Abstract

We show that Martin’s Maximum$^{++}$ implies Woodin’s $P_{\text{max}}$ axiom ($\ast$).

1 Introduction.

Cantor’s Continuum Problem, which later became Hilbert’s first Problem (see [7]), asks how many real numbers there are. This question got a non-answer through the discovery of the method of forcing by Paul Cohen: CH, Cantor’s Continuum Hypothesis, is independent from ZFC (see [3]), the standard axiom system for set theory which had been isolated by Zermelo and Fraenkel. CH states that every uncountable set of reals has the same size as $\mathbb{R}$.

Ever since Cohen, set theorists have been searching for natural new axioms which extend ZFC and which settle the Continuum Problem. See e.g. [21], [22], [9], and the discussion in [4]. There are two prominent such axioms which decide CH in the negative and which in fact both prove that there are $\aleph_2$ reals: Martin’s Maximum (MM, for short) or variants thereof on the one hand (see [6]), and Woodin’s axiom ($\ast$) on the other hand (see [20]). See e.g. [12], [19], and [14].

Both of these axioms may be construed as maximality principles for the theory of the structure $(H_{\omega_2}; \in)$, but up to this point the relationship between MM and ($\ast$) was a bit of a mystery, which led M. Magidor to call ($\ast$) a “competitor” of MM ([12, p. 18]). Both MM and ($\ast$) are inspired by and formulated in the language of forcing,
and they both have “the same intuitive motivation: Namely, the universe of sets is rich” ([12, p. 18]).

This paper resolves the tension between MM and (⁎) by proving that MM+++, a strengthening of MM, actually implies (⁎), see Theorem 2.1 below, so that MM and (⁎) are actually compatible with each other. This answers [20, Question (18) a) on p. 924], see also [20, p. 846], [12, Conjecture 6.8 on p. 19], and [14, Problem 14.7].

2 Preliminaries.

Martin’s Maximum++, abbreviated by MM++, see [6] (cf. also [20, Definition 2.45 (2)]), is the statement that if \( P \) is a forcing which preserves stationary subsets of \( \omega_1 \), if \( \{ D_i : i < \omega_1 \} \) is a collection of dense subsets of \( P \), and if \( \{ \tau_i : i < \omega_1 \} \) is a collection of \( P \)-names for stationary subsets of \( \omega_1 \), then there is a filter \( g \subset P \) such that for every \( i < \omega_1 \),

(i) \( g \cap D_i \neq \emptyset \) and  
(ii) \( (\tau_i)^g = \{ \xi < \omega_1 : \exists p \in g \ p \vDash P \xi \in \tau_i \} \) is stationary.

Woodin’s \( P_{\text{max}} \) axiom (⁎), see [20, Definition 5.1], is the statement that

(i) \( \text{AD} \) holds in \( L(\mathbb{R}) \) and

(ii) there is some \( g \) which is \( P_{\text{max}} \)-generic over \( L(\mathbb{R}) \) such that \( \mathcal{P}(\omega_1) \cap V \subset L(\mathbb{R})[g] \).

Already PFA, the Proper Forcing Axiom, which is weaker than MM++, implies \( \text{AD}^{L(\mathbb{R})} \) and much more, see [17], [8], and [15, Chapter 12]. This paper produces a proof of the following result.

**Theorem 2.1** Assume Martin’s Maximum++. Then Woodin’s \( P_{\text{max}} \)-axiom (⁎) holds true.

Our key new idea is (Σ.8) on page 9 below.

P. Larson, see [10] and [11], has shown that \( \text{MM}^{+\omega} \) is consistent with \( \neg(\ast) \) relative to a supercompact limit of supercompact cardinals.

Throughout our entire paper, “\( \omega_1 \)” will always denote \( \omega_1^V \), the \( \omega_1 \) of \( V \).

Let us fix throughout this paper some \( A \subset \omega_1 \) such that \( \omega_1^{L[A]} = \omega_1 \). Let us define \( g_A \) as the set of all \( P_{\text{max}} \) conditions \( p = (N; \in, I, a) \) such that there is a generic iteration

\[
(N_i, \sigma_{ij} : i \leq j \leq \omega_1)
\]
of \( p = N_0 \) of length \( \omega_1 + 1 \) such that if we write \( N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*) \),\(^1\) then \( I^* = (NS_{\omega_1})^V \cap N_{\omega_1} \) and \( a^* = A \).

**Lemma 2.2 (Woodin)** Assume that \( NS_{\omega_1} \) is saturated and that \( \mathcal{P}(\omega_1)^# \) exists.

1. \( g_A \) is a filter.

2. If \( g_A \) is \( P^{\text{max}} \)-generic over \( L(\mathbb{R}) \), then \( \mathcal{P}(\omega_1) \subset L(\mathbb{R})[g] \).

**Proof.** This routinely follows from the proof of [20, Lemma 3.12 and Corollary 3.13] and from [20, Lemma 3.10]. \( \square \)

One may also use BMM plus “\( NS_{\omega_1} \) is precipitous” to show that \( g_A \) is a filter, this is by the proof from [2].

Let \( \Gamma \subset \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \). We say that \( \Gamma \) is productive iff for all \( k < \omega \) and all \( D \in \Gamma \cap \mathcal{P}(\mathbb{R}^{k+2}) \), if \( D \) is universally Baire (see [5]) as being witnessed by the trees \( T \) and \( U \) on \( k+2, \omega \times \text{OR} \), i.e., \( D = p[T] \) and for all posets \( \mathbb{P} \),

\[
\models_{\mathbb{P}} p[U] = \mathbb{R}^{k+2} \setminus p[T],
\]

and if

\[
\bar{U} = \{(s \upharpoonright (k+1), (s(k+1), t)): (s, t) \in U\},
\]

so that \( (x_0, \ldots, x_k) \in p(\bar{U}) \) iff there is some \( y \) such that \( (x_0, \ldots, x_k, y) \in p[U] \), then there is a tree \( \bar{T} \) on \( k+4, \omega \times \text{OR} \) such that for all posets \( \mathbb{P} \),

\[
\models_{\mathbb{P}} p[\bar{U}] = \mathbb{R}^{k+4} \setminus p[\bar{T}].
\]

Let us denote by \( \Gamma^\infty \) the collection of all \( D \in \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \) which are universally Baire. If \( D \in \Gamma^\infty \), then there is an unambiguous version of \( D \) in any forcing extension of \( V \), which as usual we denote by \( D^* \). (2) then means that if \( D = p[U] \) and \( E = p[\bar{U}] \), then in any forcing extension of \( V \), \( E^* = \exists^\mathbb{R} D^* \).

If \( \Gamma \subset \Gamma^\infty \) is productive and if \( D \in \Gamma \), then any projective statement about \( D \) is absolute between \( V \) and any forcing extension of \( V \),\(^2\) i.e., if \( \varphi \) is projective, \( x_1, \ldots, x_k \in \mathbb{R} \), and \( \mathbb{P} \) is any poset, then

\[
V \models \varphi(x_1, \ldots, x_k, D) \iff \models_{\mathbb{P}} \varphi(\hat{x}_1, \ldots, \hat{x}_k, D^*).
\]

\(^1\)Here and elsewhere we often confuse a model with its underlying universe.

\(^2\)This seems to be wrong if we just assume \( \Gamma \subset \Gamma^\infty \), but the hypothesis that \( \Gamma \) be productive is crossed out.
By a theorem of Woodin, see e.g. [18, Theorem 1.2], combined with the key result of Martin and Steel in [13], the pointclass $\Gamma^\infty$ is productive under the hypothesis that there is a proper class of Woodin cardinals.

The proof from [17] produces the result that under PFA, the universe is closed under the operation $X \mapsto M^\#(X)$, which implies that every set of reals in $L(\mathbb{R})$ is universally Baire and that $\bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\mathbb{R})$ is productive. See e.g. [16, Section 3, pp. 187f.] on the relevant argument. Therefore, in the light of Lemma 2.2, Theorem 2.1 follows from the following more general statement.

**Theorem 2.3** Let $\Gamma \subset \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k)$. Assume that

1. $\Gamma = \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,
2. $\Gamma \subset \Gamma^\infty$,
3. $\Gamma$ is productive, and
4. Martin’s Maximum++ holds true.

Then $g_A$ is $\mathbb{P}_{\text{max}}$-generic over $L(\Gamma, \mathbb{R})$.\(^3\)

In the light of Lemma 2.2 and [6, Corollary 17], Theorem 2.3 readily follows from the following via a standard application of MM++.

**Lemma 2.4** Let $\Gamma \subset \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k)$. Assume that

1. $\Gamma = \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,
2. $\Gamma \subset \Gamma^\infty$,
3. $\Gamma$ is productive, and
4. $\text{NS}_{\omega_1}$ is saturated.\(^4\)

Let $D \subset \mathbb{P}_{\text{max}}$ be open dense, $D \in L(\Gamma, \mathbb{R})$.\(^5\) There is then a stationary set preserving forcing $\mathbb{P}$ such that in $V^\mathbb{P}$ there is some $p = (N; \in, I^*, a^*) \in D^*$ and some generic iteration

$$(N_i, \sigma_{ij}; i \leq j \leq \omega_1)$$

of $p = N_0$ of length $\omega_1 + 1$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, then $I^* = (\text{NS}_{\omega_1} V^\mathbb{P}) \cap N_{\omega_1}$ and $a^* = A$.

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\(^3\)In the presence of a proper class of Woodin cardinals, hypotheses (i), (ii), and (iv) then give $(*)_\Gamma$, see [16, Definition 4.1].

\(^4\)We could weaken this hypothesis to “$\text{NS}_{\omega_1}$ is precipitous.”

\(^5\)By hypothesis, $D$ is then universally Baire in the codes, so that there is an unambiguous version of $D$ in any forcing extension of $V$, which again we denote by $D^*$. 

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The attentive reader will notice that we don’t need the full power of $\text{MM}^{++}$ in order to derive Theorem 2.3 from Lemma 2.4, the hypothesis that $D\text{-BMM}^{++}$ holds true for all $D \in \Gamma^\infty$ would suffice, see [20, Definition 10.123]. By the proof of [1, Theorem 2.7], (*) is then actually equivalent to a version of $\text{BMM}$; we state this as follows.

**Theorem 2.5** Let $\Gamma \subset \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k)$. Assume that

(i) $\Gamma = \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) $\Gamma \subset \Gamma^\infty$,

(iii) $\Gamma$ is productive, and

(iv) $\text{NS}_{\omega_1}$ is saturated.\(^6\)

The following statements are then equivalent.

(1) $D\text{-BMM}^{++}$ holds true for all $D \in \Gamma$.

(2) $g_A$ is $\mathbb{P}_{\text{max}}$-generic over $L(\Gamma, \mathbb{R})$.

**Theorem 2.6** Assume that there is a proper class of Woodin cardinals. The following statements are then equivalent.

(1) $D\text{-BMM}^{++}$ holds true for all $D \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$.

(2) (*)

Our next section is entirely devoted to a proof of Lemma 2.4.

The authors thank Andreas Lietz for his comments on earlier versions of this paper.

### 3 The forcing.

Let us assume throughout the hypotheses of Lemma 2.4. We aim to verify its conclusion.

Let us fix $D \subset \mathbb{P}_{\text{max}}$, an open dense set in $L(\Gamma, \mathbb{R})$. By hypotheses (ii) and (iii) in the statement of Lemma 2.4 we will have that\(^6\)

\(^6\)Again, we could weaken this hypothesis to “$\text{NS}_{\omega_1}$ is precipitous.”
Let us identify $D$ with a canonical set of reals coding the elements of $D$,\footnote{We will have to spell out a bit more precisely below in which way we aim to have the elements of $p[T]$ code the elements of $D$, see (Σ.5) below.} and let $T \in V$ be a tree on $\omega \times 2^{\aleph_2}$ such that

$V^{\text{Col}(\omega, \omega_2)} \models "D^* \text{ is an open dense subset of } P_{\text{max}}."
$

Let us write $\kappa = (2^{\aleph_2})^+$, \hspace{1cm} (4)

so that $T \in H_\kappa$. Let $d$ be $\text{Col}(\kappa, \kappa)$-generic over $V$. In $V[d]$, let $(\bar{A}_\lambda : \lambda < \kappa)$ be a $\Diamond_\kappa$-sequence, i.e., for all $\bar{A} \subset \kappa$, $\{ \lambda < \kappa : \bar{A} \cap \lambda = \bar{A}_\lambda \}$ is stationary. Also, let $c : \kappa \to H_\kappa^V = H_\kappa^{V[d]}$, $c \in V[d]$, be bijective. For $\lambda < \kappa$, let

$Q_\lambda = c" \lambda$ and $A_\lambda = c" \bar{A}_\lambda$. \hspace{1cm} (5)

Let $C \subset \kappa$ be club such that for all $\lambda \in C$,

(i) $Q_\lambda$ is transitive,

(ii) $\{ T, ((H_\omega^V; \in, (\mathsf{NS}_{\omega_1})^V, A)) \cup 2^{\aleph_2} \subset Q_\lambda, \}

(iii) $Q_\lambda \cap \text{OR} = \lambda$ (so that $c \upharpoonright \lambda : \lambda \to Q_\lambda$ is bijective), and

(iv) $(Q_\lambda; \in) \prec (H_\kappa; \in)$.

In $V[d]$, for all $P$, $B \subset H_\kappa$, the set of all $\lambda \in C$ such that

$(Q_\lambda; \in, P \cap Q_\lambda, B \cap Q_\lambda) \prec (H_\kappa; \in, P, B)$

is club, and the set of all $\lambda \in C$ such that $B \cap Q_\lambda = A_\lambda$ is stationary, so that

$(\Diamond)$ For all $P$, $B \subset H_\kappa$ the set

$\{ \lambda \in C : (Q_\lambda; \in, P \cap Q_\lambda, A_\lambda) \prec (H_\kappa; \in, P, B) \}$

is stationary.
We shall sometimes also write \( Q_\kappa = H_\kappa \).

We shall now go ahead and produce a stationary set preserving forcing \( P \in V[d] \) which adds some \( p \in D^* \) and some generic iteration

\[
(N_i, \sigma_{ij} : i \leq j \leq \omega_1)
\]

of \( p = N_0 \) such that if we write \( N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*) \), then \( I^* = (NS_{\omega_1})^{V[d]^p \cap N_{\omega_1}} \) and \( a^* = A \). As the forcing \( \text{Col}(\kappa, \kappa) \) which added \( d \) is certainly stationary set preserving, this will verify Lemma 2.4.

\( NS_{\omega_1} \) is still saturated in \( V[d] \) and (D.1) and (D.2) are still true in \( V[d] \), so that in order to simplify our notation, we shall in what follows confuse \( V[d] \) with \( V \), i.e., pretend that in addition to “\( NS_{\omega_1} \) is saturated” plus (D.1) and (D.2), \( (\diamond) \) is also true in \( V \).

Working under these hypotheses, we shall now recursively define a \( \subset \)-increasing and continuous chain of forcings \( P_\lambda \) for all \( \lambda \in C \cup \{ \kappa \} \). The forcing \( P \) will be \( P_\kappa \).

Assume that \( \lambda \in C \cup \{ \kappa \} \) and \( P_\mu \) has already been defined in such a way that \( P_\mu \subset Q_\mu \) for all \( \mu \in C \cap \lambda \).

We shall be interested in objects \( \mathcal{C} \) which exist in some outer model\(^8\) and which have the following properties.

\[
\mathcal{C} = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) : n < \omega \rangle, \langle \lambda_\delta, X_\delta : \delta \in K \rangle, \tag{6}
\]

and

\( C.1 \) \( M_0, N_0 \in \mathbb{P}_{\text{max}}, \)

\( C.2 \) \( x = \langle k_n : n < \omega \rangle \) is a real code for \( N_0 \) and \( \langle (k_n, \alpha_n) : n < \omega \rangle \in [T], \)

\( C.3 \) \( \langle M_i, \pi_{ij} : i \leq j \leq \omega_1^N \rangle \in N_0 \) is a generic iteration of \( M_0 \) which witnesses that \( N_0 < M_0 \) in \( \mathbb{P}_{\text{max}}, \)

\( C.4 \) \( \langle N_i, \sigma_{ij} : i \leq j \leq \omega_1 \rangle \) is a generic iteration of \( N_0 \) such that if

\[
N_{\omega_1} = (N_{\omega_1}; \in, I^*, A^*),
\]

then \( A^* = A \).\(^9\)

\(^8\)W is an outer model iff \( W \) is a transitive model of \( \text{ZFC} \) with \( W \supset V \) and which has the same ordinals as \( V \); in other words, \( W \) is an outer model iff \( V \) is an inner model of \( W \).

\(^9\)There is no requirement on \( I^* \) matching the non-stationary ideal of some model in which \( \mathcal{C} \) exists.
\[(C.5) \langle M_i, \pi_{ij} : i \leq j \leq \omega_1 \rangle = \sigma_{0\omega_1}(\langle M_i, \pi_{ij} : i \leq j \leq \omega_1^{N_0} \rangle) \] and \[M_{\omega_1} = ((H_{\omega_2})^V, \in, (\text{NS}_{\omega_1})^V, A), \] (7)

\[(C.6) K \subset \omega_1, \]
and for all \(\delta \in K, \)
\[(C.7) \lambda_\delta < \lambda, \text{ and if } \gamma < \delta \text{ is in } K, \text{ then } \lambda_\gamma < \lambda_\delta \text{ and } X_\gamma \cup \{\lambda_\gamma\} \subset X_\delta, \text{ and} \]
\[(C.8) X_\delta \prec (Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \text{ and } X_\delta \cap \omega_1 = \delta. \]

We need to define a language \(\mathcal{L}\) (independently from \(\lambda\)) whose formulae will be able to describe \(\mathfrak{C}\) with the above properties by producing the models \(M_i\) and \(N_i, i < \omega_1\), as term models out of equivalence classes of terms of the form \(\hat{n}\), \(n < \omega\). The language \(\mathcal{L}\) will have the the following constants.

\(\hat{T}\) intended to denote \(T\)
\(x\) for every \(x \in H_{\kappa}\) intended to denote \(x\) itself
\(\hat{n}\) for every \(n < \omega\) as terms for elements of \(M_i\) and \(N_i, i < \omega_1\)
\(\hat{M}_i\) for \(i < \omega_1\) intended to denote \(M_i\)
\(\hat{\pi}_{ij}\) for \(i \leq j \leq \omega_1\) intended to denote \(\pi_{ij}\)
\(\hat{\vec{M}}\) intended to denote \((M_j, \pi_{jj'} : j \leq j' \leq \omega_1^{N_1})\) for \(i < \omega_1\)
\(\hat{N}_i\) for \(i < \omega_1\) intended to denote \(N_i\)
\(\hat{\sigma}_{ij}\) for \(i \leq j < \omega_1\) intended to denote \(\sigma_{ij}\)
\(\hat{a}\) intended to denote the distinguished \(a\)-predicate of \(M_i, N_i, i < \omega_1\)
\(\hat{I}\) intended to denote the distinguished ideal of \(N_i, i < \omega_1\)
\(\hat{X}_\delta\) for \(\delta < \omega_1\) intended to denote \(X_\delta\).

Formulae of \(\mathcal{L}\) will be of the following form.

\[\Gamma \vdash C \text{ for } C \text{ of the following form.} \]

\[\Gamma \vdash \varphi(\xi_1, \ldots, \xi_k, \hat{n}_1, \ldots, \hat{n}_\ell, \hat{a}, \hat{I}, \hat{M}_j, \ldots, \hat{M}_{j_m}, \hat{\pi}_{q_1r_1}, \ldots, \hat{\pi}_{q_s r_s}, \hat{\vec{M}}) \]

\[\text{for } i < \omega_1, \xi_1, \ldots, \xi_k < \omega_1, n_1, \ldots, n_\ell < \omega, j_1, \ldots, j_m < \omega_1, q_1 \leq r_1 < \omega_1, \ldots, q_s \leq r_s < \omega_1\]

\[\Gamma \vdash \hat{\pi}_{i\omega_1}(\hat{n}) = x \] for \(i < \omega_1\) and \(x \in H_{\omega_2}\)
\[\Gamma \vdash \hat{\pi}_{\omega_1 \omega_1}(x) = x \] for \(x \in H_{\omega_2}\)
\[\Gamma \vdash \hat{\sigma}_{ij}(\hat{n}) = \hat{m} \] for \(i \leq j < \omega_1, n, m < \omega\]
\[ \gamma(\vec{u}, \vec{a}) \in \hat{T}^- \quad \text{for} \quad \vec{u} \in <^\omega \omega \text{ and } \vec{a} \in <^\omega (2^{\aleph_2}) \]

\[ \gamma \delta \mapsto \hat{\lambda}^- \quad \text{for} \quad \delta < \omega_1, \hat{\lambda} < \kappa \]

\[ \gamma x \in \hat{X}_\delta^- \quad \text{for} \quad \delta < \omega_1, x \in H_\kappa \]

Let us write \( \mathcal{L}^\lambda \) for the collection of all \( \mathcal{L} \)-formulae except for the formulae which mention elements outside of \( Q_\lambda \), i.e., except for the formulae of the form \( \gamma \delta \mapsto \hat{\lambda}^- \) for \( \delta < \omega_1 \) and \( \lambda \leq \hat{\lambda} < \kappa \) and \( \gamma x \in \hat{X}_\delta^- \) for \( \delta < \omega_1 \) and \( x \in H_\kappa \setminus Q_\lambda \). We may and shall assume that \( \mathcal{L} \) is built in a canonical way so that \( \mathcal{L}^\lambda \subset Q_\lambda \).

We say that the objects \( \mathcal{C} \) as in (6) are pre-certified by a collection \( \Sigma \) of \( \mathcal{L}^\lambda \)-formulae if and only if (C.1) through (C.8) are satisfied by \( \mathcal{C} \) and there are surjections \( e_i : \omega \rightarrow N_i \) for \( i < \omega_1 \) such that the following hold true.

(\( \Sigma.1 \)) \( \gamma N_i \models \varphi(\xi_1, \ldots, \xi_k, \hat{n}_1, \ldots, \hat{n}_\ell, \hat{\alpha}, \hat{I}, \hat{M}_1, \ldots, \hat{M}_m, \hat{\pi}_{q_1r_1}, \ldots, \hat{\pi}_{q_rr}, \hat{\bar{M}})^\gamma \in \Sigma \) iff \( i < \omega_1, \xi_1, \ldots, \xi_k \leq \omega_{N_i}, n_1, \ldots, n_\ell < \omega, j_1, \ldots, j_m \leq \omega_{N_i}, q_1 \leq r_1 \leq \omega_{N_i}, \ldots, q_\delta \leq r_\delta \leq \omega_{N_i} \) and \( N_i \models \varphi(\xi_1, \ldots, \xi_k, e_i(n_1), \ldots, e_i(n_\ell), A \cap \omega_{N_i}, I^N_i, M_j, \ldots, M_m, \pi_{q_1r_1}, \ldots, \pi_{q_rr}, \bar{M}) \), where \( I^N_i \) is the distinguished ideal of \( N_i \) and \( \bar{M} = \langle M_j, \pi_{jj'} : j \leq j' \leq \omega_{N_i}^1 \rangle \).

(\( \Sigma.2 \)) \( \gamma \pi_{i\omega_1}(\hat{n}) = x^\gamma \in \Sigma \) iff \( i < \omega_1, n < \omega \), and \( \pi_{i\omega_1}(e_i(n)) = x \),

(\( \Sigma.3 \)) \( \gamma \pi_{\omega_1}(x) = x^\gamma \in \Sigma \) iff \( x \in H_\omega \),

(\( \Sigma.4 \)) \( \gamma \sigma_{ij}(\hat{n}) = \hat{m}^\gamma \in \Sigma \) iff \( i < j < \omega_1, n, m < \omega \), and \( \sigma_{ij}(e_i(n)) = e_j(m) \),

(\( \Sigma.5 \)) letting \( F \) with \( \text{dom}(F) = \omega \) be the monotone enumeration of the Gödel numbers of all \( \gamma N_0 \models \varphi(\hat{n}_1, \ldots, \hat{n}_\ell, \hat{\alpha}, \hat{I})^\gamma \) with \( \gamma N_0 \models \varphi(\hat{n}_1, \ldots, \hat{n}_\ell, \hat{\alpha}, \hat{I})^\gamma \in \Sigma \), we have that \( \gamma (\vec{u}, \vec{a}) \in T^\gamma \in \Sigma \) iff there is some \( n < \omega \) such that \( \langle \vec{u}, \vec{a} \rangle = \langle (F(m), \alpha_m) : m < n \rangle \) and \( F(m) = k_m \) for all \( m < n \),

(\( \Sigma.6 \)) \( \gamma \delta \mapsto \hat{\lambda}^\gamma \in \Sigma \) iff \( \delta \in K \) and \( \hat{\lambda} = \lambda_\delta \), and

(\( \Sigma.7 \)) \( \gamma x \in \hat{X}_\delta^- \in \Sigma \) iff \( \delta \in K \) and \( x \in X_\delta \).

We say that the objects \( \mathcal{C} \) as in (6) are certified by a collection \( \Sigma \) of formulae if and only if \( \mathcal{C} \) is pre-certified by \( \Sigma \) and in addition,

(\( \Sigma.8 \)) if \( \delta \in K \), then \( [\Sigma]^{<\omega} \cap X_\delta \cap E \neq \emptyset \) for every \( E \subset \mathbb{P}_{\lambda_\delta} \) which is dense in \( \mathbb{P}_{\lambda_\delta} \) and definable over the structure \((Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, \lambda_\delta)\)

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from parameters in \(X_\delta\).\(^{10}\)

By way of definition, we call \(C\) as in (6) a **semantic certificate** iff there is a collection \(\Sigma\) of formulae such that \(C\) is certified by \(\Sigma\). We call \(\Sigma\) a **syntactic certificate** iff there is a semantic certificate \(C\) such that \(C\) is certified by \(\Sigma\). Given a syntactic certificate \(\Sigma\), there is a unique semantic certificate \(C\) such that \(C\) is certified by \(\Sigma\). Even though it is obvious how to construct \(C\) from \(\Sigma\), in the proof of Lemma 3.3 below we will provide details on how to derive a semantic certificate from a given \(\Sigma\).

Let \(\Sigma \cup p\) be a set of formulae, where \(p\) is finite. We say that \(p\) is certified by \(\Sigma\) if and only if there is some (unique) \(C\) as in (6) such that \(C\) is certified by \(\Sigma\) and

\[(\Sigma.9) \ p \in [\Sigma]^{<\omega}.
\]

We may also say that \(p\) is certified by \(C\) as in (6) iff there is some \(\Sigma\) such that \(C\) and \(p\) are both certified by \(\Sigma\) – and we will then also refer to \(\Sigma\) as a syntactical certificate for \(p\) and to \(C\) as the associated semantical certificate.

We are then ready to define the forcing \(P_\lambda\). We say that \(p \in P_\lambda\) if and only if

\[V^{\text{Col}(\omega, \lambda)} \models \text{"There is a set } \Sigma \text{ of } \mathcal{L}_\lambda \text{-formulae such that } p \text{ is certified by } \Sigma." \quad (8)
\]

Let \(p\) be a finite set of formulae of \(\mathcal{L}_\lambda\). By the homogeneity of \(\text{Col}(\omega, \lambda)\), if there is some \(h\) which is \(\text{Col}(\omega, \lambda)\)-generic over \(V\) and there is some \(\Sigma \in V[h]\) such that \(p\) is certified by \(\Sigma\), then for all \(h\) which are \(\text{Col}(\omega, \lambda)\)-generic over \(V\) there is some \(\Sigma \in V[h]\) such that \(p\) is certified by \(\Sigma\). It is then easy to see that \(\langle P_\lambda : \lambda \in C \cup \{\kappa\} \rangle\) is definable over \(V\) from \(\langle A_\lambda : \lambda < \kappa \rangle\) and \(C\), and is hence an element of \(V\).

Again let \(p\) be a finite set of formulae of \(\mathcal{L}_\lambda\). By \(\Sigma_1^1\) absoluteness, if there is any outer model in which there is some \(\Sigma\) which certifies \(p\), then there is some \(\Sigma \in V^{\text{Col}(\omega, \lambda)}\) which certifies \(p\).\(^{11}\) This simple observation is important in the verification that \(P_\lambda\) is actually non-empty, cf. Lemma 3.2, and in the proof of Lemma 3.8.

It is easy to see that

1. \(P = \mathbb{P}_\kappa \subset H_\kappa\),
2. if \(\bar{\lambda} < \lambda\) are both in \(C \cup \{\kappa\}\), then \(\mathbb{P}_{\bar{\lambda}} \subset \mathbb{P}_\lambda\), and

\(^{10}\)Equivalently, \([\Sigma]^{<\omega} \cap E \neq \emptyset\) for every \(E \subset \mathbb{P}_\lambda \cap X_\delta\) which is dense in \(\mathbb{P}_\lambda \cap X_\delta\) and definable over the structure \((X_\delta; \in, \mathbb{P}_\lambda \cap X_\delta, A_{\lambda_\delta} \cap X_\delta)\)

\(^{11}\)In fact, if \(P\) is a transitive model of KP plus the axiom Beta with \((Q_\lambda; \langle A_\lambda : \bar{\lambda} < \lambda \rangle) \in P\) and if \(p \in P_\lambda\), then there is some \(\Sigma \in P^{\text{Col}(\omega, \lambda)}\) which certifies \(p\).
(iii) if $\lambda \in C \cup \{\kappa\}$ is a limit point of $C \cup \{\kappa\}$, then $\mathbb{P}_\lambda = \bigcup_{\lambda \in C \cap \lambda} \mathbb{P}_\lambda$, so that there is some club $D \subset C$ such that for all $\lambda \in D$,

$$\mathbb{P}_\lambda = \mathbb{P} \cap Q_\lambda.$$ 

Hence $(\diamondsuit)$ gives us the following.

$(\diamondsuit(\mathbb{P}))$ For all $B \subset H_\kappa$ the set

$$\{\lambda \in C : (Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\kappa; \in, \mathbb{P}, B)\}$$

is stationary.

The first one of the following lemmas is entirely trivial.

**Lemma 3.1** Let $\Sigma$ be a syntactic certificate, and let $p, q \in [\Sigma]^{<\omega}$. Then $p$ and $q$ are compatible conditions in $\mathbb{P}$.

**Lemma 3.2** Let $\lambda \in C \cup \{\kappa\}$. Then $\emptyset \in \mathbb{P}_\lambda$.

**Proof.** See the proof of [1, Theorem 2.8], or the proof of [16, Theorem 4.2]. Notice that for all $\lambda \in C \cup \{\kappa\}$, $\emptyset \in \mathbb{P}_\lambda$ iff $\emptyset \in \mathbb{P}$.

Let $h$ be Col($\omega, \omega_2$)-generic over $V$, and write $\rho = \omega_3^V = \omega_1^{V[h]}$. Inside $V[h]$,

$$((H_{\omega_2})^V; \in, (NS_{\omega_1})^V, A)$$

is a $\mathbb{P}_{\max}$ condition, call it $p$. Let $q \in (\mathbb{P}_{\max})^{V[h]}$, $q < p$, $q \in D^*$, cf. (D.1). Let $q = N_0 = (N_0; \in, I, a)$. Let $(M_i, \pi_{ij} : i \leq j \leq \omega^N_0) \in N_0$ be the unique generic iteration of $p$ which witnesses $q < p$.

Let $(N_i, \sigma_{ij} : i \leq j \leq \rho) \in V[h]$ be a generic iteration of $N_0$ such that $\rho = \omega^N_1$.\footnote{If we wished, then we could even arrange that writing $N_\rho = (N_\rho; \in, I^*, a^*)$, we have that $I^* = (NS_\rho)^{V[h]} \cap N_\rho$, but this is not relevant here; cf. footnote 9.}

Let

$$(M_i, \pi_{ij} : i \leq j \leq \rho) = \sigma_0((M_i, \pi_{ij} : i \leq j \leq \omega^N_0))$$

We may lift (10) to a generic iteration

$$(M_i^+, \pi_{ij}^+ : i \leq j \leq \rho)$$

of $V$. Let us write $M = M_\rho^+$ and $\pi = \pi_0^+$. 

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Let \( \langle k_n, \alpha_n : n < \omega \rangle \) be such that \( x = \langle k_n : n < \omega \rangle \) is a real code for \( N_0 \) à la (\( \Sigma, 5 \)) and \( \langle (k_n, \alpha_n) : n < \omega \rangle \in \langle T \rangle \). We then clearly have that \( \langle (k_n, \pi(\alpha_n)) : n < \omega \rangle \in [\pi(T)] \).

It is now easy to see that

\[
C = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} : i \leq j \leq \rho \rangle, \langle (k_n, \pi(\alpha_n)) : n < \omega \rangle, \rangle \tag{11}
\]
certifies \( \emptyset \), construed as the empty set of \( \pi(L^\kappa) \) formulae: as the third component \( \langle \rangle \) of \( C \) in (11) is empty, any set of surjections \( \epsilon_i : \omega \to N_i, i < \omega, \) will induce a syntactic certificate for \( \emptyset \) whose associated semantic certificate is \( C \). By \( \Sigma^1_1 \) absoluteness, there is then some \( C \in M^{Col(\omega, \pi(\omega))} \) as in (11) which certifies \( \emptyset \) so that \( \emptyset \in \pi(P) \). By the elementarity of \( \pi \), then, there is some \( C \in V^{Col(\omega, \omega_2)} \) which certifies \( \emptyset \), construed as the empty set of \( L^\kappa \) formulae. Hence \( \emptyset \in \mathbb{P} \). □

**Lemma 3.3** Let \( \lambda \in C \cup \{ \kappa \} \). Let \( g \subset P_\lambda \) be a filter such that \( g \cap E \neq \emptyset \) for all dense \( E \subset P_\lambda \) which are definable over \( (Q_\lambda; e, P_\lambda) \) from elements of \( Q_\lambda \). Then \( \bigcup g \) is a syntactic certificate.

**Proof.** It is obvious how to read off from \( \bigcup g \) a candidate

\[
C = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) : n < \omega \rangle, \langle \lambda_\delta, X_\delta : \delta \in K \rangle
\]
like in (6) for a semantical certificate for \( \bigcup g \). Let us be somewhat explicit, though. A variant of what is to come shows how to derive \( C \) from a given syntactic certificate \( \Sigma \), where \( C \) is unique such that \( \Sigma \) certifies \( C \), cf. the remark on p. 10.

For \( i, j < \omega_1 \) and \( \tau, \sigma \in \{ \dot{n} : n < \omega \} \cup \omega_1 \) define

\[
\tau \sim_{i} \sigma \quad \text{iff} \quad \Gamma \dot{N}_i \models \tau = \sigma^\gamma \in \bigcup g
\]
\[
(i, \tau) \sim_{\omega_1} (j, \sigma) \quad \text{iff} \quad i \leq j \wedge \exists \rho \{ \Gamma \dot{\sigma}_{ij}(\tau) = \rho^\gamma, \Gamma \dot{N}_j \models \rho = \sigma^\gamma \} \subset \bigcup g
\]
or
\[
\text{or} \quad j \leq i \wedge \exists \rho \{ \Gamma \dot{\sigma}_{ji}(\sigma) = \rho^\gamma, \Gamma \dot{N}_j \models \rho = \tau^\gamma \} \subset \bigcup g
\]

\[
[j, \tau] = \{ (i, \sigma) : (i, \tau) \sim_{\omega_1} (j, \sigma) \}
\]
\[
M_i = \{ [\tau]_i : \tau \in \{ \dot{n} : n < \omega \} \cup \omega_1 \wedge \Gamma \dot{N}_i \models \tau \in \dot{M}_i \} \in \bigcup g \}
\]
\[
M_{\omega_1} = (H_{\omega_2})^V
\]
\[
N_i = \{ [\tau]_i : \tau \in \{ \dot{n} : n < \omega \} \cup \omega_1 \}
\]
\[
N_{\omega_1} = \{ [i, \tau]_{\omega_1} : i < \omega_1 \wedge \tau \in \dot{M}_i \} \in \bigcup g \}
\]

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For each $i < \omega_1$, let us first show:

**Proposition 3.4.** Straightforward arguments using (Σ.1) show that $\langle k, \alpha \rangle = (k^\theta_n, \alpha^\theta_n)$ if $\exists \bar{u} \exists \bar{\alpha} \langle \bar{u}, \bar{\alpha} \rangle \in \mathcal{T} \forall k = \bar{u}(n) \land \alpha = \bar{\alpha}(n)$

$\delta \in K^\theta$ if $\exists \bar{\lambda} \Rightarrow \bar{\lambda} \forall \in \mathcal{U}$

$\bar{\lambda} = \lambda^\theta_n$ if $\delta \in K^\theta \Rightarrow \bar{\lambda} \forall \in \mathcal{U}$

$x \in X^\theta_n$ if $\delta \in K^\theta \Rightarrow x \in \mathcal{X} \forall \in \mathcal{U}$

We will first have that $\bar{e}_0$ is wellfounded and that in fact (the transitive collapse of) the structure $N_0 = \langle N_0; \bar{e}_0, a^{N_0}, I^{N_0} \rangle$ is an iterable $\mathfrak{P}_{\max}$ condition. This is true because straightforward density arguments give (C.2), i.e., that $\langle (k_n, \alpha_n) : n < \omega \rangle \in [T]$ and $\langle k_n : n < \omega \rangle$ will code the theory of $N_0 \amb{à la} (\Sigma.5)$.

Another set of easy density arguments will give that $(N_i, \sigma_{ij} : i \leq j \leq \omega_1)$ is a generic iteration of $N_0$, were we identify $N_i$ with the structure $(N_i; \bar{e}_i, a^{N_i}, I^{N_i})$. To verify this, let us first show:

**Claim 3.4** For each $i < \omega_1$ and for each $\xi \leq \omega_1^{N_i}$, $[\xi]_i$ represents $\xi$ in (the transitive collapse of the well-founded part of) the term model for $N_i$; moreover, $a^{N_i} = A \cap \omega_1^{N_i}$. Hence $a^{N_{\omega_1}} = A$.

**Proof of Claim 3.4.** Straightforward arguments using (Σ.1) show that $[\xi]_i$ must always represent $\xi$ in $N_i$ as given by any certificate. Claim 3.4 then follows by straightforward density arguments. \(\square\) (Claim 3.4)
Similarly:

**Claim 3.5** Let $i < \omega_1$. $N_{i+1}$ is generated from $\text{ran}(\sigma_{i+1}) \cup \{\omega_1^{N_i}\}$ in the sense that for every $x \in N_{i+1}$ there is some function $f \in \omega_1^{N_i} (N_i) \cap N_i$ such that $x = \sigma_{i+1}(f)(\omega_1^{N_i})$.

**Claim 3.6** Let $i < \omega_1$. \{X \in \mathcal{P}(\omega_1^{N_i}) \cap N_i: \omega_1^{N_i} \in \sigma_{i+1}(X)\} is generic over $N_i$ for the forcing given by the $\mathcal{P}_i$-positive sets.

**Claim 3.7** Let $i \leq \omega_1$ be a limit ordinal. For every $x \in N_i$ there is some $j < i$ and some $\bar{x} \in N_j$ such that $x = \sigma_{ji}(\bar{x})$.

$(N_i, \sigma_{ij}; i \leq j \leq \omega_1)$ is then indeed a generic iteration of $N_0$. As $N_0$ is iterable, we may and shall identify $N_i$ with its transitive collapse, so that (C.4) holds true.

Another round of density arguments will show that $\mathcal{C}$ satisfies (C.1), (C.3), (C.5), (C.6), and (C.7), where we identify $M_i$ with the structure $(M_i; \in, (\mathcal{N}_j)_{\omega_1^M_i}, A \cap \omega_1^{M_i})$. Let us now verify (C.8) and (C.9).

As for (C.8), $X_\delta \cap \omega_1 = \delta$ for $\delta \in K$ is easy. Let $x_1, \ldots, x_k \in X_\delta$, $\delta \in K$. Suppose that

\[(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \vDash \exists v \varphi(v, x_1, \ldots, x_k). \tag{12}\]

Let $p \in g$ be such that \{\[\gamma x_1 \in \hat{X}_\delta, \ldots, \gamma x_k \in \hat{X}_\delta, \gamma \delta \mapsto \lambda_\delta \}\} \subset p$. Let $q \leq p$, and let $\Sigma$ be a syntactical certificate for $q$ whose associated semantical certificate is

$\mathcal{C}' = \langle M'_i, \pi'_i, N'_i, \sigma'_ij; i \leq j \leq \omega_1; \{(k'_n, \alpha'_n) : n < \omega\}; (\lambda'_\delta, X'_\delta; \delta \in K')$.

Then $\delta \in K'$ and

\[\{x_1, \ldots, x_k\} \subset X'_\delta \prec (Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta})\]

so that by (12) we may choose some $x \in X'_\delta$ with

\[(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \vDash \varphi(x, x_1, \ldots, x_k)\].

Let $r = q \cup \{\gamma x \in \hat{X}_\delta \}$. By density, there is then some $y \in X_\delta$ such that

\[(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \vDash \varphi(y, x_1, \ldots, x_k)\].

The proof of (C.9) is similar. Let again $\delta \in K$. Let $E \subset \mathbb{P}_{\lambda_\delta} \cap X_\delta$ be dense in $\mathbb{P}_{\lambda_\delta} \cap X_\delta$, and $r \in E$ iff $r \in \mathbb{P}_{\lambda_\delta} \cap X_\delta$ and

\[(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \vDash \varphi(r, x_1, \ldots, x_k). \tag{13}\]
Let \( p \in g \) be such that \( \{ \forall x_1 \in \hat{X}_\delta \gamma, \ldots, \forall x_k \in \hat{X}_\delta \gamma, \forall \delta \rightarrow \lambda_\delta \gamma \} \subset p \). Let \( q \leq p \), and again let \( \Sigma \) be a syntactical certificate for \( q \) whose associated semantical certificate is

\[
\mathcal{C}' = \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) : n < \omega \rangle, \langle \lambda'_\delta, X'_\delta : \delta \in K' \rangle.
\]

Then \( [\Sigma]^{\omega} \cap X'_\delta \) has an element, say \( r \), such that (13) holds true. Let \( s = q \cup r \cup \{ \forall r \in \hat{X}_\delta \gamma \} \).

By density, then, \( g \cap X_\delta \cap E \neq \emptyset \).

\[ \square \]

**Lemma 3.8** Let \( g \) be \( \mathbb{P} \)-generic over \( V \). Let

\[
\mathcal{C} = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) : n < \omega \rangle, \langle \lambda_\delta, X_\delta : \delta \in K \rangle
\]

be the semantic certificate associated with the syntactic certificate \( \bigcup g \). Let

\[ N_{\omega_1} = (N_{\omega_1}; \in, A, I^*) \]

and let \( T \in (P(\omega_1) \cap N_{\omega_1}) \setminus I^* \). Then \( T \) is stationary in \( V[g] \).

If \( \mathcal{C}, I^* \), and \( T \) are as in the statement of Lemma 3.8, then by Lemma 3.3 and (C.3) and (C.4) we will have that \( (\text{NS}_{\omega_1})^V = I^* \cap V \), so that the conclusion of Lemma 3.8 also gives that \( \mathbb{P} \) preserves the stationarity of \( T \). In other words:

**Corollary 3.9** \( \mathbb{P} \) preserves stationary subsets of \( \omega_1 \).

**Proof** of Lemma 3.8. Let \( \hat{N}_{\omega_1} \in V^{\mathbb{P}} \) be a canonical name for \( N_{\omega_1} \), and let \( \hat{I}^* \in V^{\mathbb{P}} \) be a canonical name for \( I^* \). Let \( \bar{p} \in g, \hat{C}, \hat{S} \in V^{\mathbb{P}}, \) and \( i < \omega_1 \) and \( n < \omega \) be such that

(i) \( T = \hat{S}^\gamma \),

(ii) \( \bar{p} \models \text{"} \hat{C} \subset \omega_1 \text{ is club,"} \)

(iii) \( \bar{p} \models \text{"} \hat{S} \in (P(\omega_1) \cap \hat{N}_{\omega_1}) \setminus \hat{I}^* \text{,"} \) and

(vi) \( \bar{p} \models \text{"} \hat{S} \text{ is represented by } [i, n] \text{ in the term model producing } \hat{N}_{\omega_1} \text{."} \)

We may and shall also assume that

\[ \hat{N}_i \models n \text{ is a subset of the first uncountable cardinal, yet } \hat{n} \notin \hat{I}^\gamma \in \bar{p}. \]  \[ (14) \]

Let \( p \leq \bar{p} \) be arbitrary. We aim to produce some \( q \leq p \) and some \( \delta < \omega_1 \) such that \( q \models \delta \in \check{C} \cap \check{S} \), see Claim 3.11 below.
For $\xi < \omega_1$, let
\[
D_\xi = \{ q \leq p : \exists \eta \geq \xi (\eta < \omega_1 \land q \models \tilde{\eta} \in \dot{C}) \},
\]
so that $D_\xi$ is open dense below $p$. Let
\[
E = \{ (q, \eta) \in \mathbb{P} \times \omega_1 : q \models \tilde{\eta} \in \dot{C} \}.
\]
Let us write
\[
\tau = ((D_\xi : \xi < \omega_1), E).
\]
We may and shall identify $\tau$ with some subset of $H_\kappa$ which codes $\tau$.

By ($\diamond (\mathbb{P})$), we may pick some $\lambda \in C$ such that $p \in \mathbb{P}_\lambda$ and
\[
(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\kappa; \in, \mathbb{P}, \tau).
\] (15)

Let $h$ be Col$(\omega, 2^{\aleph_2})$-generic over $V$, and let $g' \in V[h]$ be a filter on $\mathbb{P}_\lambda$ such that $p \in g'$ and $g'$ meets every dense set which is definable over $(N_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$ from parameters in $N_\lambda$. By Lemma 3.3, $\bigcup g'$ is a syntactic certificate for $p$, and we may let
\[
\langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) : n < \omega \rangle, \langle \lambda'_\delta, X'_\delta : \delta \in K' \rangle
\]
be the associated semantic certificate. In particular, $K' \subset \lambda$.

Let $S$ denote the subset of $\omega_1$ which is represented by $[i, \tilde{n}]$ in the term model giving $N'_{\omega_1}$, so that if $N'_{\omega_1} = (N'_{\omega_1}, \in, A, I')$, then by (14),
\[
S \in (\mathcal{P}(\omega_1) \cap N'_{\omega_1}) \setminus I'.
\] (16)

Let us also write $\rho = \omega_1^{V[h]} = (2^{\aleph_2})^{+V}$. Inside $V[h]$, we may extend $\langle N'_i, \sigma'_{ij} : i \leq j \leq \omega_1 \rangle$ to a generic iteration
\[
\langle N'_i, \sigma'_{ij} : i \leq j \leq \rho \rangle
\]
such that
\[
\omega_1 \in \sigma'_{\omega_1, \omega_1+1}(S).
\] (17)

This is possible as $\omega_1^{N'_{\omega_1}} = \sup \{ \omega_1^{N_j} : j < \omega_1 \} = \omega_1$ and by (16). Let
\[
\langle M'_i, \pi'_{ij} : i \leq j \leq \rho \rangle = \sigma_{0,\rho}(\langle M'_i, \pi'_{ij} : i \leq j \leq \omega_1^{N'_{\omega_1}} \rangle),
\]
so that $\langle M'_i, \pi'_{ij} : i \leq j \leq \rho \rangle$ is an extension of $\langle M'_i, \pi'_{ij} : i \leq j \leq \omega_1 \rangle$. 

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Recalling (7), we may lift \((M^*_i, \pi^*_{ij}: \omega_1 \leq i \leq j \leq \rho)\) to a generic iteration
\[(M^+_i, \pi^+_ij: \omega_1 \leq i \leq j \leq \rho)\]
of \(V\). Let us write \(M = M^*_\rho\) and \(\pi = \pi^*_\omega,\rho^*\).

The key point is now that \((M^*_i, \pi^*_{ij}, N^*_i, \sigma^*_{ij}: i \leq j \leq \rho)\) may be used to extend \(\pi'' \cup g'\) to a syntactic certificate
\[\Sigma \supset \pi'' \cup g'\] (18)
for \(\pi(p)\) in the following manner. Let \(K^* = K' \cup \{\omega_1\}\). For \(\delta \in K'\), let \(\lambda^*_\delta = \pi(\lambda^*_\delta)\) and \(X^*_\delta = \pi'' X^*_\delta\). Also, write \(\lambda^*_\omega = \pi(\lambda)\) and \(X^*_\omega = \pi'' Q\lambda\). Let
\[\mathfrak{C}^* = \langle M^*_i, \pi^*_{ij}, N^*_i, \sigma^*_{ij}: i \leq j \leq \rho \rangle, ((k^*_n, \pi(\alpha^*_n)): n < \omega), (\lambda^*_\delta, X^*_\delta: \delta \in K^*)\].
It is then straightforward to verify that \(\mathfrak{C}^*\) is a semantic certificate for \(\pi(p)\), and that in fact there is some syntactic certificate \(\Sigma\) as in (18) such that \(\mathfrak{C}^*\) is certified by \(\Sigma\).

Now let \(\hat{m}|_{\omega_1+1}\) represent \(\sigma'_{\omega_1,\omega_1+1}(S)\) in the term model for \(N'_{\omega_1+1}\) provided by \(\Sigma\), so that\(^\text{13}\)
\[\{\hat{\sigma}_{\omega_1+1}(\hat{n}) = \hat{m}, \hat{N}_{\omega_1+1} \models \omega_1 \in \hat{m}\}\} \subset \Sigma\],
in other words,
\[\pi(p) \cup \{\hat{\sigma}_{\omega_1+1}(\hat{n}) = \hat{m}, \hat{N}_{\omega_1+1} \models \omega_1 \in \hat{m}\}\} \text{ is certified by } \Sigma\]. (19)

Let us now define
\[q^* = \pi(p) \cup \{\hat{\sigma}_{\omega_1+1}(\hat{n}) = \hat{m}, \hat{N}_{\omega_1+1} \models \omega_1 \in \hat{m}, \hat{\omega}_1 \mapsto \pi(\lambda)\}\}.\] (20)

We thus established the following.

Claim 3.10 \(q^* \in \pi(\mathbb{P})\), as being certified by \(\Sigma\).

The elementarity of \(\pi: V \rightarrow M^+\) then gives some \(\delta < \omega_1\) and some \(\mu < \kappa\) such that
\[q = p \cup \{\hat{\sigma}_{\delta+1}(\hat{n}) = \hat{m}, \hat{N}_{\delta+1} \models \delta \in \hat{m}, \hat{\omega}_1 \mapsto \pi(\lambda)\} \in \mathbb{P}\]. (21)

Claim 3.11 \(q \models \check{\delta} \in \hat{C} \cap \hat{S}\).

\(^\text{13}\)Here, \(\hat{\sigma}_{\omega_1+1}\) and \(\hat{N}_{\omega_1+1}\) are terms of the language associated with \(\pi(\mathbb{P}_\lambda)\) and \(\hat{\sigma}_{\omega_1+1}(\hat{n}) = \hat{m}\) and \(\hat{N}_{\omega_1+1} \models \omega_1 \in \hat{m}\) are formulae of that language.
Proof of Claim 3.11. \( q \models \delta \in \check{S} \) readily follows from \( \{ \check{\sigma}_{i\delta+1}(\check{n}) = \check{m}, \check{N}_{\delta+1} \models \delta \in \check{m} \} \subset q \), the fact that \( \check{p} \geq p \) forces that \( \check{S} \) is represented by \([i, \check{n}]\) in the term model giving \( \check{N}_{\omega_1}, \) and the fact that by Claim 3.4, \( [\check{\delta}]_{\delta+1} \) represents \( \delta \) in the model \( N_{\delta+1} \) of any semantic certificate for \( q \).

Let us now show that \( q \models \check{\delta} \in \check{C} \). We will in fact show that \( q \) forces that \( \check{\delta} \) is a limit point of \( \check{C} \). Otherwise there is some \( r \leq q \) and some \( \eta < \delta \) such that

\[ r \models \check{C} \cap \check{\delta} \subset \check{\eta}. \tag{22} \]

Let

\[ \langle M_i', \pi_{ij}', N_i', \sigma_{ij}': i \leq j \leq \omega_1 \rangle, \langle (k_n', \alpha_n)': n < \omega \rangle, \langle \lambda', X': \check{\delta} \in K' \rangle \tag{23} \]

certify \( r \). We must have that

(a) \( \delta \in K' \),

(b) \( X' \delta < (Q; \in, P, A) \),

(c) \( X' \delta \cap \omega_1 = \delta \), and

(d) for some \( \Sigma \) such that the objects from (23) are certified by \( \Sigma \), \( [\Sigma]^{\omega} \cap X' \delta \cap E \neq \emptyset \) for every \( E \subset P \) which is dense in \( P \cap X' \delta \) and definable over the structure \( (Q; \in, P, A) \)

from parameters in \( X' \delta \).

Notice that \( A = \tau \cap Q \), and hence \( A \) may be identified with \((D \cap Q; \xi < \omega_1), E \cap Q \)). As \( \eta < \delta \subset X' \delta \), \( D \eta \) is definable over the structure

\[ (Q; \in, P, A) \]

from a parameter in \( X' \delta \). By (15), \( D \eta \cap Q \) is dense in \( P \). By (d) above, there is then some \( s \in [\Sigma]^{\omega} \cap X' \delta \cap D \eta \cap Q \).

By (15) again, the unique smallest \( \eta' \succeq \eta \) with \( s \models \check{\eta'} \in \check{C} \) must be in \( X' \delta \), hence \( \eta' < \delta \) by (c) above. By Lemma 3.1, \( s \) is compatible with \( r \). We have reached a contradiction with (22). \( \square \)
References


[16] R. Schindler, *Woodin’s axiom (\ast), or Martin’s Maximum, or both?*, in: Foundations of mathematics, essays in honor of W. Hugh Woodin’s 60th birthday, Harvard University (Caicedo et al., eds.), pp. 177-204.


