Martin’s Maximum++ implies Woodin’s Axiom (*)&

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Abstract
We show that Martin’s Maximum++ implies Woodin’s $P_{\text{max}}$ axiom (*). This answers a question from the 1990’s and amalgamates two prominent axioms of set theory which were both known to imply that there are $\aleph_2$ many real numbers.

1 Introduction.

Cantor’s Continuum Problem, which later became Hilbert’s first Problem (see [15]), asks how many real numbers there are. After having proved his celebrated theorem according to which $\mathbb{R}$ is uncountable, i.e., $2^{\aleph_0} > \aleph_0$, see [3], Cantor conjectured that every uncountable set of reals has the same size as $\mathbb{R}$, i.e., $2^{\aleph_0} = \aleph_1$. This statement is known as Cantor’s Continuum Hypothesis (CH). Gödel [11] proved in the 1930’s that CH is consistent with the standard axiom system for set theory, ZFC, by showing that CH holds in his constructible universe $L$, the minimal transitive model of ZFC containing all the ordinals. The axiom $V = L$, saying that the universe $V$ of all sets is simply identical with $L$, has often been rejected, however, as an undesirable minimalistic assumption about $V$. For instance, $L$ cannot have measurable cardinals by a result of Scott [32]. Gödel himself believed that CH would be shown not to follow from ZFC, and at least for part of his life he held the view that CH is indeed false and that actually

$$2^{\aleph_0} = \aleph_2.$$ (1)

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Luzin [24] proposed a related hypothesis which also refutes CH, namely

\[
2^{\aleph_0} = 2^{\aleph_1}.
\]  

(2)

That CH does not follow from ZFC was confirmed by Cohen in 1963 through the discovery of the method of forcing: Every model of ZFC can be generically extended to a model of ZFC in which CH fails, see [5]. In fact, using forcing one can show that it is relatively consistent with ZFC that the cardinality of the continuum is \(\aleph_1\), \(\aleph_2\), \(\aleph_{155}\), \(\aleph_{\omega^2+17}\), or \(\aleph_\alpha\) for many other values of \(\alpha\), see [33].

Ever since Cohen’s work, set-theorists have been searching for natural new axioms which extend ZFC and which settle the Continuum Problem (see e.g. [43], [44], [20], and the discussion in [8]). One family of such axioms is the hierarchy of large cardinal axioms. It was realized early on, though, that these axioms cannot settle the Continuum Problem: one can always force CH to hold or be false by small forcing notions, and all large cardinals which exist in \(V\) will retain their large cardinal properties in the respective extensions, see [23].

One axiom which does settle the Continuum Problem is CH itself; after all, CH looks natural in that it gives the least possible value to \(2^{\aleph_0}\) consistent with Cantor’s theorem, \(2^{\aleph_0} > \aleph_0\). CH allows the “diagonal” construction of objects of size \(\aleph_1\) with specific combinatorial properties, e.g. Luzin or Sierpiński sets. In 1985, Woodin proved his \(\Sigma^2_1\) absoluteness result conditioned on CH. Namely, if CH holds true, there is a proper class of measurable Woodin cardinals, and \(\sigma\) is a statement of the form “There is a set of reals \(X\) such that \(\varphi(X, r)\),” where \(r\) is a real and \(\varphi(X)\) is a formula of set theory all of whose quantifiers are restricted to reals, such that \(\sigma\) can be forced over \(V\), then \(\sigma\) actually holds true in \(V\), see e.g. [7, Theorem 4.1]. Over the last decade, Woodin has developed a sophisticated scenario for set theory according to which CH is true, see e.g. [45] and [46].

Nevertheless, and despite the appeal of \(\Sigma^2_1\) absoluteness, CH is often regarded as a minimalistic assumption on a par with its parent, \(V = L\). To give an illustrative example, under CH one can easily find sets \(X\) and \(Y\) of reals without endpoints which are both \(\aleph_1\)-dense—in the sense that every interval of points contains exactly \(\aleph_1\) many points—but such that \(X\) and \(Y\) are not order-isomorphic. On the other hand, by a theorem of Baumgartner [2], given any such \(X\) and \(Y\), there is a nicely behaved forcing notion which adds an order-isomorphism between \(X\) and \(Y\). Thus, adopting CH precludes the existence of sufficiently generic filters for such forcing notions—which may consistently exist.

A dual approach to CH is then to formulate axioms stipulating the existence of objects which may possibly exist, i.e., to look for “maximality principles” express-
ing some form of saturation of the universe of all sets with respect to its generic extensions. Such principles are known as forcing axioms.

Shortly after the discovery of forcing, it was realized that it is possible to iterate the process of forming generic extensions $V \subset V[g_0] \subset V[g_1] \subset \ldots \subset V[g_\alpha]$ of $V$ in any length $\alpha$ in such a way that the final model is itself a generic extension of $V$. By “closing off” one may then try to get to final models which are saturated with respect to the existence of certain (partial) generics.

The first forcing axiom shown to be consistent was Martin’s Axiom at $\omega_1$, $\text{MA}_{\omega_1}$, see [34]. $\text{MA}_{\omega_1}$ says that for every partial order $\mathbb{P}$ with the countable chain condition (i.e., there is no uncountable family of pairwise incompatible conditions in $\mathbb{P}$) and for every collection $\mathcal{D}$ of size $\aleph_1$ consisting of dense subsets of $\mathbb{P}$ there is a filter $g \subset \mathbb{P}$ which is $\mathcal{D}$-generic (i.e., is such that $g \cap D \neq \emptyset$ for each $D \in \mathcal{D}$). Over the following years, generalizations of $\text{MA}_{\omega_1}$ got isolated. This line of research culminated in the proof by Foreman-Magidor-Shelah of the consistency of Martin’s Maximum, $\text{MM}$, see [10].

Martin’s Maximum is the statement that if $\mathbb{P}$ is a forcing notion such that forcing with $\mathbb{P}$ preserves stationary subsets of $\omega_1$ in $V$ and if $\mathcal{D}$ is a collection of size $\aleph_1$ consisting of dense subsets of $\mathbb{P}$, then there is a $\mathcal{D}$-generic filter $g \subset \mathbb{P}$. $\text{MM}$ is provably maximal in the sense that the forcing axiom for any forcing notion destroying some stationary subset $S$ of $\omega_1$ necessarily fails. The conclusion of Baumgartner’s result [2] holds true under (a weakening of) $\text{MM}$, and $\text{MM}$ admits many other features of maximality.

At the same time, $\text{MM}$ can be forced by means of a forcing iteration $\mathbb{P} \subset V_\kappa$, assuming that $\kappa$ is a supercompact cardinal. The natural such forcing $\mathbb{P}$ actually produces a model of a strengthening of $\text{MM}$, called $\text{MM}^{++}$. This is the statement that if $\mathbb{P}$ is a forcing notion preserving stationary subsets of $\omega_1$ in $V$ and if $\mathcal{D}$ is a collection of size $\aleph_1$ consisting of dense subsets of $\mathbb{P}$, and $\{\tau_\alpha : \alpha < \omega_1\}$ is a collection of $\mathbb{P}$-names for stationary subsets of $\omega_1$, then there is a filter $g \subset \mathbb{P}$ which is $\mathcal{D}$-generic and which, furthermore, interprets every $\tau_\alpha$, $\alpha < \omega_1$, as a truly stationary set in $V$ (i.e., $\{\nu < \omega_1 : \exists p \in g \mid p \Vdash \nu \in \tau_\alpha\}$ is stationary for every $\alpha < \omega_1$).

Already $\text{MA}_{\omega_1}$ contradicts $\text{CH}$, and it even proves Luzin’s hypothesis (2), i.e., $2^{\aleph_0} = 2^{\aleph_1}$. More interestingly, $\text{MM}$ (in contrast to $\text{MA}_{\omega_1}$) decides the cardinality of the continuum, and in fact it confirms Gödel’s conjecture (1), $2^{\aleph_0} = \aleph_2$. This is shown by producing an affirmative answer to Friedman’s Problem under $\text{MM}$, see [10, Theorem 10].$^1$ $\text{MM}^{++}$ is—by its very definition and the fact that no strictly stronger forcing axiom can be consistent—a prototype maximality principle for $V$.

$^1$It was later verified by Moore that already the much weaker forcing axiom $\text{BPFA}$ implies (1), see [27].
Remarkably, the empirical evidence seems to suggest that \( \text{MM}^{++} \) provides a complete theory of the initial segment \( H_{\omega_2} \) of the universe of sets, at least with respect to natural questions. Here, \( H_{\omega_2} \) is the collection of all sets which are hereditarily of size \( < \aleph_2 \).

There is another maximality principle, though, which Magidor called a “competitor” of \( \text{MM} \), see [25, p. 18], and which is denoted by (\( * \)). In much the same way as \( \text{MM} \), (\( * \)) is inspired by and formulated in the language of forcing, and they both have “the same intuitive motivation: Namely, the universe of sets is rich” ([25, p. 18]). (\( * \)), introduced by Woodin in [42, Definition 5.1], is the conjunction of the following two statements.

(i) \( \text{AD} \), the Axiom of Determinacy,\(^2\) holds in \( L(\mathbb{R}) \), and

(ii) there is some \( g \) which is \( P_{\max} \)-generic over \( L(\mathbb{R}) \) such that \( P(\omega_1) \cap V \subset L(\mathbb{R})[g] \).

Item (i), that \( \text{AD} \) holds in \( L(\mathbb{R}) \), follows from the existence of large cardinals, e.g. from the existence of infinitely many Woodin cardinals with a measurable cardinal above them all, see [26, p. 91]. Item (ii) is the part of (\( * \)) which goes beyond assuming the existence of large cardinals. \( P_{\max} \) is a forcing notion which was isolated by Woodin, see [42, Definition 4.33]. It arose out of earlier work by Steel-Van Wesep [37] and by himself [41] on the size of \( \delta_{13} \) and the question if \( \text{NS}_{\omega_1} \), the nonstationary ideal on \( \omega_1 \), can be saturated.

\( P_{\max} \) consists of countable transitive structures, and membership in \( P_{\max} \) is uniformly \( \Pi^1_2 \) in the codes. \( P_{\max} \) is \( \omega \)-closed and homogeneous, see [42, Lemma 4.43]. The homogeneity of \( P_{\max} \) yields that under (\( * \)), the theory of \( L(P(\omega_1)) \) becomes part of the theory of \( L(\mathbb{R}) \). If \( \text{AD} \) holds in \( L(\mathbb{R}) \), then there is no well-order of the reals in \( L(\mathbb{R}) \) (see e.g. [30, Lemma 12.2]), but if \( g \) is \( P_{\max} \)-generic over \( L(\mathbb{R}) \), then \( \text{ZFC} \) is true in \( L(\mathbb{R})[g] \) (see [42, Theorem 4.54]), and moreover \( \text{NS}_{\omega_1} \) is saturated in \( L(\mathbb{R})[g] \) (see [42, Theorem 4.50]) and \( L(\mathbb{R})[g] \) provides an effective failure of \( \text{CH} \) in that \( \delta_{12} = \omega_2 \) is true in \( L(\mathbb{R})[g] \) (see [42, Theorem 4.53]).

Like the “classical” forcing axioms culminating with \( \text{MM}^{++} \), (\( * \)) is also a maximality principle, as it implies (and is in fact equivalent with) what is dubbed “\( \Pi^1_2 \) maximality.” A sentence \( \sigma \) (in the language of set theory, possibly augmented with some additional predicates) is said to be \( \Pi^1_2 \) if it is of the form \( \forall x \exists y \varphi(x,y) \), with \( \varphi(x,y) \) being a formula with only restricted quantifiers. There is a whole family of interesting statements which are \( \Pi^1_2 \) in the language for the structure

\[
(H_{\omega_2}; \in, \text{NS}_{\omega_1}),
\]

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\(^2\)See e.g. [30, Chapter 12].
see e.g. the discussion in [6]. The formulation of “Π₂ maximality” involves the concept of Ω-logic, see [42, Section 10.4]; for a sentence σ to be “Ω-consistent” is stronger than it just being consistent in that σ needs to be true in models which are closed under arbitrarily complicated universally Baire operations, see [42, Definition 10.144]. The Π₂ maximality theorem, see [42, Theorem 10.150], then runs as follows.

**Theorem 1.1 (Woodin)** Suppose there is a proper class of Woodin cardinals. Then the following statements are equivalent.

(1) (∗).

(2) Let σ be a Π₂ sentence in the language for the structure

\[(H_{ω_2}; ∈, NS_{ω_1}, A: A ∈ P(ℝ) ∩ L(ℝ)).\]

If σ is Ω-consistent, then σ is true.

One specific instance of σ in (2) of Theorem 1.1 is called ψ_{AC}, see [42, Definition 5.12]. It is in spirit a local version of an affirmative solution to Friedman’s Problem. Woodin showed, see [42, Theorem 5.14, Lemmata 5.15 and 5.18], that ψ_{AC} follows from both MM and (∗) and that ψ_{AC} implies Gödel’s conjecture (1), i.e. 2^{ℵ_0} = ℵ_2.

The homogeneity of \(P_{max}\) gives that in the presence of large cardinals, (∗) yields a complete theory for \(L(P(ω_1))\) modulo set-forcing: all set-generic extensions of \(V\) in which (∗) holds agree on the theory of this inner model.

Despite its nice properties, in order for (∗) to be a convincing candidate for a natural axiom, it would have to be compatible with all consistent large cardinal axioms. While \(L(ℝ)[g]\) is trivially a model of (∗), provided that \(g\) is \(P_{max}\)-generic over \(L(ℝ)\), Scott’s result [32] carries over from \(L\) to \(L(ℝ)[g]\) and shows that \(L(ℝ)[g]\) cannot have measurable cardinals either. [42] and subsequent work had left open the problem if (∗) would be compatible with large cardinals beyond the level of Woodin cardinals.

Prior to the current paper, the relation between classical forcing axioms like MM, which could be forced by iterated forcing over models of ZFC with large cardinals, and the axiom (∗), whose models were obtained by forcing over models satisfying the Axiom of Determinacy, remained a complete mystery. It had been known by a result of P. Larson [21] that even \(MM^{+ω}\), an axiom strictly between MM and \(MM^{++}\), does not imply (∗).³ One can build models of \(MM^{+ω}\) with a well-order of \(H_{ω_2}\) which is

³ \(MM^{+ω}\) is the strengthening of MM obtained by replacing, in the formulation of \(MM^{++}\), collections of \(ℵ_1\)-many names for stationary sets with collections of only countably many such names.
definable over $(H_{\omega_2}; \in)$ by a formula without parameters, and the existence of such a well-order is incompatible with ($*$) by the homogeneity of $\mathbb{P}_{\text{max}}$. It remained even unclear whether classical strong forcing axioms would be compatible at all with ($*$), see [42, p. 846]. See also [42, Question (18) a) on p. 924], [25, Conjecture 6.8 on p. 19], and [28, Problem 14.7].

This paper resolves the tension between MM and ($*$). We prove:

**Theorem 1.2** Assume Martin’s Maximum++. Then Woodin’s $\mathbb{P}_{\text{max}}$-axiom ($*$) holds true.

In particular, MM and ($*$) are compatible with one another, and ($*$) is compatible with all consistent large cardinal axioms: If $\kappa$ is a supercompact cardinal and $\mathbb{P} \subset V_\kappa$ is the partial order from [10] to force MM$^{++}$, then by Theorem 1.2 the axiom ($*$) holds in $V^\mathbb{P}$, but all the large cardinals of $V$ above $\kappa$ are preserved by $\mathbb{P}$.

It also follows from Theorem 1.2 that MM$^{++}$ implies (2) of Theorem 1.1, and that ($*$) can be characterized, in the presence of large cardinals, by a statement which on the face of its formulation is weaker than (2) of Theorem 1.1. We will prove the following theorem at the end of the next section.

**Theorem 1.3** Suppose there is a supercompact cardinal. Then the following statements are equivalent.

(1) ($*$).

(2) Let $\sigma$ be a $\Pi_2$ sentence in the language for the structure

$(H_{\omega_2}; \in, \text{NS}_{\omega_1}, A: A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$.

If there is a stationary set preserving forcing $\mathbb{P}$ such that $\sigma$ holds in $V^\mathbb{P}$, then $\sigma$ is true in $V$.

This equivalence of ($*$) is a variant of one which we are going to state below, see Theorem 2.10, and which characterizes ($*$) as a strong version of a bounded forcing axiom.

In 1947, Gödel [12] had written:

[...] one may on good reason suspect that the role of the continuum problem in set theory will be this, that it will finally lead to the discovery of new axioms which will make it possible to disprove Cantor’s conjecture.

In light of our unifying result, Theorem 1.2, one could make the case that with MM$^{++}$ and ($*$), natural and strong such axioms have already been found.
2 Preliminaries.

Let us first restate \( \text{MM}^{++} \). Martin’s Maximum\(^{++} \), abbreviated by \( \text{MM}^{++} \) and isolated by Foreman-Magidor-Shelah \[10\] (cf. also \[42, Definition 2.45 (2)\]). \( \text{MM}^{++} \) is the statement that if \( P \) is a forcing which preserves stationary subsets of \( \omega_1 \), if \( \{D_i : i < \omega_1\} \) is a collection of dense subsets of \( P \), and if \( \{\tau_i : i < \omega_1\} \) is a collection of \( P \)-names for stationary subsets of \( \omega_1 \), then there is a filter \( g \subset P \) such that for every \( i < \omega_1 \),

1. \( g \cap D_i \neq \emptyset \) and
2. \( \left( \tau_i \right)^g = \{ \xi < \omega_1 : \exists p \in g P \vdash \xi \in \tau_i \} \) is stationary.

The forcing \( P_{\text{max}} \) was designed by W. Hugh Woodin, see \[42, Chapter 4\]. The conditions in \( P_{\text{max}} \) (see \[42, Definition 4.33\]) are countable transitive models of a sufficiently large fragment of \( \text{ZFC} \) plus \( \text{MA}_{\omega_1} \) of the form \( (M; \in, I, a) \), where

1. \( (M; I) \) is amenable\(^4 \) and \( (M; I) \models \text{“} I \text{ is a normal uniform ideal on } \omega_1 \text{”} \),
2. \( a \in \mathcal{P}(\omega_1^M) \cap M \) and \( M \models \omega_1 = \omega_1^{L[a,x]} \) for some real \( x \), and
3. \( (M; \in, I) \) is generically iterable.

The last item, generic iterability, refers to generic iterations of \( (M; \in, I) \). Given an ordinal \( \gamma \leq \omega_1 \), \( \langle \langle (M_\alpha; \in, I_\alpha) : \alpha \leq \gamma \rangle \rangle, \langle \pi_{\alpha,\beta} : \alpha \leq \beta \leq \gamma \rangle, \langle g_\alpha : \alpha < \gamma \rangle \rangle \) is a generic iteration of \( (M; \in, I) \) if the following hold true.

- \( (M_0; \in, I_0) = (M; \in, I) \),
- for \( \alpha < \gamma \), \( g_\alpha \) is a \( \mathcal{P}(\omega_1^{M_\alpha}) \setminus I_\alpha \)-generic filter over \( M_\alpha \), \( M_{\alpha+1} \) is the ultrapower of \( M_\alpha \) by \( g_\alpha \), and \( \pi_{\alpha,\alpha+1} : (M_\alpha; \in, I_\alpha) \rightarrow (M_{\alpha+1}; \in, I_{\alpha+1}) \) is the corresponding generic elementary embedding,
- \( \pi_{\alpha_0,\alpha_2} = \pi_{\alpha_1,\alpha_2} \circ \pi_{\alpha_0,\alpha_1} \) for all \( \alpha_0 \leq \alpha_1 \leq \alpha_2 \), and
- if \( \beta \) is a nonzero limit ordinal \( \leq \gamma \), then \( (M_\beta, (\pi_{\alpha,\beta} : \alpha < \beta)) \) is the direct limit of \( (M_\alpha, \pi_{\alpha,\alpha'} : \alpha \leq \alpha' < \beta) \).

\( (M; \in, I) \) being generically iterable means that all models in any generic iteration iteration of \( (M; \in, I) \) are well-founded, irrespective of the filters \( g_\alpha \) chosen at any stage \( \alpha \), see \[42, Definition 4.1\].

\(^4\)i.e., \( x \cap I \in M \) for all \( x \in M \)
The current paper will only consider such generic iterations rather than iterations of mice as being studied in inner model theory. 

\( \mathbb{P}_{\text{max}} \) becomes a partial order by declaring \((N; \in, J, b) < (M; \in, I, a)\), “\((N; \in, J, b)\) is stronger than \((M; \in, I, a)\),” iff \((M; \in, I, a) \in N\) and inside \(N\) there is a generic iteration of \((M; \in, I, a)\) of length \(\omega_1^N + 1\) with last model \((M^*; \in, I^*, a^*)\) such that \(I^* = J \cap M^*\) and \(a^* = b\).

Most of [42] studies the effect of forcing with \(\mathbb{P}_{\text{max}}\) or variants thereof over a model of the Axiom of Determinacy. Let us restate Woodin’s \(\mathbb{P}_{\text{max}}\) axiom (\(\ast\)), see [42, Definition 5.1]. (\(\ast\)) says that

(i) AD holds in \(L(\mathbb{R})\) and

(ii) there is some \(g\) which is \(\mathbb{P}_{\text{max}}\)-generic over \(L(\mathbb{R})\) such that \(\mathcal{P}(\omega_1) \cap V \subset L(\mathbb{R})[g] \).

Already PFA, the Proper Forcing Axiom, which is weaker than \(\text{MM}^{++}\), implies \(\text{AD}^{L(\mathbb{R})}\) and much more, see [35], [17], and [29, Chapter 12].

The current paper produces a proof of Theorem 1.2. Our key new idea is (\(\Sigma.8\)) on page 20 below. We try to give an overview of the proof of Theorem 1.2 at the end of this section.

Theorem 1.2 is optimal in that P. Larson [21] and [22] has shown that \(\text{MM}^{+\omega}\) is consistent with \(\neg(\ast)\) relative to a supercompact limit of supercompact cardinals. Our proof is also optimal in that the forcing which we will use to verify Theorem 1.2 has size \(2^{\aleph_2}\), while Woodin has shown that \(\text{MM}^{++}\) for forcings of size \(2^{\aleph_0}\) does not imply \(\ast\), see [42, Theorem 10.90], and it is consistent with \(\text{MM}^{++}\) that \(2^{\aleph_2} = \aleph_3\).

Throughout our entire paper, “\(\omega_1\)” will always denote \(\omega_1^V\), the \(\omega_1\) of \(V\).

Convention 2.1 Let us fix throughout this paper some \(A \subset \omega_1\) such that \(\omega_1^{L[A]} = \omega_1\). Let us define \(g_A\) as the set of all \(\mathbb{P}_{\text{max}}\) conditions \(p = (N; \in, I, a)\) such that there is a generic iteration

\( (N_i, \sigma_{ij} : i \leq j \leq \omega_1) \)

of \(p = N_0\) of length \(\omega_1 + 1\) such that if we write \(N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)\), then \(I^* = (\text{NS}_{\omega_1})^V \cap N_{\omega_1}\) and \(a^* = A\).

Lemma 2.2 (Woodin) Assume that \(\text{NS}_{\omega_1}\) is saturated and that \(\mathcal{P}(\omega_1)\)# exists.

(1) \(g_A\) is a filter. 

\[5\text{Here and elsewhere we often confuse a model with its underlying universe.}\]
If $g_A$ is $\mathbb{P}_{\text{max}}$-generic over $L(\mathbb{R})$, then $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[g]$.

**Proof.** This routinely follows from the proof of [42, Lemma 3.12 and Corollary 3.13] and from [42, Lemma 3.10]. □

One may also use BMM (Bounded Martin’s Maximum) plus “$\text{NS}_{\omega_1}$ is precipitous” to show that $g_A$ is a filter, this is by the proof from [4].

Let $1 \leq k < \omega$, and let $D \in \mathcal{P}(\mathbb{R}^k)$. We say that the trees $T$ and $U$ on $k \omega \times \text{OR}$ witness that $D$ is universally Baire iff $D = p[T]$ and for all posets $\mathbb{P}$,

$$\forall \mathbb{P} \ p[U] = \mathbb{R}^k \setminus p[T].$$  \hfill (3)

$D$ is called universally Baire iff there are trees $T$ and $U$ witnessing that $D$ is universally Baire. The concept of universally Baire set was isolated by Feng-Magidor-Woodin in [9, Section 2]; see also [30, Definition 8.6].

Let us denote by $\Gamma^\infty$ the collection of all $D \in \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k)$ which are universally Baire. If $D \in \Gamma^\infty$, then there is an unambiguous version of $D$ in any forcing extension $V[g]$ of $V$, which as usual we denote by $D^*$ and which is equal to $p[T] \cap V[g]$ for some/all trees $T$ and $U$ which witness that $D$ is universally Baire. See [30, p. 149f.].

**Definition 2.3** Let $\Gamma \subset \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k)$. We say that $\Gamma$ is productive iff

(a) $\Gamma \subset \Gamma^\infty$,

(b) for all $k < \omega$ and all $D \in \Gamma \cap \mathcal{P}(\mathbb{R}^{k+1})$, $\mathbb{R}^{k+1} \setminus D \in \Gamma$ and if $k > 0$, then $\exists \mathbb{R}D = \{(x_0, \ldots, x_{k-1}) : \exists x_k(x_0, \ldots, x_{k-1}, x_k) \in D\} \in \Gamma$, and

(c) for all $k < \omega$ and all $D \in \Gamma \cap \mathcal{P}(\mathbb{R}^{k+2})$, if the trees $T$ and $U$ on $k+2 \omega \times \text{OR}$ witness that $D$ is universally Baire and if

$$\tilde{U} = \{(s \upharpoonright (k+1), (s(k+1), t)) : (s, t) \in U\},$$  \hfill (4)

then there is a tree $\tilde{T}$ on $k+1 \omega \times \text{OR}$ such that for all posets $\mathbb{P}$,

$$\forall \mathbb{P} \ p[\tilde{U}] = \mathbb{R}^{k+1} \setminus p[\tilde{T}].$$  \hfill (5)

Hence a pointclass consisting of universally Baire sets is productive iff it is closed under complements and projections in the sense of (b) of Definition 2.3 and for all $k < \omega$ and $D \in \Gamma \cap \mathbb{R}^{k+2}$,

$$\text{(3R}D)^* = \{\bar{x} \in \mathbb{R}^{k+1} : \exists y \in \mathbb{R} (\bar{x}, y) \in D^*\}$$  \hfill (6)
will be true in every generic extension. On the other hand, at least on the face of its
definition, \( \Gamma \) being productive is stronger than having that \( \Gamma \subset \Gamma^\infty \) and \( \Gamma \) is closed
under complements and projections (this is exactly [9, Question 3]).

If \( \Gamma \) is productive and if \( D \in \Gamma \), then any projective statement about \( D \) is absolute
between \( V \) and any forcing extension of \( V \), i.e., if \( \varphi \) is projective, \( x_1, \ldots, x_k \in \mathbb{R} \),
and \( \mathbb{P} \) is any poset, then

\[
V \models \varphi(x_1, \ldots, x_k, D) \iff \|\mathbb{P}\| \varphi(\bar{x}_1, \ldots, \bar{x}_k, D^*).
\]

This is shown by a trivial induction on the complexity of \( \varphi \). [9, Question 3] is
cconcerned with the question about the connection of, on the one hand, projective
absoluteness with respect to forcing extensions and, on the other hand, having that
every projective set is universally Baire (see [9, Questions 1 and 7]).

**Theorem 2.4 (Woodin)** Assume that there is a proper class of Woodin cardinals.
Then \( \Gamma^\infty \) is productive.

**Proof.** This is true by a theorem of Woodin, see e.g. [36, Theorem 1.2], combined
with the key result of Martin and Steel from [26]. \( \square \)

For any set \( X \), \( M_\#(X) \) denotes the least active \( X \)-mouse which has infinitely
many Woodin cardinals. See [38].

**Theorem 2.5 (Steel)** Assume PFA. Then the universe is closed under the operation
\( X \mapsto M_\#(X) \). In particular, every set of reals in \( L(\mathbb{R}) \) is universally Baire, and
\( \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\mathbb{R}) \) is productive.

**Proof.** The proof from [35] produces the result that under PFA, the universe is
closed under the operation \( X \mapsto M_\#(X) \). The rest is given by standard inner model
theoretic arguments, see e.g. [31, Section 3, pp. 187f.]. \( \square \)

By Lemma 2.2 and Theorem 2.5, Theorem 1.2 follows from the following more
general statement.

**Theorem 2.6** Let \( \Gamma \subset \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k) \). Assume that

1. \( \Gamma = \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R}) \),

2. \( \Gamma \) is productive, and

3. Martin’s Maximum++ holds true.
Then $g_A$ is $\mathbb{P}_{\text{max}}$-generic over $L(\Gamma, \mathbb{R})$.

The abbreviation $(\ast)_\Gamma$ was introduced in [31, Definition 4.1] to denote a straightforward generalization of $(\ast)$ to larger pointclasses.

**Corollary 2.7** Assume that there is a proper class of Woodin cardinals. Let $\Gamma \subset \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k) \cap L^\infty(\Gamma, \mathbb{R})$. Suppose that (i)-(iii) from the statement of Theorem 2.6 are satisfied. Then $(\ast)_\Gamma$ holds true.

Let $e: \mathbb{R} \to \text{HC}$ be a fixed simple coding of hereditarily countable sets by reals, see e.g. [31, p. 179]. A set $D \subset \text{HC}$ is then called universally Baire in the codes iff the code set $\{x \in \mathbb{R} : e(x) \in D\}$ of $D$ is universally Baire. If this is the case, then every forcing extension of $V$ will have its unique new version of $D$, which we denote by $D^*$. If the code set of $D$ is a member of a productive pointclass, then for every forcing $\mathbb{P}$,

$$(\text{HC}; \in, D) \prec (\text{HC}^{V^\mathbb{P}}; \in, D^*).$$  \hfill (7)

Here and in what follows, $(M; \in, \ldots) \prec (N; \in, \ldots)$ means that $(M; \in, \ldots)$ is an elementary substructure of $(N; \in, \ldots)$, i.e., $M \subset N$, the two models have the same first order language associated to it, and for all formulae $\varphi$ of that common language and all $x_1, \ldots, x_k \in M$,

$$(M; \in, \ldots) \models \varphi(x_1, \ldots, x_k) \iff (N; \in, \ldots) \models \varphi(x_1, \ldots, x_k).$$

Theorem 2.6 readily follows from the following Lemma via a standard application of $\text{MM}^{++}$.

**Lemma 2.8** Let $\Gamma \subset \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k)$. Assume that

(i) $\Gamma = \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) $\Gamma$ is productive, and

(iii) $\text{NS}_{\omega_1}$ is saturated.

Let $D \subset \mathbb{P}_{\text{max}}$ be open dense, $D \in L(\Gamma, \mathbb{R})$. There is then a stationary set preserving forcing $\mathbb{P}$ of size $2^{\aleph_2}$ such that in $V^\mathbb{P}$ there is some $p = (N; \in, I, a) \in D^*$ and some generic iteration

$$(N_i, \sigma_{ij} : i \leq j \leq \omega_1)$$

of $p = N_0$ of length $\omega_1 + 1$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, then $I^* = (\text{NS}_{\omega_1})^{V^\mathbb{P}} \cap N_{\omega_1}$ and $a^* = A$. 

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Proof of Theorem 2.6 from Lemma 2.8. MM implies that $\text{NS}_{\omega_1}$ is saturated, see [10, Theorem 12]. By Lemma 2.2, it remains to show that $D \cap g_A \neq \emptyset$ for every open dense $D \subset P_{\text{max}}, D \in \Gamma$.

Let us fix such $D$. The statement that there is a $p$ as in the conclusion of Lemma 2.8, which is tantamount to saying that there is a $p \in D \cap g_A$, is easily seen to be $\Sigma_1$ expressible over the structure $(H_{\omega_2}; \in, \text{NS}_{\omega_1}, A, D)$. By the conclusion of Lemma 2.8, the existence of such a $p$ may be forced by a stationary set preserving forcing. Hence by $\text{MM}^{++}$, cf. [42, Theorem 10.124], there is such a $p$ in $V$. \hfill $\Box$

The attentive reader will notice that with virtually the same proof that is to come we could weaken the hypothesis (iii) of Lemma 2.8 to “$\text{NS}_{\omega_1}$ is precipitous” at the cost of increasing the size of $\mathbb{P}$ to become $(2^{2^{\omega_1}})^+$. The same remark applies to hypothesis (iii) in Theorem 2.9. We decided to present our proof working under the stronger hypothesis, though, as doing so simplifies the notation, and in practice the weaker versions of Lemma 2.8 and Theorem 2.9 are as interesting as their stronger counterparts.

As the proof of Theorem 2.6 from Lemma 2.8 shows, we don’t need the full power of $\text{MM}^{++}$ in order to derive Theorem 2.6 from Lemma 2.8, the hypothesis that $D$-$\text{BMM}^{++}$ holds true for all $D \in \Gamma$ would suffice, see [42, Definition 10.123]. (*) is then actually equivalent to a version of $\text{BMM}$. Let us first state a more general fact, Theorem 2.9, which gives the characterization of (*), i.e. Theorem 2.10, as a special case.

**Theorem 2.9** Assume that there is a proper class of Woodin cardinals. Let $\Gamma \subset \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k)$. Assume that

(i) $\Gamma = \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) $\Gamma$ is productive.

The following statements are then equivalent.

(1) $D$-$\text{BMM}^{++}$ holds true for all $D \in \Gamma$.

(2) $g_A$ is $P_{\text{max}}$-generic over $L(\Gamma, \mathbb{R})$.

Proof. (2) $\implies$ (1): This is exactly by the proof of (A) $\implies$ (B) of [1, Theorem 2.7].

(1) $\implies$ (2): We may first force $\text{NS}_{\omega_1}$ to be saturated by a stationary set preserving forcing, see e.g. [42, Theorem 2.64]. The rest is then by the proof of Theorem 2.6 from Lemma 2.8 which was given above. \hfill $\Box$
Theorem 2.10 Assume that there is a proper class of Woodin cardinals. The following statements are then equivalent.

(1) $D$-BMM$^{++}$ holds true for all $D \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$.

(2) $\ast$.

Let us now give a Proof of Theorem 1.3. $(1) \implies (2)$ is weaker than $(1) \implies (2)$ of Theorem 1.1. Let us now assume (2) and show (1). Fix $D \subset \mathbb{P}_{\text{max}}$, any open dense set in $L(\mathbb{R})$. As the statement of the theorem assumes a supercompact cardinal to exist, there is a semi-proper (and hence stationary set preserving) forcing $\mathbb{P}$ such that MM$^{++}$ holds true in $V^\mathbb{P}$. Inside $V^\mathbb{P}$, we will have that (the proof of) Lemma 2.8 that for all $A' \subset \omega_1$ with $\omega_1^{L[A']} = \omega_1$ there will be some $p = (N; \epsilon, I^*, a^*) \in D^*$ and some generic iteration $(N_i, \sigma_{ij}; i \leq j \leq \omega_1)$ of $p = N_0$ of length $\omega_1 + 1$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \epsilon, I^*, a^*)$, then $I^* = (\text{NS}_{\omega_1})^{V^\mathbb{P}} \cap N_{\omega_1}$ and $a^* = A'$. This is a statement which is $\Pi_2$ over the structure mentioned in (2). This statement will therefore be true in $V$, which readily implies that $g_A$ is $\mathbb{P}_{\text{max}}$-generic over $L(\mathbb{R})$ and $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[g_A]$. \hfill \Box

The forcing which we designed in order to produce Lemma 2.8 is a souped up version of the forcings from [4] and [6], which are in turn variants of the $\mathcal{L}$-forcing of Jensen as being developed e.g. in [16].\textsuperscript{6} All these forcings may be construed as building uncountable models as term models of a given language, $\mathcal{L}$, with the forcing conditions being finite fragments of a consistent and complete $\mathcal{L}$-theory which will give those term models, augmented by “side conditions” which will guarantee that the forcing only collapses cardinals in a controlled way. Our forcing will change the cofinalities of $\omega_2$ and $\omega_3$ to $\omega$ and $\omega_1$, respectively, and does not collapse any other cardinal outside of the (possibly empty) half-open interval $(\omega_3, 2^{\aleph_2}]$. To prove Lemma 2.8, we aim to build a stationary set preserving forcing $\mathbb{P}$ which adds a generic iteration of some $\mathbb{P}_{\text{max}}$-condition $(N; \epsilon, I, a)$ coded by a real in the projection of a tree $T$ projecting to the set of codes for conditions in our given dense set $D$. Moreover, we want this iteration to send the distinguished set $a$ of $(N; \epsilon, I, a)$ to $A$, and we want every $I^*$-positive set in the final model $(N^*; \epsilon, I^*, A)$ to be a stationary subset of $\omega_1$ in $V^\mathbb{P}$. Our approach is to think of all the relevant

\textsuperscript{6}The referee informs us that J. Keisler in [18] and [19] developed forcings which work in a similar fashion.
objects – \((N; \in, I, a)\), a branch through \(T\) projecting to a real coding \((N; \in, I, a)\), and the generic iteration of \((N; \in, I, a)\) of length \(\omega_1 + 1\) – as being given by “term models” in a suitable language, \(\mathcal{L}\), and add them via finite approximations. Thus, the working parts of our forcing will be finite sets \(p\) of sentences from \(\mathcal{L}\) providing partial information about the above objects. We will require these finite bits of information \(p\) to be realized in some outer model. The existence of such an outer models will be absolute to any generic extensions of \(V\) via \(\text{Col}(\omega, \omega_2)\).

In \(V^{\text{Col}(\omega, \omega_2)}\),
\[
(H^V_{\omega_2}; \in, \text{NS}^V_{\omega_1}, A)
\]
becomes a \(\mathbb{P}_{\text{max}}\)-condition, and \(p[T] = D^*\) is still dense, so that in \(V^{\text{Col}(\omega, \omega_2)}\) there is a \(\mathbb{P}_{\text{max}}\)-condition \((N; \in, I, a) \in D^*\) which is stronger than \((H^V_{\omega_2}; \in, \text{NS}^V_{\omega_1}, A)\). We may now iterate \((N; \in, I, a)\) in length \(\omega_1^{\text{Col}(\omega, \omega_2)} + 1 = \omega_3^V + 1\) so as to produce
\[
\sigma: (N; \in, I, a) \rightarrow (N^*; \in, I^*, a^*)
\]
If \((M_i, \pi_{ij}: i \leq j \leq \omega_1^N) \in N\) is the generic iteration of \((H^V_{\omega_2}; \in, \text{NS}^V_{\omega_1}, A)\) witnessing that \((N; \in, I, a)\) is stronger than \((H^V_{\omega_2}; \in, \text{NS}^V_{\omega_1}, A)\), then \(j((M_i, \pi_{ij}: i \leq j \leq \omega_1^N)) = (M_i, \pi_{ij}: i \leq j \leq \omega_3^V)\) is an extension of that iteration such that
\[
\pi_{\omega_1^V, \omega_3^V}: M_{\omega_1^V} = (H^V_{\omega_2}; \in, \text{NS}^V_{\omega_1}, A) \rightarrow M_{\omega_3^V}
\]
may be lifted to a generic iteration
\[
\tilde{\pi}: V \rightarrow M
\]
of \(V\) in a way that \(M\) is transitive. One can now verify that \(V^{\text{Col}(\omega, \omega_2)}\) contains the objects we intend to approximate, albeit not defined relative to the parameters \(T\) and \(A\), but relative to \(\tilde{\pi}(T)\) and \(\tilde{\pi}(A)\). By absoluteness, such objects will also exist in \(M^{\text{Col}(\omega, \tilde{\pi}(\omega_2))}\), so that by pulling back via \(\tilde{\pi}\), the objects we intend to approximate, this time with the right parameters \(T\) and \(A\), will exist in \(V^{\text{Col}(\omega, \omega_2)}\).

For technical reasons, not only the objects we are ultimately interested in obtaining, but also the iteration \((M_i, \pi_{ij}: i \leq j \leq \omega_1^N) \in N\) will be among the objects our forcing \(\mathbb{P}\) will need to approximate. We will think of the objects themselves, which exist in \(V^{\text{Col}(\omega, \omega_2)}\), as “certificates” of some finite piece of information about them. The idea is then to have our forcing consist of finite sets of \(\mathcal{L}\)-sentences for which there is a “certificate” in \(V^{\text{Col}(\omega, \omega_2)}\).

The problem with the above strategy is that, although a forcing \(\mathbb{P}\) like the one we have described would in fact add the desired objects, one would still need to show that it preserves stationary subsets of \(\omega_1\) and that every positive set in the final
model of the iteration being added by $\mathbb{P}$ is in fact stationary in that extension. Our forcing $\mathbb{P}$ will be a subset of $H_{\omega_3}$, and one tool for taking care of these issues is the use of a diamond sequence $\langle (Q_\lambda, A_\lambda) : \lambda < \omega_3 \rangle$ consisting of transitive structures in $H_{\omega_3}$ in order to guess $(H_{\omega_3}, \dot{C})$, where $\dot{C}$ is a $\mathbb{P}$-name for a club in $\omega_1$, $\dot{C} \subset H_{\omega_3}$. That $(H_{\omega_3}, \dot{C})$ be guessed means that there are stationarily many $\lambda < \omega_3$ such that $(Q_\lambda, A_\lambda)$ is an elementary substructure of $(H_{\omega_3}, \dot{C})$.

Imagine that $\mathbb{P}$ is a forcing which adds the desired objects, but which also preserves stationary subsets of $\omega_1$. Let $\dot{C}$ be a $\mathbb{P}$-name for a club in $\omega_1$, $\dot{C} \subset H_{\omega_3}$, and let $S \subset \omega_1$ be stationary in $V$. Let $g$ be $\mathbb{P}$-generic over $V$. There will be some $\lambda < \omega_3$ such that $(H_{\omega_3}, \dot{C})$ is guessed by $(Q_\lambda, A_\lambda)$ and in $V[g]$ there will be some countable elementary substructure $X$ of $(Q_\lambda, A_\lambda)$ such that

(a) $X \cap \omega_1 \in S$, and

(b) $X[g] \cap Q_\lambda = X \cap Q_\lambda$.

Here, $X[g] = \{ \tau^g : \tau \in V^\mathbb{P} \cap X \}$. That $(Q_\lambda, A_\lambda)$ is an elementary substructure of $(H_{\omega_3}, \dot{C})$ will then mean in practice that $X \cap \omega_1 \in \dot{C}^g$, so that $X \cap \omega_1$ witnesses that $S$ is still stationary in $V[g]$. Calling some $g$ with (b) “$(\mathbb{P}, X)$-generic” there is, however, no reason to expect an $X$ such that $g$ is $(\mathbb{P}, X)$-generic to exist in $V$ (in fact, $V$ won’t have such $X$).

When defining $\mathbb{P}$, we will turn this around and have our conditions also approximate finite bits of information about such elementary substructures $X$.

Our key tool for taking care of the above issue is then to define $\mathbb{P}$ as the last forcing from a recursively defined $\subset$-increasing sequence $\vec{\mathbb{P}} = (\mathbb{P}_\lambda : \lambda \leq \omega_3)$. Each $\mathbb{P}_\lambda$ will be a subset of $Q_\lambda$. Hence, when defining $\mathbb{P}_\lambda$, $\lambda \leq \omega_3$, if $\delta < \lambda$, then we already know what it means for some $g$ to be (partially) $\mathbb{P}_\delta$-generic over $(Q_\delta, A_\delta)$, and if $X$ is a countable elementary substructure of $(Q_\delta, A_\delta)$, then $X[g]$ may be assigned a meaningful interpretation as $X[g] = \{ \tau^g : \tau \in V^\mathbb{P}_\delta \cap X \}$. We will maintain that at each stage $\lambda$ in the construction of $\vec{\mathbb{P}}$ we define $\mathbb{P}_\lambda$ by saying that a finite set $p$ of $\mathcal{L} \cap Q_\lambda$-sentences is in $\mathbb{P}_\lambda$ if and only if there is a certificate for $p$ which, when intersected with each of the side conditions $X_\delta$ (also given by the certificate), for $\delta < \lambda$, is generic over $X_\delta$ for the already defined forcing $\mathbb{P}_\delta$.

Our next section is entirely devoted to a proof of Lemma 2.8.

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3 The forcing.

Recall our Convention 2.1. Let us assume throughout the hypotheses of Lemma 2.8. We aim to verify its conclusion.

Let us fix $D \subset \mathbb{P}_{\text{max}}$, an open dense set in $L(\Gamma, \mathbb{R})$. By hypotheses (ii) in the statement of Lemma 2.8 (i.e., $\Gamma \subset \Gamma^\infty$ is productive) we will have that

$$V^{\text{Col}(\omega, \omega_2)} \models "D^* \text{ is an open dense subset of } \mathbb{P}_{\text{max}}."$$

Let us identify $D$ with a canonical set of reals coding the elements of $D$,\footnote{We will later have to spell out a bit more precisely in which way we aim to have the elements of $p[T]$ code the elements of $D$, see (C.2) and (Σ.5) below.} and let $\tilde{T} \in V$ be a tree on $\omega \times 2^{\aleph_2}$ such that

$$V^{\text{Col}(\omega, \omega_2)} \models D^* = p[\tilde{T}].$$

Let $h$ be $\text{Col}(\omega, \omega_2)$-generic over $V$. Inside $V[h]$,

$$((H_{\omega_2})^V; \in, (\text{NS}_{\omega_1})^V, A)$$

is a $\mathbb{P}_{\text{max}}$ condition, call it $p$. Let $q^* \in (\mathbb{P}_{\text{max}})^{V[h]}, q^* < p, q^* \in D^*$, cf. (D.1). Let $q^* = (N^*; \in, I^*, a^*)$. Identifying $q^*$ with some real coding it, we have that $q^* \in p[\tilde{T}]$, cf. (D.2). Let $T \in V$ be a tree on $\omega \times \omega_2$ such that

$$q^* \in p[T] \subset p[\tilde{T}].$$

Let us write

$$\kappa = \aleph_3,$$

so that $T \in H_\kappa$. Let $d$ be $\text{Col}(\kappa, \kappa)$-generic over $V$. In $V[d]$, let $(\bar{A}_\lambda; \lambda < \kappa)$ be a $\Diamond_\kappa$-sequence, i.e., for all $\bar{A} \subset \kappa$, $\{\lambda < \kappa; \bar{A} \cap \lambda = \bar{A}_\lambda\}$ is stationary. Also, let $c: \kappa \to H_\kappa^V = H_\kappa^{V[d]}, c \in V[d]$, be bijective. For $\lambda < \kappa$, let

$$Q_\lambda = c"\lambda \text{ and } A_\lambda = c"\bar{A}_\lambda.$$
(iii) \( Q_\lambda \cap \text{OR} = \lambda \) (so that \( c \upharpoonright \lambda : \lambda \to Q_\lambda \) is bijective), and

(iv) \( (Q_\lambda ; \in) \prec (H_\kappa; \in) \).

We will fix a \( C \) with the above properties from now on.

In \( V[d] \), for all \( P, B \subset H_\kappa \), the set of all \( \lambda \in C \) such that

\[
(Q_\lambda; \in, P \cap Q_\lambda, B \cap Q_\lambda) \prec (H_\kappa; \in, P, B)
\]

is club, and the set of all \( \lambda \in C \) such that \( B \cap Q_\lambda = A_\lambda \) is stationary, so that

\[
(\diamond) \text{ For all } P, B \subset H_\kappa \text{ the set } \{ \lambda \in C : (Q_\lambda; \in, P \cap Q_\lambda, A_\lambda) \prec (H_\kappa; \in, P, B) \}
\]

is stationary.

We shall sometimes also write \( Q_\kappa = H_\kappa \). It is easy to see that the principle which we refer to as \( (\diamond) \) is actually equivalent to \( \diamond_\kappa \). We shall use \( (\diamond) \) to guess information about names for club subsets of \( \omega_1 \); this will play a crucial role in the verification that our forcing preserves stationary subsets of \( \omega_1 \).

We shall now go ahead and produce a stationary set preserving forcing \( \mathbb{P} \in V[d] \) of size \( \kappa \) which adds some \( p \in D^* \) and some generic iteration

\[
(N_i, \sigma_{ij} : i \leq j \leq \omega_1)
\]

of \( p = N_0 \) such that if we write \( N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*) \), then \( I^* = (\text{NS}_{\omega_1})^{V[d]^p} \cap N_{\omega_1} \) and \( a^* = A \). As the forcing \( \text{Col}(\kappa, \kappa) \) which added \( d \) is certainly stationary set preserving, this will verify Lemma 2.8.

\( \text{NS}_{\omega_1} \) is still saturated in \( V[d] \). This is true simply because forcing with \( \text{Col}(\kappa, \kappa) \) doesn’t add any sequences of elements of \( V \) of length \( \aleph_2 \). Moreover, (D.1) and (D.2) are true with \( V \) being replaced by \( V[d] \), so that in order to simplify our notation, we shall in what follows confuse \( V[d] \) with \( V \), i.e., pretend that, in addition to “\( \text{NS}_{\omega_1} \) is saturated” plus (D.1) and (D.2), \( (\diamond) \) is also true in \( V \).

Working under these hypotheses, we shall now recursively define a \( \subset \)-increasing and continuous chain of forcings \( \mathbb{P}_\lambda \) for all \( \lambda \in C \cup \{ \kappa \} \). The forcing \( \mathbb{P} \) will be \( \mathbb{P}_\kappa \). Each \( \mathbb{P}_\lambda \) will consist of a finite set of formulae of an associated first order language, \( \mathcal{L}_\lambda \), which will be defined below. The order of each \( \mathbb{P}_\lambda \) will be just reverse inclusion, i.e., \( q \leq p \) if and only if \( q \supset p \) for \( p, q \in \mathbb{P}_\lambda \).

Assume that \( \lambda \in C \cup \{ \kappa \} \) and \( \mathbb{P}_\mu \) has already been defined in such a way that \( \mathbb{P}_\mu \subset Q_\mu \) for all \( \mu \in C \cap \lambda \).
We shall be interested in objects \( \mathcal{C} \) which exist in some outer model\(^8\) and which have the following properties.

\[
\mathcal{C} = \langle \langle M_i, \pi_{ij}, N_i, \sigma_{ij}, : i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) : n < \omega \rangle, \langle \lambda_\delta, X_\delta : \delta \in K \rangle \rangle, \tag{11}
\]

where

\begin{itemize}
  \item[(C.1)] \( M_0, N_0 \in \mathbb{P}_{\text{max}} \),
  \item[(C.2)] \( x = \langle k_n : n < \omega \rangle \) is a real code for \( N_0 = (N_0; \in, I, a) \), by which we mean that there is some surjection \( f : \omega \to N_0 \) such that \( x \) is the monotone enumeration of the Gödel numbers of all expressions of the form \( \gamma \mathcal{N}_0 \models \varphi(n_1, \ldots, n_\ell, \dot{a}, \dot{I}) \) with \( \varphi \) being a first order formula of the language associated to \( (N_0; \in, I, a) \), \( N_0 \models \varphi(f(n_1), \ldots, f(n_\ell), a, I) \), and \( \langle (k_n, \alpha_n) : n < \omega \rangle \in [T] \),
  \item[(C.3)] \( \langle M_i, \pi_{ij} : i \leq j \leq \omega_1 \rangle \) is a generic iteration of \( M_0 \) which witnesses that \( N_0 < M_0 \) in \( \mathbb{P}_{\text{max}} \),
  \item[(C.4)] \( \langle N_i, \sigma_{ij} : i \leq j \leq \omega_1 \rangle \) is a generic iteration of \( N_0 \) such that if \( N_\omega \) is \( (N_\omega; \in, I^*, A^*) \), then \( A^* = A \),\(^9\)
  \item[(C.5)] \( \langle M_i, \pi_{ij} : i \leq j \leq \omega_1 \rangle = \sigma_{0 \omega_1}((M_i, \pi_{ij} : i \leq j \leq \omega_{N_0})) \) and \( M_{\omega_1} = ((H_{\omega_2})^V; \in, (\text{NS}_{\omega_1})^V, A) \),
  \item[(C.6)] \( K \subset \omega_1 \),
  \item[(C.7)] \( \lambda_\delta \in \lambda \cap C \), and if \( \gamma < \delta \) is in \( K \), then \( \lambda_\gamma < \lambda_\delta \) and \( X_\gamma \cup \{ \lambda_\gamma \} \subset X_\delta \), and
  \item[(C.8)] \( X_\delta \prec (Q_{\lambda_\delta}; \in, P_{\lambda_\delta}, A_{\lambda_\delta}) \) and \( X_\delta \cap \omega_1 = \delta \).
\end{itemize}

\(^8\)W is an outer model iff \( W \) is a transitive model of \( \text{ZFC} \) with \( W \supset V \) and which has the same ordinals as \( V \); in other words, \( W \) is an outer model iff \( V \) is an inner model of \( W \).

\(^9\)There is no requirement on \( I^* \) matching the non-stationary ideal of some model in which \( \mathcal{C} \) exists.
For future purposes, let us refer to any object \( \mathcal{C} \) as in (11) which satisfies the above properties (C.1) through (C.8) as a potential certificate.

We need to define a first order language \( \mathcal{L} \) (independently from \( \lambda \)) whose formulae will be able to describe \( \mathcal{C} \) with the above properties by producing the models \( M_i \) and \( N_i, i < \omega_1 \), as term models out of equivalence classes of terms of the form \( \hat{n}, n < \omega \). The language \( \mathcal{L} \) will have the the following constants.

\[
\begin{align*}
\hat{T} & \quad \text{intended to denote } T \\
x & \quad \text{for every } x \in H_\kappa \quad \text{intended to denote } x \text{ itself} \\
\hat{n} & \quad \text{for every } n < \omega \quad \text{as terms for elements of } M_i \text{ and } N_i, i < \omega_1 \\
\hat{M}_i & \quad \text{for } i < \omega_1 \quad \text{intended to denote } M_i \\
\hat{\pi}_{ij} & \quad \text{for } i \leq j \leq \omega_1 \quad \text{intended to denote } \pi_{ij} \\
\hat{\tilde{M}} & \quad \text{intended to denote } (M_j, \pi_{jj'}: j \leq j' \leq \omega_1^{N_i}) \text{ for } i < \omega_1 \\
\hat{N}_i & \quad \text{for } i < \omega_1 \quad \text{intended to denote } N_i \\
\hat{\sigma}_{ij} & \quad \text{for } i \leq j < \omega_1 \quad \text{intended to denote } \sigma_{ij} \\
\hat{a} & \quad \text{intended to denote the distinguished } a\text{-predicate of } M_i, N_i, i < \omega_1 \\
\hat{I} & \quad \text{intended to denote the distinguished ideal of } N_i, i < \omega_1 \\
\hat{X}_\delta & \quad \text{for } \delta < \omega_1 \quad \text{intended to denote } X_\delta.
\end{align*}
\]

The formulae of \( \mathcal{L} \) will be exactly the expressions of the following form.\(^{10}\)

\[
\begin{align*}
\Gamma \hat{N}_i & \models \varphi(\xi_1, \ldots, \xi_k, \hat{n}_1, \ldots, \hat{n}_\ell, \hat{a}, \hat{I}, \hat{M}_j, \ldots, \hat{M}_{j_m}, \hat{\pi}_{q_1 r_1}, \ldots, \hat{\pi}_{q_s r_s}, \hat{\tilde{M}}) \gamma \\
\text{for } i < \omega_1, \xi_1, \ldots, \xi_k < \omega_1, n_1, \ldots, n_\ell < \omega, j_1, \ldots, j_m < \omega_1, q_1 \leq r_1 < \omega_1, \ldots, q_s \leq r_s < \omega_1 \quad \text{and } \varphi \text{ being a first order formula of the language of set theory}
\end{align*}
\]

associated with \( \mathbb{P}_{\max}\)-structures

\[
\begin{align*}
\Gamma \hat{\pi}_{i \omega_1}(\hat{n}) & = x \gamma \quad \text{for } i < \omega_1 \text{ and } x \in H_{\omega_2} \\
\Gamma \hat{\pi}_{\omega_1 \omega_1}(x) & = x \gamma \quad \text{for } x \in H_{\omega_2} \\
\Gamma \hat{\sigma}_{ij}(\hat{n}) & = \hat{m} \gamma \quad \text{for } i \leq j < \omega_1, n, m < \omega \\
\Gamma (\vec{u}, \vec{\alpha}) & \in \hat{T} \gamma \quad \text{for } \vec{u} \in <^\omega \omega \text{ and } \vec{\alpha} \in <^\omega \omega_2 \\
\Gamma \delta \mapsto \bar{\lambda} \gamma & \quad \text{for } \delta < \omega_1, \lambda < \kappa \\
\Gamma x & \in \hat{X}_\delta \gamma \quad \text{for } \delta < \omega_1, x \in H_\kappa
\end{align*}
\]

\(^{10}\)There are no dots on top of \( \xi_1, \ldots, \xi_k \) in the first formula, as those are constants of the type which intend to denote themselves. The term \( \hat{n} \) in the second formula is of the type denoting an element of \( M_i \), etc.
Let us write $\mathcal{L}^\lambda$ for the collection of all $\mathcal{L}$-formulae except for the formulae which mention elements outside of $Q_\lambda$, i.e., except for the formulae of the form $\gamma \delta \mapsto \lambda^\gamma$ for $\delta < \omega_1$ and $\lambda \leq \lambda < \kappa$ as well as $\gamma x \in X_\delta^\gamma$ for $\delta < \omega_1$ and $x \in H_\kappa \setminus Q_\lambda$. We may and shall assume that $\mathcal{L}$ is built in a canonical way so that $\mathcal{L}^\lambda = \mathcal{L} \cap Q_\lambda$.

We say that a potential certificate $\mathcal{C}$ as in (11) is pre-certified by a collection $\Sigma$ of $\mathcal{L}^\lambda$-formulae if and only if (C.1) through (C.8) are satisfied by $\mathcal{C}$ and there are surjections $e_i: \omega \to N_i$ for $i < \omega_1$ such that the following hold true.

(S.1) $\gamma N_i \models \varphi(\xi_1, \ldots, \xi_k, \dot{n}_1, \ldots, \dot{n}_\ell, \dot{a}, \dot{I}, \dot{M}_{j_1}, \ldots, \dot{M}_{j_m}, \dot{\pi}_{q_1 r_1}, \ldots, \dot{\pi}_{q_r r_r}, \dot{\mathcal{M}})^\gamma \in \Sigma$ iff $i < \omega_1$, $\xi_1, \ldots, \xi_k \leq \omega_1^{N_i}$, $n_1, \ldots, n_\ell < \omega$, $j_1, \ldots, j_m \leq \omega_1^{N_i}$, $r_1 \leq \omega_1^{N_i}$, $q_r \leq \omega_1^{N_i}$, and $N_i \models \varphi(\xi_1, \ldots, \xi_k, e_i(n_1), \ldots, e_i(n_\ell), A \cap \omega_1^{N_i}, I^{N_i}, M_{j_1}, \ldots, M_{j_m}, \pi_{q_1 r_1}, \ldots, \pi_{q_r r_r}, \mathcal{M})$.

(S.2) $\gamma \pi_{\omega_1}(\dot{n}) = x^\gamma \in \Sigma$ iff $i < \omega_1$, $n < \omega$, and $\pi_{\omega_1}(e_i(n)) = x$.

(S.3) $\gamma \pi_{\omega_1}(x) = x^\gamma \in \Sigma$ iff $x \in H_{\omega_2}$.

(S.4) $\gamma \sigma_{ij}(\dot{n}) = \dot{m}^\gamma \in \Sigma$ iff $i < j < \omega_1$, $m < \omega$, and $\sigma_{ij}(e_i(n)) = e_j(m)$.

(S.5) $\gamma \delta \mapsto \lambda^\gamma \in \Sigma$ iff $\delta \in K$ and $\lambda = \lambda_\delta$, and

(S.6) $\gamma \delta \mapsto \lambda^\gamma \in \Sigma$ iff $\delta \in K$ and $\lambda = \lambda_\delta$, and

(S.7) $\gamma x \in X_\delta^\gamma \in \Sigma$ iff $\delta \in K$ and $x \in X_\delta$.

We say that a potential certificate $\mathcal{C}$ as in (11) is certified by a collection $\Sigma$ of formulae if and only if $\mathcal{C}$ is pre-certified by $\Sigma$ and, in addition,

(S.8) if $\delta \in K$, then $[\Sigma]^{<\omega} \cap X_\delta \cap E \neq \emptyset$ for every $E \subset \mathbb{P}_{\lambda_\delta}$ which is dense in $\mathbb{P}_{\lambda_\delta}$ and definable over the structure $(Q_{\lambda_\delta}, \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta})$ from parameters in $X_\delta$.

\[11\] Equivalently, $[\Sigma]^{<\omega} \cap E \neq \emptyset$ for every $E \subset \mathbb{P}_{\lambda_\delta} \cap X_\delta$ which is dense in $\mathbb{P}_{\lambda_\delta} \cap X_\delta$ and definable over the structure $(X_\delta; \in, \mathbb{P}_{\lambda_\delta} \cap X_\delta, A_{\lambda_\delta} \cap X_\delta)$ from parameters in $X_\delta$.  

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By way of definition, we call a potential certificate $C$ as in (11) a \textit{semantic certificate} iff there is a collection $\Sigma$ of formulae such that $C$ is certified by $\Sigma$. We call $\Sigma$ a \textit{syntactic certificate} iff there is a semantic certificate $C$ such that $C$ is certified by $\Sigma$. Given a syntactic certificate $\Sigma$, there is a \textit{unique} semantic certificate $C$ such that $C$ is certified by $\Sigma$. Even though it is obvious how to construct $C$ from $\Sigma$, in the proof of Lemma 3.3 below we will provide details on how to derive a semantic certificate from a given $\Sigma$.

It is worth stressing that not every collection of $L^\lambda$-formulae which is merely consistent is already a syntactic certificate. The requirement that the constant $x \in H_\kappa$ is to be interpreted by itself (cf. (Σ.2), (Σ.3), and (Σ.7)) may be restated as saying that for a consistent $L^\lambda$-theory to be a syntactic certificate it is to be true that certain types are omitted.

Let $\Sigma \cup p$ be a set of formulae, where $p$ is finite. We say that $p$ is \textit{certified by} $\Sigma$ if and only if there is some (unique) $C$ as in (11) such that $C$ is certified by $\Sigma$ and

$$\tag{Σ.9} p \in [\Sigma]^{<\omega}.$$  

We may also say that $p$ is \textit{certified by} $\Sigma$ as in (11) iff there is some $\Sigma$ such that $C$ and $p$ are both certified by $\Sigma$—and we will then also refer to $\Sigma$ as a syntactical certificate for $p$ and to $C$ as the associated semantic certificate.

We are then ready to define the forcing $P_\lambda$. We say that $p \in P_\lambda$ if and only if

$$V^{Col(\omega, \lambda)} \models \text{“There is a set } \Sigma \text{ of } L^\lambda\text{-formulae such that } p \text{ is certified by } \Sigma.” \quad (13)$$

Let $p$ be a finite set of formulae of $L^\lambda$. By the homogeneity of $Col(\omega, \lambda)$, if there is some $h$ which is $Col(\omega, \lambda)$-generic over $V$ and there is some $\Sigma \in V[h]$ such that $p$ is certified by $\Sigma$, then for all $h$ which are $Col(\omega, \lambda)$-generic over $V$ there is some $\Sigma \in V[h]$ such that $p$ is certified by $\Sigma$. It is then easy to see that $\langle P_\lambda : \lambda \in C \cup \{\kappa}\rangle$ is definable over $V$ from $\langle A_\lambda : \lambda < \kappa\rangle$ and $C$, and is hence an element of $V$.\footnote{To remind the reader, $C$ is the club from p. 16.}

Again let $p$ be a finite set of formulae of $L^\lambda$. By $\Sigma^1_1$ absoluteness (see [30, Corollary 7.21]), if there is any outer model in which there is some $\Sigma$ which certifies $p$, then there is some $\Sigma \in V^{Col(\omega, \lambda)}$ which certifies $p$.\footnote{In fact, if $P$ is a transitive model of KP plus the axiom $Beta$ with $\langle Q_\lambda : A_\lambda \in P \rangle \in P$ and if $p \in P_\lambda$, then there is some $\Sigma \in P^{Col(\omega, \lambda)}$ which certifies $p$.} This simple observation is important in the verification that $P_\lambda$ is actually non-empty, cf. Lemma 3.2, and in the proof of Lemma 3.8.

It is easy to see that

$$(i) \ P = P_\kappa \subset H_\kappa,$$
(ii) if $\bar{\lambda} < \lambda$ are both in $C \cup \{\kappa\}$, then $\mathbb{P}_{\bar{\lambda}} \subset \mathbb{P}_\lambda$, and

(iii) if $\lambda \in C \cup \{\kappa\}$ is a limit point of $C \cup \{\kappa\}$, then $\mathbb{P}_\lambda = \bigcup_{\bar{\lambda} \in C \cap \lambda} \mathbb{P}_{\bar{\lambda}}$, so that there is some club $D \subset C$ such that for all $\lambda \in D$, $\mathbb{P}_\lambda = \mathbb{P} \cap Q_\lambda$.

Hence $(\diamondsuit)$ gives us the following.

$(\diamondsuit(\mathbb{P}))$ For all $B \subset H_\kappa$ the set

$$\{\lambda \in C: (Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\kappa; \in, \mathbb{P}, B)\}$$

is stationary.

The first one of the following lemmas is entirely trivial.

**Lemma 3.1** Let $\Sigma$ be a syntactic certificate, and let $p, q \in [\Sigma]^{<\omega}$. Then $p$ and $q$ are compatible conditions in $\mathbb{P}$.

**Lemma 3.2** Let $\lambda \in C \cup \{\kappa\}$. Then $\emptyset \in \mathbb{P}_\lambda$.

**Proof.** This is a simple variant of the proofs of [1, Theorem 2.8] and of [31, Theorem 4.2]. What needs to be done is to construct a semantic/syntactic certificate (for $\emptyset$) in some outer model. Notice that for all $\lambda \in C \cup \{\kappa\}$, $\emptyset \in \mathbb{P}_\lambda$ iff $\emptyset \in \mathbb{P}$.

Let $h$ be $\text{Col}(\omega, \omega_2)$-generic over $V$. Let $q^* = (N^*; \in, I^*, a^*) \in (\mathbb{P}_{\text{max}})^{V[h]}$ be as in the paragraph preceding (8) and such that (8) is true. Let $(M_i, \pi_{ij} : i \leq j \leq \omega_1^{N_0}) \in N^*$ be the unique generic iteration of $p$ which witnesses $q^* \prec p = M_0 = (H_{\omega_2}; \in, (\text{NS}_{\omega_1})^V, A)$.

Let $(N_i, \sigma_{ij} : i \leq j \leq \kappa) \in V[h]$ be a generic iteration of $N_0 = N^*$ such that $\kappa = \omega_1^{N_0}$. Let

$$(M_i, \pi_{ij} : i \leq j \leq \kappa) = \sigma_0((M_i, \pi_{ij} : i \leq j \leq \omega_1^{N_0})) \quad (14)$$

Since $M_0 = ((H_{\omega_2})^V, \in, (\text{NS}_{\omega_1})^V, A)$ and $(\text{NS}_{\omega_1})^V$ is assumed to be saturated in $V$, every maximal antichain in $V$ consisting of stationary subsets of $\omega_1$ is an element of $M_0$. We may hence lift the generic ultrapower map $\pi_{01} : M_0 \to M_1$ to act on all of $V$, and inductively we may lift the entire generic iteration (14) to a generic iteration

$$(M_i^+, \pi_{ij}^+ : i \leq j \leq \kappa) \quad (15)$$

If we wished, then we could even arrange that writing $N_\kappa = (N_\kappa; \in, I', a')$, we have that $I' = (\text{NS}_{\kappa})^{V[h]} \cap N_\kappa$, but this is not relevant here; cf. footnote 9.
of $V$ in such a way that all $M_i^+, i \leq \kappa$, are transitive. Cf. [42, Lemma 3.8]. Let us write $M = M_\kappa^+$ and $\pi = \pi^\kappa_0$.

Let $\langle k_n, \alpha_n : n < \omega \rangle$ be such that $x = \langle k_n : n < \omega \rangle$ is a real code for $N_0$ à la (C:2) and $\langle (k_n, \alpha_n) : n < \omega \rangle \in [T]$. We then clearly have that $\langle (k_n, \pi(\alpha_n)) : n < \omega \rangle \in [\pi(T)]$.

It is now easy to see that

$$C = \langle \langle M_i, \pi ij, N_i, \sigma ij : i \leq j \leq \kappa \rangle, \langle (k_n, \pi(\alpha_n)) : n < \omega \rangle, \langle \rangle \rangle \quad (16)$$

certifies $\emptyset$, construed as the empty set of $\pi(L^K)$ formulae: as the third component $\langle \rangle$ of $C$ in (16) is empty, any set of surjections $e_i : \omega \to N_i$, $i < \omega_1$, will induce a syntactic certificate for $\emptyset$ whose associated semantic certificate is $C$. By $\Sigma^1_1$ absoluteness, there is then some $C \in M^{\Col(\omega, \pi(\omega_2))}$ as in (16) which certifies $\emptyset$, so that $\emptyset \in \pi(\mathbb{P})$. By the elementarity of $\pi$, then, $\emptyset \in \mathbb{P}$. \hfill \Box

Lemma 3.3 Let $\lambda \in C \cup \{\kappa\}$. Let $g \subseteq \mathbb{P}_\lambda$ be a filter such that $g \cap E \neq \emptyset$ for all dense $E \subseteq \mathbb{P}_\lambda$ which are definable over $(Q_\lambda ; \in, \mathbb{P}_\lambda)$ from elements of $Q_\lambda$. Then $\bigcup g$ is a syntactic certificate.

**Proof.** Let us first describe how to read off from $\bigcup g$ a candidate

$$C = \langle \langle M_i, \pi ij, N_i, \sigma ij : i \leq j \leq \omega_1 \rangle, \langle (k_n, \pi(\alpha_n)) : n < \omega \rangle, \langle \lambda_\delta, X_\delta : \delta \in K \rangle \rangle$$

for a semantic certificate for $\bigcup g$. A variant of what is to come shows how to derive $C$ from a given syntactic certificate $\Sigma$, where $C$ is unique such that $\Sigma$ certifies $C$, cf. the remark on p. 21.

For $i, j < \omega_1$ and $\tau, \sigma \in \{\dot{n} : n < \omega\} \cup \omega_1$ define

$$\tau \sim_i \sigma \quad \text{iff} \quad \dot{N}_i \vDash \tau = \sigma \vDash \bigcup g$$

$$(i, \tau) \sim_\omega (j, \sigma) \quad \text{iff} \quad i \leq j \land \exists \rho \{\dot{\sigma} ij(\tau) = \rho \vDash \dot{N}_1 \vDash \rho = \sigma \vDash \bigcup g \} \subset \bigcup g$$

or $j \leq i \land \exists \rho \{\dot{\sigma} ji(\sigma) = \rho \vDash \dot{N}_i \vDash \rho = \tau \vDash \bigcup g \} \subset \bigcup g$

$$[\tau]_i = \{\sigma : \tau \sim_i \sigma\}$$

$$[(i, \tau)] = \{(j, \sigma) : (i, \tau) \sim_\omega (j, \sigma)\}$$

$$M_i = \{[\tau]_i : \tau \in \{\dot{n} : n < \omega\} \cup \omega_1 \land \dot{N}_i \vDash \tau \in M_i \vDash \bigcup g\}$$

$$M_{\omega_1} = (H_{\omega_2})^V$$

$$N_i = \{[\tau]_i : \tau \in \{\dot{n} : n < \omega\} \cup \omega_1\}$$
Hence a straightforward density arguments give (C.2), i.e., that
\[ \langle \text{generic iteration of } N \rangle \text{ of) the term model for } N \]
Claim 3.4
For each verify this, let us first show:

\[ \text{Proof of Claim 3.4. A straightforward induction on } N \text{ will code the theory of } N_0 \text{ à la (C.2).} \]

Another set of easy density arguments will give that \( (\xi, N_i) \in \) is an iterable \( \mathbb{P}_{\text{max}} \) condition. This is true because straightforward density arguments give (C.2), i.e., that \( \langle (k_n, \alpha_n) : n < \omega \rangle \in [T] \) and \( \langle k_n : n < \omega \rangle \) will code the theory of \( N_0 \).

We will first have that \( \bar{\epsilon}_0 \) is wellfounded and that in fact (the transitive collapse of) the structure \( N_0 = (N_0; \bar{\epsilon}_0, a^{N_0}, I^{N_0}) \) is an iterable \( \mathbb{P}_{\text{max}} \) condition. This is true because straightforward density arguments give (C.2). Thus, we identify \( N_i \) with the structure \( (N_i; \bar{\epsilon}_i, a^{N_i}, I^{N_i}) \). To verify this, let us first show:

Claim 3.4 For each \( i < \omega_1 \) and for each \( \xi \leq \omega_1^{N_i} \), \( [\xi]_i \) represents \( \xi \) in (the transitive collapse of the well-founded part of) the term model for \( N_i \); moreover, \( a^{N_i} = A \cap \omega_1^{N_i} \).

\[ \text{Hence } a^{\omega_1^{N_i}} = A. \]

**Proof of Claim 3.4.** A straightforward induction on \( \xi \leq \omega_1^{N_i} \) using (Σ.1) shows
that $[ξ]_i$ must always represent $ξ$ in $N_i$ as given by any certificate. Claim 3.4 then follows by straightforward density arguments. □ (Claim 3.4)

Similarly:

Claim 3.5 Let $i < ω_1$. $N_{i+1}$ is generated from $\text{ran}(σ_{i+1})∪\{ω_1^{N_i}\}$ in the sense that for every $x ∈ N_{i+1}$ there is some function $f ∈ ω_1^{N_i}(N_i) ∩ N_i$ such that $x = σ_{i+1}(f)(ω_1^{N_i})$.

Claim 3.6 Let $i < ω_1$. $\{X ∈ P(ω_1^{N_i}) ∩ N_i: ω_1^{N_i} ∈ σ_{i+1}(X)\}$ is generic over $N_i$ for the forcing given by the $P_i$-positive sets.

Claim 3.7 Let $i ≤ ω_1$ be a limit ordinal. For every $x ∈ N_i$ there is some $j < i$ and some $\bar{x} ∈ N_j$ such that $x = σ_{ji}(\bar{x})$.

$(N_i, σ_{ij}: i ≤ j ≤ ω_1)$ is then indeed a generic iteration of $N_0$. As $N_0$ is iterable, we may and shall identify $N_i$ with its transitive collapse, so that (C.4) holds true.

Another round of density arguments will show that $C$ satisfies (C.1), (C.3), (C.5), (C.6), and (C.7), where we identify $M_i$ with the structure $(M_i; ∈, (NS_{ω_1^{M_i}})^{M_i}, A∩ω_1^{M_i})$. Let us now verify (C.8) and (Σ.8).

As for (C.8), its second part, $X_δ ∩ ω_1 = δ$ for $δ ∈ K$, is easy. We will now use the Tarski-Vaught test to verify the first part of (C.8). Let $ϕ$ be any formula, and let $x_1, ..., x_k ∈ X_δ$, $δ ∈ K$. Suppose that

$$(Q_{λ_δ}; ∈ P_{λ_δ}, A_{λ_δ}) \models \exists v ϕ(v, x_1, ..., x_k). \tag{17}$$

Let $p ∈ g$ be such that $\{^γ x_1 ∈ X_δ, ..., ^γ x_k ∈ X_δ, ^γ δ \rightarrow λ_δ \}$ $⊂ p$. Let $q ≤ p$, and let $Σ$ be a syntactical certificate for $q$ whose associated semantic certificate is

$$(C') = (⟨⟨M_{i'}, π_{ij}', N_{i'}, σ_{ij}': i ≤ j ≤ ω_1⟩, ⟨⟨k_n', α_n'⟩: n < ω⟩, ⟨X_{δ}', X_{δ'}: δ ∈ K'⟩⟩).$$

Then $δ ∈ K'$ and

$$\{x_1, ..., x_k\} ⊂ X_{δ'} \prec (Q_{λ_δ}; ∈ P_{λ_δ}, A_{λ_δ}),$$

so that by (17) we may choose some $x ∈ X_{δ'}$ with

$$(Q_{λ_δ}; ∈ P_{λ_δ}, A_{λ_δ}) \models ϕ(x, x_1, ..., x_k).$$

Let $r = q ∪ \{^γ x ∈ X_δ\}$.

By density, there is then some $y ∈ X_δ$ such that

$$(Q_{λ_δ}; ∈ P_{λ_δ}, A_{λ_δ}) \models ϕ(y, x_1, ..., x_k).$$

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The proof of (Σ.8) is similar. Let again $\delta \in K$. Let $E \subset P_{\lambda_\delta} \cap X_{\delta}$ be dense in $P_{\lambda_\delta} \cap X_\delta$, and $r \in E$ iff $r \in P_{\lambda_\delta} \cap X_\delta$ and

$$\langle Q_{\lambda_\delta}; \in P_{\lambda_\delta}, A_{\lambda_\delta} \rangle \vdash \varphi(r, x_1, \ldots, x_k). \quad (18)$$

Let $p \in g$ be such that $\{\neg x_1 \in \dot{X}_{\delta}, \ldots, \neg x_k \in \dot{X}_{\delta}, \delta \mapsto \lambda_\delta \} \in p$. Let $q \leq p$, and again let $\Sigma$ be a syntactical certificate for $q$ whose associated semantic certificate is

$$\mathcal{C}' = \langle \langle M_i', \pi_{ij}, N_i'; \sigma_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k_n', \alpha_n') : n < \omega \rangle, \langle \lambda'_\delta, X'_\delta : \delta \in K' \rangle \rangle.$$

Then $[\Sigma]^{<\omega} \cap X'_\delta$ has an element, say $r$, such that (18) holds true. Let $s = q \cup r \cup \{\neg r \in \dot{X}_{\delta}\}$. By density, then, $g \cap X_\delta \cap E \neq \emptyset$. \hfill $\square$

**Lemma 3.8** Let $g$ be $\mathbb{P}$-generic over $V$. Let

$$\mathcal{C} = \langle \langle M_i, \pi_{ij}, N_i, \sigma_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) : n < \omega \rangle, \langle \lambda_\delta, X_\delta : \delta \in K \rangle \rangle$$

be the semantic certificate associated with the syntactic certificate $\bigcup g$. Let

$$N_{\omega_1} = (N_{\omega_1}; \in, A, I^*)$$

Then every element of $(\mathcal{P}(\omega_1) \cap N_{\omega_1}) \setminus I^*$ is stationary in $V[g]$.

**Corollary 3.9** $\mathbb{P}$ preserves stationary subsets of $\omega_1$.

**Proof** of Corollary 3.9 from Lemma 3.8. If $\mathcal{C}$ and $I^*$ are as in the statement of Lemma 3.8, then by Lemma 3.3 and (C.3) and (C.4) we will have that $(\text{NS}_{\omega_1})^V = I^* \cap V$, so that the conclusion of Lemma 3.8 also gives that $\mathbb{P}$ preserves stationary subsets of $\omega_1$. \hfill $\square$

**Proof** of Lemma 3.8. Let $\dot{N}_{\omega_1} \in V^\mathbb{P}$ be a canonical name for $N_{\omega_1}$, and let $\dot{I}^* \in V^\mathbb{P}$ be a canonical name for $I^*$. Let $p \in g$, $\dot{C}, \dot{S} \in V^\mathbb{P}$, and $i < \omega_1$ and $n < \omega$ be such that

(i) $p \Vdash \text{"$\dot{C} \subset \omega_1$ is club,"}$

(ii) $p \Vdash \text{"$\dot{S} \in (\mathcal{P}(\omega_1) \cap \dot{N}_{\omega_1}) \setminus \dot{I}^*$,\"}$ and

(iii) $p \Vdash \text{"$\dot{S}$ is represented by $[i, \dot{n}]$ in the term model producing $\dot{N}_{\omega_1}$.\"}$

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We may and shall also assume that
\[ \Gamma \dot{N}_i \models \dot{n} \text{ is a subset of the first uncountable cardinal, yet } \dot{n} \notin \dot{\mathcal{I}} \in \dot{p}, \] (19)
because the \( \mathcal{L} \)-formula in (19) must belong to every syntactic certificate for \( \dot{p} \), as \( \dot{p} \) satisfies (ii) and (iii).

Let \( p \leq \dot{p} \) be arbitrary, \( p \in \mathbb{P} \). We aim to produce some \( q \leq p \) and some \( \delta < \omega_1 \) such that \( q \models \dot{\delta} \in \dot{\mathcal{C}} \cap \dot{S} \), see Claim 3.11 below.

For \( \xi < \omega_1 \), let
\[ D_\xi = \{ q \leq p : \exists \eta \geq \xi (\eta < \omega_1 \land q \models \dot{\eta} \in \dot{\mathcal{C}}) \}, \]
so that \( D_\xi \) is open dense below \( p \). Let
\[ E = \{ (q, \eta) \in \mathbb{P} \times \omega_1 : q \models \dot{\eta} \in \dot{\mathcal{C}} \}. \]

Let us write
\[ \tau = ((D_\xi : \xi < \omega_1), E). \]
We may and shall identify \( \tau \) with some subset of \( H_\kappa \) which codes \( \tau \).

By (3(P)), we may pick some \( \lambda \in C \) such that \( p \in \mathbb{P}_\lambda \) and
\[ (Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\kappa; \in, \mathbb{P}, \tau). \] (20)

Let \( h \) be \( \text{Col}(\omega, \omega_2) \)-generic over \( V \), and let \( g' \in V[h] \) be a filter on \( \mathbb{P}_\lambda \) such that \( p \in g' \) and \( g' \) meets every dense set which is definable over \( (Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \) from parameters in \( Q_\lambda \). By Lemma 3.3, \( \bigcup g' \) is a syntactic certificate for \( p \), and we may let
\[ \langle (M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} : i \leq j \leq \omega_1), ((k'_n, \alpha'_n) : n < \omega), (\lambda'_\delta, X'_\delta : \delta \in K') \rangle \]
be the associated semantic certificate. In particular, \( K' \subset \lambda \).

Let \( S \) denote the subset of \( \omega_1 \) which is represented by \([i, \dot{n}]\) in the term model giving \( N'_{\omega_1} \), so that if \( N'_{\omega_1} = (N'_{\omega_1}, \in, A, I') \), then by (19),
\[ S \in \left( \mathcal{P}(\omega_1) \cap N'_{\omega_1} \right) \setminus I'. \] (21)

Let us also write \( \rho = \omega_1^{V[h]} = \omega_1^{V} \). Inside \( V[h] \), we may extend \( \langle N'_i, \sigma'_{ij} : i \leq j \leq \omega_1 \rangle \)
to a generic iteration
\[ \langle N'_i, \sigma'_{ij} : i \leq j \leq \rho \rangle \]
such that
\[ \omega_1 \in \sigma'_{\omega_1, \omega_1+1}(S). \] (22)
This is possible as \( \omega_1^{N_0} = \sup\{\omega_1^{N_j} : j < \omega_1\} = \omega_1 \) and by (21). Let

\[
\langle M'_i, \pi'_{ij} : i \leq j \leq \rho \rangle = \sigma_{0,\rho}(\langle M'_i, \pi'_{ij} : i \leq j \leq \omega_1^{N'_i} \rangle),
\]

so that \( \langle M'_i, \pi'_{ij} : i \leq j \leq \rho \rangle \) is an extension of \( \langle M'_i, \pi'_{ij} : i \leq j \leq \omega_1 \rangle \).

Since \( M'_\omega = ((H_{\omega_1})^V; \in, (\text{NS}_{\omega_1})^V, A) \), cf. (12), and \((\text{NS}_{\omega_1})^V \) is assumed to be saturated in \( V \), every maximal antichain in \( V \) consisting of stationary subsets of \( \omega_1 \) is an element of \( M'_\omega \). We may hence lift the generic ultrapower map \( \pi'_{\omega_1+1} : M'_\omega \rightarrow M'_\omega+1 \) to act on all of \( V \), and inductively we may lift the entire generic iteration \( \langle M'_i, \pi'_{ij} : \omega_1 \leq i \leq j \leq \rho \rangle \) to a generic iteration

\[
\langle M_+^i, \pi_+^i : \omega_1 \leq i \leq j \leq \rho \rangle \]

of \( V \) with all \( M_+^i \), \( \omega_1 \leq i \leq \rho \), being transitive. Cf. [42, Lemma 3.8]. Let us write \( M = M_+^\rho \) and \( \pi = \pi_+^\rho \).

The key point is now that \( \langle M_+^i, \pi_+^i, N_+^i, \sigma_+^i : i \leq j \leq \rho \rangle \) may be used to extend \( \pi'' \cup g' \) to a syntactic certificate

\[
\Sigma \supset \pi'' \cup g'
\]

for \( \pi(p) \) in the following manner. Let \( K^* = K' \cup \{\omega_1\} \). For \( \delta \in K' \), let \( \lambda_\delta^* = \pi(\lambda_\delta') \) and \( X_\delta^* = \pi''X_\delta' \). Also, write \( \lambda_\omega^* = \pi(\lambda) \) and \( X^*_\omega = \pi''Q_\lambda \). Notice that \( \omega_1 \in \pi(C) \), so that \( K^* \subset \pi(C) \). Let

\[
\mathcal{E}^* = \langle \langle M'_i, \pi'_{ij}, N'_i, \sigma'_ij : i \leq j \leq \rho \rangle, (k'_n, \pi(\alpha'_n)) : n < \omega \rangle, (\lambda_\delta^*, X_\delta^* : \delta \in K^*) \rangle.
\]

We claim that \( \mathcal{E}^* \) is a semantic certificate for \( \pi(p) \) as an element of \( \pi(\mathbb{P}) \). First notice that \( \langle (k'_n, \pi(\alpha'_n)) : n < \omega \rangle \in [\pi(T)] \). Next, if \( \delta \in K' \), then \( X_\delta^* = \pi''X_\delta' \prec (\pi(Q_\lambda'); \in, \pi(\mathbb{P}_{\lambda'}), \pi(A_{\lambda'})) \), and \( \pi''g' \cap X_\delta^* = \pi''(g' \cap X'_\delta) \); as \( \cup g' \) is a syntactic certificate for \( p \), we thus have that \( \pi''g' \cap X_\delta^* \cap E \neq \emptyset \) for every \( E \subset \pi(\mathbb{P}_{\lambda'}) \) which is dense in \( \pi(\mathbb{P}_{\lambda'}) \) and definable over the structure \( (\pi(Q_\lambda'); \in, \pi(\mathbb{P}_{\lambda'}), \pi(A_{\lambda'})) \) from parameters in \( X_\delta^* \). Finally, \( X_\omega^* = \pi''Q_\lambda \) and the choice of \( g' \) imply that \( \pi''g' \cap X_\omega^* \cap E \neq \emptyset \) for every \( E \subset \pi(\mathbb{P}_{\lambda}) \) which is dense in \( \pi(\mathbb{P}_{\lambda}) \) and definable over the structure \( (\pi(Q_\lambda); \in, \pi(\mathbb{P}_{\lambda}), \pi(A_{\lambda})) \) from parameters in \( X_\omega^* \). This buys us that \( \mathcal{E}^* \) is indeed a semantic certificate for \( \pi(p) \) as an element of \( \pi(\mathbb{P}) \), and that moreover there is some syntactic certificate \( \Sigma \) as in (23) such that \( \mathcal{E}^* \) is certified by \( \Sigma \).

Now let \([\hat{m}]_{\omega_1+1} \) represent \( \sigma'_{\omega_1+1}(S) \) in the term model for \( N'_{\omega_1+1} \) provided by \( \Sigma \), so that

\[
\{\langle \hat{\sigma}_{\omega_1+1}(\hat{n}) = \hat{m} \rangle, \langle \check{N}_{\omega_1+1} \models \omega_1 \in \hat{m} \rangle \} \subset \Sigma.
\]

\[\tag{15}\]

Here, \( \hat{\sigma}_{\omega_1+1} \) and \( \check{N}_{\omega_1+1} \) are terms of the language associated with \( \pi(\mathbb{P}_{\lambda}) \), and \( \langle \hat{\sigma}_{\omega_1+1}(\hat{n}) = \hat{m} \rangle \) and \( \langle \check{N}_{\omega_1+1} \models \omega_1 \in \hat{m} \rangle \) are formulae of that language.
in other words,
\[ \pi(p) \cup \{\varGamma \sigma_{\omega_1+1}(\dot{n}) = \dot{m}_\gamma, \varGamma \dot{N}_{\omega_1+1} \models \omega_1 \in \dot{m}_\gamma\} \text { is certified by } \Sigma. \] (24)

Let us now define
\[ q^* = \pi(p) \cup \{\varGamma \sigma_{\omega_1+1}(\dot{n}) = \dot{m}_\gamma, \varGamma \dot{N}_{\omega_1+1} \models \omega_1 \in \dot{m}_\gamma, \varGamma \omega_1 \mapsto \pi(\lambda)^\gamma\}. \] (25)

We thus established the following.

**Claim 3.10** \( q^* \in \pi(\mathbb{P}) \), as being certified by \( \Sigma \).

The elementarity of \( \pi: V \to M \) then gives some \( \delta < \omega_1 \) and some \( \mu < \kappa \) such that
\[ q = p \cup \{\varGamma \sigma_{\delta+1}(\dot{n}) = \dot{m}_\gamma, \varGamma \dot{N}_{\delta+1} \models \delta \in \dot{m}_\gamma, \varGamma \delta \mapsto \lambda\} \in \mathbb{P}. \] (26)

**Claim 3.11** \( q \vDash \tilde{\delta} \in \tilde{C} \cap \tilde{S} \).

**Proof** of Claim 3.11. \( q \vDash \tilde{\delta} \in \tilde{S} \) readily follows from \( \{\varGamma \sigma_{\delta+1}(\dot{n}) = \dot{m}_\gamma, \varGamma \dot{N}_{\delta+1} \models \delta \in \dot{m}_\gamma\} \subset q \), the fact that \( \tilde{p} \geq p \) forces that \( \tilde{S} \) is represented by \([i, \dot{n}] \) in the term model giving \( \dot{N}_{\omega_1} \), and the fact that by Claim 3.4, \( \delta_{\delta+1} \) represents \( \delta \) in the model \( \dot{N}_{\delta+1} \) of any semantic certificate for \( q \).

Let us now show that \( q \vDash \tilde{\delta} \in \tilde{C} \). We will in fact show that \( q \) forces that \( \tilde{\delta} \) is a limit point of \( \tilde{C} \). Otherwise there is some \( r \leq q \) and some \( \eta < \delta \) such that
\[ r \vDash \tilde{C} \cap \tilde{\delta} \subset \eta. \] (27)

Let
\[ \langle \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) : n < \omega \rangle, \langle \lambda'_\delta, X'_\delta : \tilde{\delta} \in K' \rangle \rangle \] (28)
certify \( r \). We must have that

(a) \( \delta \in K' \),

(b) \( X'_\delta \prec (Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \),

(c) \( X'_\delta \cap \omega_1 = \delta \), and

(d) for some \( \Sigma \) such that the objects from (28) are certified by \( \Sigma \), \( [\Sigma]^\omega \cap X'_\delta \cap E \neq \emptyset \) for every \( E \subset \mathbb{P}_\lambda \) which is dense in \( \mathbb{P}_\lambda \cap X'_\delta \) and definable over the structure
\[ (Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \]
from parameters in \( X'_\delta \).
We have that $A_\lambda = \tau \cap Q_\lambda$, and hence $A_\lambda$ may be identified with $((D_\xi \cap Q_\lambda : \xi < \omega_1), E \cap Q_\lambda)$. As $\eta < \delta \subset X'_\delta$, $D_\eta$ is definable over the structure

$$(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$$

from a parameter in $X'_\delta$. By (20), $D_\eta \cap Q_\lambda$ is dense in $\mathbb{P}_\lambda$. By (d) above, there is then some $s \in [\Sigma]^{<\omega} \cap X'_\delta \cap D_\eta$.

By (20) again, the unique smallest $\eta' \geq \eta$ with $s \Vdash \check{\eta}' \in \dot{C}$ must be in $X'_\delta$, hence $\eta' < \delta$ by (c) above. By Lemma 3.1, $s$ is compatible with $r$. We have reached a contradiction with (27). □

4 Open questions.

Woodin [42] also introduced the axiom $(*^+)$ as a strengthening of $(*)$. $(*)^+$ says that there is some pointclass $\Gamma \subset \mathcal{P}(\mathbb{R})$ and some filter $g \subset \mathbb{P}_{\text{max}}$ such that

1. $L(\Gamma, \mathbb{R}) \models \text{AD}^+$,
2. $g$ is $\mathbb{P}_{\text{max}}$-generic over $L(\Gamma, \mathbb{R})$, and
3. $\mathcal{P}(\mathbb{R}) \subset L(\Gamma, \mathbb{R})[g]$.

See [42, p. 908]. While the main result of the current paper gives a new twist to the question if $\text{MM}$ is compatible with $(*)^+$, see [42, p. 923, Question (15) a)], it also leaves this question wide open.

There is a strengthening of $\text{MM}^{++}$, isolated by Viale [40], which has strong completeness properties modulo forcing similar to those of $(*)$. This is the axiom $\text{MM}^{+++}$. It says that a class $\mathcal{T}$ of towers of ideals with certain nice structural properties is dense in the category of stationary set preserving forcings; in other words, for every stationary set preserving forcing $\mathbb{P}$ there is a tower $\mathcal{T}$ in $\mathcal{T}$ such that $\mathbb{P}$ completely embeds into $\mathcal{T}$ in such a way that the quotient forcing preserves stationary sets in $V^\mathbb{P}$. $\text{MM}^{+++}$ implies $\text{MM}^{++}$, if $\kappa$ is an almost super-huge cardinal, then there is a partial order $\mathbb{P} \subset V_\kappa$ which forces $\text{MM}^{+++}$, and if there is a proper class of almost super-huge cardinals, then $\text{MM}^{+++}$ is complete for the theory of the $\omega_1$-Chang model\footnote{The $\omega_1$-Chang model is the $\subseteq$-minimal transitive model of $\text{ZF}$ containing all ordinals and closed under $\omega_1$-sequences. It can be construed as $\bigcup_{\alpha \in \text{Ord}} L([\alpha]^{\omega_1})$ and it includes $L(\mathcal{P}(\omega_1))$ as a definable submodel.} with respect to stationary set preserving partial orders forcing $\text{MM}^{+++}$.
Schindler [31, Definition 2.10] introduces \( \text{MM}^{*,++} \) as a strengthening of \( \text{MM}^{++} \) by relaxing “forceable by a stationary set preserving forcing” to “honestly consistent” in an appropriate formulation of \( \text{MM}^{++} \), see [31].

It remains open if either of \( \text{MM}^{+++} \) or \( \text{MM}^{*,++} \) is really stronger than \( \text{MM}^{++} \), and it even remains open if \( \text{MM}^{*,++} \) is consistent at all.

References


[31] R. Schindler, *Woodin’s axiom (*), or Martin’s Maximum, or both?*, in: Foundations of mathematics, essays in honor of W. Hugh Woodin’s 60th birthday, Harvard University (Caicedo et al., eds.), pp. 177-204.


