of the American Mathematical Society 2, pp. 71-125.

[27] J. Moore *Set mapping reflection*, J. Math. Logic **5** (2005), pp. 87-97.

[28] J. Moore, *What makes the continuum* $\aleph_2$, in: Foundations of mathematics, essays in honor of W. Hugh Woodin's 60th birthday, Harvard University (Caicedo et al., eds.), pp. 259-287.

[29] G. Sargsyan and N. Trang, *The largest Suslin Axiom*, available at `https://www.math.uci.edu/~ntrang/lsa.pdf`

[30] R. Schindler, *Set theory. Explorig independence and truth*, Springer-Verlag, Cham etc. 2014.

[31] R. Schindler, *Woodin's axiom* $(*)$*, or Martin's Maximum, or both?*, in: Foundations of mathematics, essays in honor of W. Hugh Woodin's 60th birthday, Harvard University (Caicedo et al., eds.), pp. 177-204.

[32] D. Scott, *Measurable cardinals and constructible sets*, Bull. Acad. Polon. Sci. **9** (1961), pp. 521-524.

[33] R. Solovay, $2^{\aleph_0}$ *can be anything it ought to be*, in: "The theory of models," Proceedings of the 1963 international symposium at Berkeley (J. Addison et al. eds.), Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam 1965, p. 435.

[34] R. Solovay and S. Tennebaum, *Iterated Cohen extensions and Souslin's problem*, Ann. of Math. **94** (1971), pp.frm-e01-245.

[35] J. Steel, PFA *implies* $\mathsf{AD}^{L(\mathbb{R})}$, J. Symb. Logic 70 (2005), pp. 1255-1296.

[36] J. Steel, *A stationary-tower-free proof of the derived model theorem*, in: "Advances of logic," Contemporary Mathematics 425, S. Gao, S. Jackson, and Y. Yang (eds.), Americ. Math. Society, Providence RI (2007), pp. 105-193.

[37] J. Steel and R. Van Wesep, *Two consequences of determinacy consistent with choice*, Trans. Amer. Math. Soc. **272** (1982), pp. 67-85.

[38] J. Steel, *Projectively well-ordered inner models*, Ann. Pure Appl. Logic **74** (1995), pp. 77-104.

[39] S. Todorcevic, *The power set of ω_1 and the Continuum Problem*, `http://logic.harvard.edu/Todorcevic_Structure4.pdf`

[40] M. Viale, *Category forcings, MM^{+++}, and generic absolutenes for the theory of strong forcing axioms*, J. Amer. Math. Soc. **29** (2016), pp. 675-728.

[41] W. H. Woodin, *Some consistency results in ZFC using AD*, Cabal Seminar 79-81, Lecture Notes in Math. **1019**, Springer-Verlag Berlin 1983, pp. 172-198.

[42] W. H. Woodin, *The axiom of determinacy, forcing axioms, and the non-stationary ideal*, de Gruyter, Berlin-New York 1999.

[43] W. H. Woodin, *The Continuum Hypothesis, Part I*, Notices Amer. Math. Soc. **48 (6)** (2001), pp. 567-576.

[44] W. H. Woodin, *The Continuum Hypothesis, Part II*, Notices Amer. Math. Soc. **48 (7)** (2001), pp. 681-690.

[45] W. H. Woodin, *Strong Axioms of Infinity and the Search for V*, in: Proceedings of the International Congress of Mathematicians 2010 (ICM 2010), pp. 504-528 (2011).

[46] W. H. Woodin, *In search of Ultimate-L: The 19th Midrasha Mathematicae Lectures*, `https://dash.harvard.edu/handle/1/34649600`