MM\textsuperscript{++} implies (\ast)

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Abstract

We show that Martin’s Maximum\textsuperscript{++} implies Woodin’s \(P_{\text{max}}\) axiom (\ast).

1 Introduction.

Cantor’s Continuum Problem, which later became Hilbert’s first Problem (see [7]), asks how many real numbers there are. This question got a non-answer through the discovery of the method of forcing by Paul Cohen: \(\text{CH}\), Cantor’s Continuum Hypothesis, is independent from \(\text{ZFC}\) (see [3]), the standard axiom system for set theory which had been isolated by Zermelo and Fraenkel. \(\text{CH}\) states that every uncountable set of reals has the same size as \(\mathbb{R}\).

Ever since Cohen, set theorists have been searching for natural new axioms which extend \(\text{ZFC}\) and which settle the Continuum Problem. See e.g. [21], [22], [9], and the discussion in [4]. There are two prominent such axioms which decide \(\text{CH}\) in the negative and which in fact both prove that there are \(\aleph_2\) reals: Martin’s Maximum (\(\text{MM}\), for short) or variants thereof on the one hand (see [6]), and Woodin’s axiom (\(\ast\)) on the other hand (see [20]). See e.g. [12], [19], and [14].

Both of these axioms may be construed as maximality principles for the theory of the structure \((H_{\omega_2}; \in)\), but up to this point the relationship between \(\text{MM}\) and (\(\ast\)) was a bit of a mystery, which led M. Magidor to call (\(\ast\)) a “competitor” of \(\text{MM}\) ([12, p. 18]). Both \(\text{MM}\) and (\(\ast\)) are inspired by and formulated in the language of forcing,

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and they both have “the same intuitive motivation: Namely, the universe of sets is rich” ([12, p. 18]).

This paper resolves the tension between MM and (⋆) by proving that MM++, a strengthening of MM, actually implies (⋆), see Theorem 2.1 below, so that MM and (⋆) are actually compatible with each other. This answers [20, Question (18) a) on p. 924], see also [20, p. 846], [12, Conjecture 6.8 on p. 19], and [14, Problem 14.7].

2 Preliminaries.

Martin’s Maximum++, abbreviated by MM++, see [6] (cf. also [20, Definition 2.45 (2)]), is the statement that if \( P \) is a forcing which preserves stationary subsets of \( \omega_1 \), if \( \{ D_i : i < \omega_1 \} \) is a collection of dense subsets of \( P \), and if \( \{ \tau_i : i < \omega_1 \} \) is a collection of \( P \)-names for stationary subsets of \( \omega_1 \), then there is a filter \( g \subset P \) such that for every \( i < \omega_1 \),

(i) \( g \cap D_i \neq \emptyset \) and

(ii) \( \langle \tau_i \rangle^g = \{ \xi < \omega_1 : \exists p \in g \ p \Vdash P \xi \in \tau_i \} \) is stationary.

Woodin’s \( P_{\text{max}} \) axiom (⋆), see [20, Definition 5.1], is the statement that

(i) \( \text{AD} \) holds in \( L(\mathbb{R}) \) and

(ii) there is some \( g \) which is \( P_{\text{max}} \)-generic over \( L(\mathbb{R}) \) such that \( P(\omega_1) \cap V \subset L(\mathbb{R})[g] \).

Already PFA, the Proper Forcing Axiom, which is weaker than MM++, implies \( \text{AD}^{L(\mathbb{R})} \) and much more, see [17], [8], and [15, Chapter 12]. This paper produces a proof of the following result.

**Theorem 2.1** Assume Martin’s Maximum++. Then Woodin’s \( P_{\text{max}} \)-axiom (⋆) holds true.

Our key new idea is (Σ.8) on page 10 below.

P. Larson, see [10] and [11], has shown that \( \text{MM}^{+\omega} \) is consistent with \( \neg(\ast) \) relative to a supercompact limit of supercompact cardinals. Our proof is also optimal in that the forcing which we will use to verify Theorem 2.1 has size \( \aleph_3 \), while W.H. Woodin has shown that \( \text{MM}^{++} \) for forcings of size \( 2^{\aleph_0} = \aleph_2 \) does not imply (⋆), see [20, Theorem 10.90].

Throughout our entire paper, “\( \omega_1 \)” will always denote \( \omega_1^V \), the \( \omega_1 \) of \( V \).
Let us fix throughout this paper some $A \subset \omega_1$ such that $\omega_1^{L[A]} = \omega_1$. Let us define $g_A$ as the set of all $\mathbb{P}_{\text{max}}$ conditions $p = (N; \in, I, a)$ such that there is a generic iteration 

$$(N_i, \sigma_{ij}; i \leq j \leq \omega_1)$$

of $p = N_0$ of length $\omega_1 + 1$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, then $I^* = (\text{NS}_{\omega_1})^V \cap N_{\omega_1}$ and $a^* = A$.

**Lemma 2.2 (Woodin)** Assume that $\text{NS}_{\omega_1}$ is saturated and that $\mathcal{P}(\omega_1)^\#$ exists.

1. $g_A$ is a filter.
2. If $g_A$ is $\mathbb{P}_{\text{max}}$-generic over $L(R)$, then $\mathcal{P}(\omega_1) \subset L(R)[g]$.

**Proof.** This routinely follows from the proof of [20, Lemma 3.12 and Corollary 3.13] and from [20, Lemma 3.10]. \qed

One may also use BMM plus “$\text{NS}_{\omega_1}$ is precipitous” to show that $g_A$ is a filter, this is by the proof from [2].

Let $\Gamma \subset \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}_k)$. We say that $\Gamma$ is **productive** iff for all $k < \omega$ and all $D \in \Gamma \cap \mathcal{P}(\mathbb{R}^{k+2})$, if $D$ is universally Baire (see [5]) as being witnessed by the trees $T$ and $U$ on $k+2 \omega \times \text{OR}$, i.e., $D = p[T]$ and for all posets $\mathbb{P}$,

$$\ forcing \quad \mathbb{P} \quad p[U] = \mathbb{R}^{k+2} \setminus p[T], \quad (1)$$

and if

$$\hat{\mathcal{U}} = \{ (s \restriction (k+1), (s(k+1), t)): (s, t) \in U \}, \quad (2)$$

so that $(x_0, \ldots, x_k) \in p[\hat{\mathcal{U}}]$ iff there is some $y$ such that $(x_0, \ldots, x_k, y) \in p[U]$, then there is a tree $\hat{T}$ on $k+1 \omega \times \text{OR}$ such that for all posets $\mathbb{P}$,

$$\ forcing \quad \mathbb{P} \quad p[\hat{\mathcal{U}}] = \mathbb{R}^{k+1} \setminus p[\hat{T}], \quad (3)$$

Let us denote by $\Gamma^\infty$ the collection of all $D \in \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}_k)$ which are universally Baire. If $D \in \Gamma^\infty$, then there is an unambiguous version of $D$ in any forcing extension of $V$, which as usual we denote by $D^*$. (2) then means that if $D = p[U]$ and $E = p[\hat{\mathcal{U}}]$, then in any forcing extension of $V$, $E^* = \exists^{\mathbb{R}} D^*$.

\footnote{Here and elsewhere we often confuse a model with its underlying universe.}
If $\Gamma \subset \Gamma^\infty$ is productive and if $D \in \Gamma$, then any projective statement about $D$ is absolute between $V$ and any forcing extension of $V$, i.e., if $\varphi$ is projective, $x_1, \ldots, x_k \in \mathbb{R}$, and $P$ is any poset, then

$$V \models \varphi(x_1, \ldots, x_k, D) \iff \Vdash_P \varphi(\check{x}_1, \ldots, \check{x}_k, D^*)$$

By a theorem of Woodin, see e.g. [18, Theorem 1.2], combined with the key result of Martin and Steel in [13], the pointclass $\Gamma^\infty$ is productive under the hypothesis that there is a proper class of Woodin cardinals.

The proof from [17] produces the result that under PFA, the universe is closed under the operation $X \mapsto M^\#_\omega(X)$, which implies that every set of reals in $L(\mathbb{R})$ is universally Baire and that $\bigcup_{k<\omega} P(\mathbb{R}^k) \cap L(\mathbb{R})$ is productive. See e.g. [16, Section 3, pp. 187f.] on the relevant argument. Therefore, in the light of Lemma 2.2, Theorem 2.1 follows from the following more general statement.

**Theorem 2.3** Let $\Gamma \subset \bigcup_{k<\omega} P(\mathbb{R}^k)$. Assume that

(i) $\Gamma = \bigcup_{k<\omega} P(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) $\Gamma \subset \Gamma^\infty$,

(iii) $\Gamma$ is productive, and

(iv) Martin’s Maximum$^{++}$ holds true.

Then $g_A$ is $\mathbb{P}_{\text{max}}$-generic over $L(\Gamma, \mathbb{R})$.

In the light of Lemma 2.2 and [6, Corollary 17], Theorem 2.3 readily follows from the following via a standard application of MM$^{++}$.

**Lemma 2.4** Let $\Gamma \subset \bigcup_{k<\omega} P(\mathbb{R}^k)$. Assume that

(i) $\Gamma = \bigcup_{k<\omega} P(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) $\Gamma \subset \Gamma^\infty$,

(iii) $\Gamma$ is productive, and

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2This seems to be wrong if we just assume $\Gamma \subset \Gamma^\infty$, but the hypothesis that $\Gamma$ be productive is crossed out.

3In the presence of a proper class of Woodin cardinals, hypotheses (i), (ii), and (iv) then give $(\ast)_\Gamma$, see [16, Definition 4.1].
(iv) $\text{NS}_{\omega_1}$ is saturated.\footnote{We could weaken this hypothesis to “$\text{NS}_{\omega_1}$ is precipitous.”}

Let $D \subset \mathbb{P}_{\text{max}}$ be open dense, $D \in L(\Gamma, \mathbb{R})$.\footnote{By hypothesis, $D$ is then universally Baire in the codes, so that there is an unambiguous version of $D$ in any forcing extension of $V$, which again we denote by $D^*$.} There is then a stationary set preserving forcing $\mathbb{P}$ such that in $V^\mathbb{P}$ there is some $p = (N; \in, I^*, a^*) \in D^*$ and some generic iteration

$$(Ni, \sigma_{ij}; i \leq j \leq \omega_1)$$

of $p = N_0$ of length $\omega_1 + 1$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, then $I^* = (\text{NS}_{\omega_1})^{V^\mathbb{P}} \cap N_{\omega_1}$ and $a^* = A$.

The attentive reader will notice that we don’t need the full power of $\text{MM}^{++}$ in order to derive Theorem 2.3 from Lemma 2.4, the hypothesis that $D\text{-BMM}^{++}$ holds true for all $D \in \Gamma^\infty$ would suffice, see [20, Definition 10.123]. By the proof of [1, Theorem 2.7], (*) is then actually equivalent to a version of $\text{BMM}$; we state this as follows.

**Theorem 2.5** Let $\Gamma \subset \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k)$. Assume that

(i) $\Gamma = \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) $\Gamma \subset \Gamma^\infty$,

(iii) $\Gamma$ is productive, and

(iv) $\text{NS}_{\omega_1}$ is saturated.\footnote{Again, we could weaken this hypothesis to “$\text{NS}_{\omega_1}$ is precipitous.”}

The following statements are then equivalent.

(1) $D\text{-BMM}^{++}$ holds true for all $D \in \Gamma$.

(2) $g_A$ is $\mathbb{P}_{\text{max}}$-generic over $L(\Gamma, \mathbb{R})$.

**Theorem 2.6** Assume that there is a proper class of Woodin cardinals. The following statements are then equivalent.

(1) $D\text{-BMM}^{++}$ holds true for all $D \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$.

(2) (*).

Our next section is entirely devoted to a proof of Lemma 2.4.

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3 The forcing.

Let us assume throughout the hypotheses of Lemma 2.4. We aim to verify its conclusion.

Let us fix $D \subset P_{\text{max}}$, an open dense set in $L(\Gamma, \mathbb{R})$. By hypotheses (ii) and (iii) in the statement of Lemma 2.4 we will have that

(D.1) $V^{\text{Col}(\omega, \omega_2)} \models "D^* \text{ is an open dense subset of } P_{\text{max}}."$

Let us identify $D$ with a canonical set of reals coding the elements of $D$, and let $\tilde{T} \in V$ be a tree on $\omega \times 2^{\omega_2}$ such that

(D.2) $V^{\text{Col}(\omega, \omega_2)} \models D^* = p[\tilde{T}]$.

Let $h$ be $\text{Col}(\omega, \omega_2)$-generic over $V$. Inside $V[h]$,

$$((H_{\omega_2})^V; \in, (\text{NS}_{\omega_1})^V, A)$$

is a $P_{\text{max}}$ condition, call it $p$. Let $q^* \in (P_{\text{max}})^V[h]$, $q^* \prec p$, $q^* \in D^*$, cf. (D.1). Let $q^* = N^* = (N^*; \in, I^*, a^*)$. Identifying $q^*$ with some real coding it, we have that $q^* \in p[\tilde{T}]$, cf. (D.2). Let $T \in V$ be a tree on $\omega \times \omega_2$ such that

$$q^* \in p[T] \subset p[\tilde{T}].$$

Let us write

$$\kappa = \aleph_3,$$

so that $T \in H_\kappa$. Let $d$ be $\text{Col}(\kappa, \kappa)$-generic over $V$. In $V[d]$, let $(\tilde{A}_\lambda; \lambda < \kappa)$ be a $\diamondsuit_\kappa$-sequence, i.e., for all $\tilde{A} \subset \kappa$, $\{\lambda < \kappa; \tilde{A} \cap \lambda = \tilde{A}_\lambda\}$ is stationary. Also, let $c: \kappa \to H_\kappa^V = H_\kappa^{V[d]}$, $c \in V[d]$, be bijective. For $\lambda < \kappa$, let

$$Q_\lambda = c^\kappa \lambda \text{ and } A_\lambda = c^\kappa \tilde{A}_\lambda.$$ (6)

Let $C \subset \kappa$ be club such that for all $\lambda \in C$,

(i) $Q_\lambda$ is transitive,

(ii) $\{T, ((H_{\omega_2})^V; \in, (\text{NS}_{\omega_1})^V, A) \cup (\omega_2 + 1) \subset Q_\lambda,$

\footnote{We will have to spell out a bit more precisely below in which way we aim to have the elements of $p[T]$ code the elements of $D$, see (Σ.5) below.}
(iii) $Q_\lambda \cap \text{OR} = \lambda$ (so that $c \restriction \lambda : \lambda \to Q_\lambda$ is bijective), and

(iv) $(Q_\lambda; \in) \prec (H_\kappa; \in)$.

In $V[d]$, for all $P, B \subset H_\kappa$, the set of all $\lambda \in C$ such that

$$(Q_\lambda; \in, P \cap Q_\lambda, B \cap Q_\lambda) \prec (H_\kappa; \in, P, B)$$

is club, and the set of all $\lambda \in C$ such that $B \cap Q_\lambda = A_\lambda$ is stationary, so that

$(\Diamond)$ For all $P, B \subset H_\kappa$ the set

$$\{ \lambda \in C : (Q_\lambda; \in, P \cap Q_\lambda, A_\lambda) \prec (H_\kappa; \in, P, B) \}$$

is stationary.

We shall sometimes also write $Q_\kappa = H_\kappa$.

We shall now go ahead and produce a stationary set preserving forcing $\mathbb{P} \in V[d]$ of size $\kappa$ which adds some $p \in D^*$ and some generic iteration

$$(N_i, \sigma_{ij} : i \leq j \leq \omega_1)$$

of $p = N_0$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, then $I^* = (\text{NS}_{\omega_1})^{V[d]} \cap N_{\omega_1}$ and $a^* = A$. As the forcing Col($\kappa, \kappa$) which added $d$ is certainly stationary set preserving, this will verify Lemma 2.4.

$\text{NS}_{\omega_1}$ is still saturated in $V[d]$ and (D.1) and (D.2) are still true in $V[d]$, so that in order to simplify our notation, we shall in what follows confuse $V[d]$ with $V$, i.e., pretend that in addition to “$\text{NS}_{\omega_1}$ is saturated” plus (D.1) and (D.2), $(\Diamond)$ is also true in $V$.

Working under these hypotheses, we shall now recursively define a $\subset$-increasing and continuous chain of forcings $\mathbb{P}_\lambda$ for all $\lambda \in C \cup \{\kappa\}$. The forcing $\mathbb{P}$ will be $\mathbb{P}_\kappa$.

Assume that $\lambda \in C \cup \{\kappa\}$ and $\mathbb{P}_\mu$ has already been defined in such a way that $\mathbb{P}_\mu \subset Q_\mu$ for all $\mu \in C \cap \lambda$.

We shall be interested in objects $\mathfrak{C}$ which exist in some outer model$^8$ and which have the following properties.

$$\mathfrak{C} = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) : n < \omega \rangle, \langle \lambda_\delta, X_\delta : \delta \in K \rangle,$$

and

$^8$W is an outer model iff $W$ is a transitive model of ZFC with $W \supset V$ and which has the same ordinals as $V$; in other words, $W$ is an outer model iff $V$ is an inner model of $W$. 

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\(M_0, N_0 \in \mathbb{P}_{\text{max}},\)
\(x = \langle k_n : n < \omega \rangle\) is a real code for \(N_0\) and \(\langle (k_n, \alpha_n) : n < \omega \rangle \in [T],\)
\(\langle M_i, \pi_{ij} : i \leq j \leq \omega_1 \rangle \in N_0\) is a generic iteration of \(M_0\) which witnesses that \(N_0 < M_0\) in \(\mathbb{P}_{\text{max}},\)
\(\langle N_i, \sigma_{ij} : i \leq j \leq \omega_1 \rangle\) is a generic iteration of \(N_0\) such that if \(N_\omega = (N_\omega; \in, I^*, A^*),\)
\(\langle M_i, \pi_{ij} : i \leq j \leq \omega_1 \rangle = \sigma_{0\omega_1}(\langle M_i, \pi_{ij} : i \leq j \leq \omega_1^{N_0} \rangle)\) and
\(M_\omega = ((H_\omega)^V; \in, (NS_\omega)^V, A),\) \hfill (8)
\(K \subset \omega_1,\)
and for all \(\delta \in K,\)
\(\lambda_\delta < \lambda,\) and if \(\gamma < \delta\) is in \(K,\) then \(\lambda_\gamma < \lambda_\delta\) and \(X_\gamma \cup \{\lambda_\gamma\} \subset X_\delta,\) and
\(X_\delta \prec (Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta})\) and \(X_\delta \cap \omega_1 = \delta.\)

We need to define a language \(\mathcal{L}\) (independently from \(\lambda\)) whose formulae will be able to describe \(\mathcal{C}\) with the above properties by producing the models \(M_i\) and \(N_i,\) \(i < \omega_1,\) as term models out of equivalence classes of terms of the form \(\hat{n}, n < \omega.\) The language \(\mathcal{L}\) will have the the following constants.

\(\hat{T}\) intended to denote \(T\)
\(\hat{x}\) for every \(x \in H_\kappa\) intended to denote \(x\) itself
\(\hat{n}\) for every \(n < \omega\) as terms for elements of \(M_i\) and \(N_i, i < \omega_1\)
\(\hat{M}_i\) for \(i < \omega_1\) intended to denote \(M_i\)
\(\hat{\pi}_{ij}\) for \(i \leq j \leq \omega_1\) intended to denote \(\pi_{ij}\)
\(\hat{M}\) intended to denote \((M_j, \pi_{jj'} : j \leq j' \leq \omega_1^{N_i})\) for \(i < \omega_1\)
\(\hat{N}_i\) for \(i < \omega_1\) intended to denote \(N_i\)

\[^{9}\text{There is no requirement on } I^*\text{ matching the non-stationary ideal of some model in which } \mathcal{C} \text{ exists.}\]
\[ \sigma_{ij} \text{ for } i \leq j < \omega_1 \text{ intended to denote } \sigma_{ij} \]
\[ a \text{ intended to denote the distinguished } a\text{-predicate of } M_i, N_i, i < \omega_1 \]
\[ \hat{I} \text{ intended to denote the distinguished ideal of } N_i, i < \omega_1 \]
\[ \hat{X}_\delta \text{ for } \delta < \omega_1 \text{ intended to denote } X_\delta \].

Formulae of \( \mathcal{L} \) will be of the following form.

\[ \gamma \hat{N}_i \models \varphi(\xi_1, \ldots, \xi_k, \bar{n}_1, \ldots, \bar{n}_\ell, \bar{a}, \hat{I}, \hat{M}_{j_1}, \ldots, \hat{M}_{j_m}, \bar{\sigma}_{q_{r_1}}, \ldots, \bar{\sigma}_{q_{r_s}}, \hat{M}) \gamma \]

for \( i < \omega_1, \xi_1, \ldots, \xi_k < \omega_1, n_1, \ldots, n_\ell < \omega, j_1, \ldots, j_m < \omega_1, q_1 \leq r_1 < \omega_1, \ldots, q_s \leq r_s < \omega_1 \)

\[ \gamma \bar{\sigma}_{i\omega_1}(\bar{n}) = x \gamma \text{ for } i < \omega_1 \text{ and } x \in H_{\omega_2} \]

\[ \gamma \bar{\sigma}_{\omega_1}\omega_1(x) = x \gamma \text{ for } x \in H_{\omega_2} \]

\[ \gamma \bar{\sigma}_{ij}(\bar{n}) = \bar{m} \gamma \text{ for } i \leq j < \omega_1, n, m < \omega \]

\[ \gamma (\bar{u}, \bar{a}) \in \hat{T} \gamma \text{ for } \bar{u} \in \omega_1^{< \omega_1} \text{ and } \bar{a} \in \omega(2^{\omega_1}) \]

\[ \gamma \delta \mapsto \lambda \gamma \text{ for } \delta < \omega_1, \lambda < \kappa \]

\[ \gamma x \in \hat{X}_\delta \gamma \text{ for } \delta < \omega_1, x \in H_\kappa \]

Let us write \( \mathcal{L}^\lambda \) for the collection of all \( \mathcal{L} \)-formulae except for the formulae which mention elements outside of \( Q_\lambda \), i.e., except for the formulae of the form \( \gamma \delta \mapsto \lambda \gamma \) for \( \delta < \omega_1 \) and \( \lambda < \kappa \) and \( \gamma x \in \hat{X}_\delta \gamma \) for \( \delta < \omega_1 \) and \( x \in H_\kappa \setminus Q_\lambda \). We may and shall assume that \( \mathcal{L} \) is built in a canonical way so that \( \mathcal{L}^\lambda \subset Q_\lambda \).

We say that the objects \( \mathcal{C} \) as in (7) are pre-certified by a collection \( \Sigma \) of \( \mathcal{L}^\lambda \)-formulae if and only if (C.1) through (C.8) are satisfied by \( \mathcal{C} \) and there are surjections \( e_i : \omega \to N_i \) for \( i < \omega_1 \) such that the following hold true.

(\( \Sigma.1 \)) \( \gamma \hat{N}_i \models \varphi(\xi_1, \ldots, \xi_k, \bar{n}_1, \ldots, \bar{n}_\ell, \bar{a}, \hat{I}, \hat{M}_{j_1}, \ldots, \hat{M}_{j_m}, \bar{\sigma}_{q_{r_1}}, \ldots, \bar{\sigma}_{q_{r_s}}, \hat{M}) \gamma \in \Sigma \text{ iff } \)

\[ i < \omega_1, \xi_1, \ldots, \xi_k \leq \omega_1^{N_i}, n_1, \ldots, n_\ell < \omega, j_1, \ldots, j_m \leq \omega_1^{N_i}, q_1 \leq r_1 \leq \omega_1^{N_i}, \ldots, q_s \leq r_s \leq \omega_1^{N_i} \text{ and } \]

\[ N_i \models \varphi(\xi_1, \ldots, \xi_k, e_i(n_1), \ldots, e_i(n_\ell), A(\omega_1^{N_i}, \bar{I}), \bar{I}, \bar{M}_{j_1}, \ldots, \bar{M}_{j_m}, \bar{\sigma}_{q_{r_1}}, \ldots, \bar{\sigma}_{q_{r_s}}, \bar{M}), \]

where \( I^{N_i} \) is the distinguished ideal of \( N_i \) and \( \bar{M} = (M_j, \pi_{jj'} : j \leq j' \leq \omega_1^{N_i}) \).

(\( \Sigma.2 \)) \( \gamma \bar{\sigma}_{i\omega_1}(\bar{n}) = x \gamma \in \Sigma \text{ iff } i < \omega_1, n < \omega, \text{ and } \pi_{i\omega_1}(e_i(n)) = x, \)

(\( \Sigma.3 \)) \( \gamma \bar{\sigma}_{\omega_1}\omega_1(x) = x \gamma \in \Sigma \text{ iff } x \in H_{\omega_2}, \)

(\( \Sigma.4 \)) \( \gamma \bar{\sigma}_{ij}(\bar{n}) = \bar{m} \gamma \in \Sigma \text{ iff } i \leq j < \omega_1, n, m < \omega, \text{ and } \sigma_{ij}(e_i(n)) = e_j(m), \)

(\( \Sigma.5 \)) letting \( F \) with \( \text{dom}(F) = \omega \) be the monotone enumeration of the Gödel numbers of all \( \gamma \hat{N}_0 \models \varphi(\bar{n}_1, \ldots, \bar{n}_\ell, \bar{a}, \hat{I}) \gamma \) with \( \gamma \hat{N}_0 \models \varphi(\bar{n}_1, \ldots, \bar{n}_\ell, \bar{a}, \hat{I}) \gamma \in \Sigma, \)
we have that $\vec{u}, \vec{\alpha} \in \hat{T}^\gamma \in \Sigma$ iff there is some $n < \omega$ such that $\langle \vec{u}, \vec{\alpha} \rangle = \langle (F(m), \alpha_m) : m < n \rangle$ and $F(m) = k_m$ for all $m < n$.

$(\Sigma.6)$ $\gamma \delta \mapsto \bar{\lambda}^\gamma \in \Sigma$ iff $\delta \in \mathcal{K}$ and $\bar{\lambda} = \lambda_\delta$, and

$(\Sigma.7)$ $\gamma x \in \hat{X}^\gamma \in \Sigma$ iff $\delta \in \mathcal{K}$ and $x \in X_\delta$.

We say that the objects $\mathcal{C}$ as in (7) are certified by a collection $\Sigma$ of formulae if and only if $\mathcal{C}$ is pre-certified by $\Sigma$ and in addition,

$(\Sigma.8)$ if $\delta \in \mathcal{K}$, then $[\Sigma]^{\omega} \cap X_\delta \cap E \neq \emptyset$ for every $E \subset \mathbb{P}_\lambda$ which is dense in $\mathbb{P}_\lambda$ and definable over the structure

$$(Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta})$$

from parameters in $X_\delta$.\(^{10}\)

By way of definition, we call $\mathcal{C}$ as in (7) a semantic certificate iff there is a collection $\Sigma$ of formulae such that $\mathcal{C}$ is certified by $\Sigma$. We call $\Sigma$ a syntactic certificate iff there is a semantic certificate $\mathcal{C}$ such that $\mathcal{C}$ is certified by $\Sigma$. Given a syntactic certificate $\Sigma$, there is a unique semantic certificate $\mathcal{C}$ such that $\mathcal{C}$ is certified by $\Sigma$. Even though it is obvious how to construct $\mathcal{C}$ from $\Sigma$, in the proof of Lemma 3.3 below we will provide details on how to derive a semantic certificate from a given $\Sigma$.

Let $\Sigma \cup p$ be a set of formulae, where $p$ is finite. We say that $p$ is certified by $\Sigma$ if and only if there is some (unique) $\mathcal{C}$ as in (7) such that $\mathcal{C}$ is certified by $\Sigma$ and

$(\Sigma.9)$ $p \in [\Sigma]^{\omega}$.

We may also say that $p$ is certified by $\mathcal{C}$ as in (7) iff there is some $\Sigma$ such that $\mathcal{C}$ and $p$ are both certified by $\Sigma$ – and we will then also refer to $\Sigma$ as a syntactical certificate for $p$ and to $\mathcal{C}$ as the associated semantical certificate.

We are then ready to define the forcing $\mathbb{P}_\lambda$. We say that $p \in \mathbb{P}_\lambda$ if and only if

$V^{\text{Col}(\omega, \lambda)} \models \text{“There is a set } \Sigma \text{ of } \mathcal{L}_\lambda \text{-formulae such that } p \text{ is certified by } \Sigma.\text{”} \quad (9)$

Let $p$ be a finite set of formulae of $\mathcal{L}_\lambda$. By the homogeneity of $\text{Col}(\omega, \lambda)$, if there is some $h$ which is $\text{Col}(\omega, \lambda)$-generic over $V$ and there is some $\Sigma \in V[h]$ such that

\(^{10}\)Equivalently, $[\Sigma]^{\omega} \cap E \neq \emptyset$ for every $E \subset \mathbb{P}_{\lambda_\delta} \cap X_\delta$ which is dense in $\mathbb{P}_{\lambda_\delta} \cap X_\delta$ and definable over the structure

$$(X_\delta; \in, \mathbb{P}_{\lambda_\delta} \cap X_\delta, A_{\lambda_\delta} \cap X_\delta)$$

from parameters in $X_\delta$. 10
is certified by $\Sigma$, then for all $h$ which are $\text{Col}(\omega, \lambda)$-generic over $V$ there is some $\Sigma \in V[h]$ such that $p$ is certified by $\Sigma$. It is then easy to see that $\langle P_\lambda : \lambda \in C \cup \{\kappa\} \rangle$ is definable over $V$ from $\langle A_\lambda : \lambda < \kappa \rangle$ and $C$, and is hence an element of $V$.

Again let $p$ be a finite set of formulae of $\mathcal{L}^\lambda$. By $\Sigma_1^1$ absoluteness, if there is any outer model in which there is some $\Sigma$ which certifies $p$, then there is some $\Sigma \in V^{\text{Col}(\omega, \lambda)}$ which certifies $p$.\textsuperscript{11} This simple observation is important in the verification that $P_\lambda$ is actually non-empty, cf. Lemma 3.2, and in the proof of Lemma 3.8.

It is easy to see that

\begin{enumerate}[(i)]
  \item $P = P_\kappa \subset H_\kappa$,
  \item if $\bar{\lambda} < \lambda$ are both in $C \cup \{\kappa\}$, then $P_{\bar{\lambda}} \subset P_\lambda$, and
  \item if $\lambda \in C \cup \{\kappa\}$ is a limit point of $C \cup \{\kappa\}$, then $P_\lambda = \bigcup_{\bar{\lambda} \in C \cap \lambda} P_{\bar{\lambda}}$,
\end{enumerate}

so that there is some club $D \subset C$ such that for all $\lambda \in D$,

$$P_\lambda = P \cap Q_\lambda.$$

Hence ($\Diamond$) gives us the following.

\begin{enumerate}[(\Diamond(P))]\setcounter{enumi}{2}
  \item For all $B \subset H_\kappa$ the set
    $$\{ \lambda \in C : (Q_\lambda; \in, P_\lambda, A_\lambda) \prec (H_\kappa; \in, P, B) \}$$
    is stationary.
\end{enumerate}

The first one of the following lemmas is entirely trivial.

**Lemma 3.1** Let $\Sigma$ be a syntactic certificate, and let $p, q \in [\Sigma]^{<\omega}$. Then $p$ and $q$ are compatible conditions in $P$.

**Lemma 3.2** Let $\lambda \in C \cup \{\kappa\}$. Then $\emptyset \in P_\lambda$.

**Proof.** See the proof of [1, Theorem 2.8], or the proof of [16, Theorem 4.2]. Notice that for all $\lambda \in C \cup \{\kappa\}$, $\emptyset \in P_\lambda$ iff $\emptyset \in P$.

Let again $h$ be $\text{Col}(\omega, \omega_2)$-generic over $V$. Let $q^* = N^* = (N^*; \in, I^*, a^*) \in (\mathbb{P}_{\text{max}})^{V[h]}$ be as in the paragraph preceding (4), so that (4) is true. Let $(M_i, \pi_{ij} : i \leq j \leq \omega_1^N) \in N^*$ be the unique generic iteration of $p$ which witnesses $q < p$.

\textsuperscript{11}In fact, if $P$ is a transitive model of $\mathcal{KP}$ plus the axiom $\text{Beta}$ with $\langle Q_\lambda; (A_\lambda : \bar{\lambda} < \lambda) \rangle \in P$ and if $p \in P_\lambda$, then there is some $\Sigma \in P^{\text{Col}(\omega, \lambda)}$ which certifies $p$.  

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Let \((N_i, \sigma_{ij} : i \leq j \leq \kappa) \in V[h]\) be a generic iteration of \(N_0 = N^*\) such that 
\[\kappa = \omega_1^{N_0}.\]  
Let

\[(M_i, \pi_{ij} : i \leq j \leq \kappa) = \sigma_{0\kappa}((M_i, \pi_{ij} : i \leq j \leq \omega_1^{N_0}))\]  

(10)
We may lift (11) to a generic iteration

\[(M_i^+, \pi_{ij}^+ : i \leq j \leq \kappa)\]  

of \(V\). Let us write

\[M = M^+_\kappa\]  

and \(\pi = \pi^+_{0\kappa}\).

Let \(\langle k_n, \alpha_n : n < \omega \rangle\) be such that \(x = \langle k_n : n < \omega \rangle\) is a real code for \(N_0\) à la \((\Sigma.5)\) and \(\langle (k_n, \alpha_n) : n < \omega \rangle \in [T]\). We then clearly have that \(\langle (k_n, \pi(\alpha_n)) : n < \omega \rangle \in [\pi(T)]\).

It is now easy to see that

\[C = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} : i \leq j \leq \kappa \rangle, \langle (k_n, \pi(\alpha_n)) : n < \omega \rangle, \langle \rangle \]  

(12)

certifies \(\emptyset\), construed as the empty set of \(\pi(L^\kappa)\) formulae: as the third component \(\langle \rangle\) of \(C\) in (12) is empty, any set of surjections \(e_i : \omega \to N_i, i < \omega_1\), will induce a syntactic certificate for \(\emptyset\) whose associated semantical certificate is \(C\). By \(\Sigma_1^1\) absoluteness, there is then some \(C \in M^\text{Col}(\omega, \pi(\omega_2))\) as in (12) which certifies \(\emptyset\) so that \(\emptyset \in \pi(P)\). By the elementarity of \(\pi\), then, there is some \(C \in V^{\text{Col}(\omega, \omega_2)}\) which certifies \(\emptyset\), construed as the empty set of \(L^\kappa\) formulae. Hence \(\emptyset \in \pi(P)\).  

\[\square\]

**Lemma 3.3** Let \(\lambda \in C \cup \{\kappa\}\). Let \(g \subset P_\lambda\) be a filter such that \(g \cap E \neq \emptyset\) for all dense \(E \subset P_\lambda\) which are definable over \((Q_\lambda : \in, P_\lambda)\) from elements of \(Q_\lambda\). Then \(\bigcup g\) is a syntactic certificate.

**Proof.** It is obvious how to read off from \(\bigcup g\) a candidate

\[C = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) : n < \omega \rangle, \langle \lambda_\delta, X_\delta : \delta \in K \rangle\]  

like in (7) for a semantical certificate for \(\bigcup g\). Let us be somewhat explicit, though. A variant of what is to come shows how to derive \(C\) from a given syntactic certificate \(\Sigma\), where \(C\) is unique such that \(\Sigma\) certifies \(C\), cf. the remark on p. 10.

For \(i, j < \omega_1\) and \(\tau, \sigma \in \{\hat{\eta} : n < \omega\} \cup \omega_1\) define

\[\tau \sim_i \sigma \iff \hat{\eta}_i \models \tau = \sigma \iff \tau \in \bigcup g\]

If we wished, then we could even arrange that writing \(N_\kappa = (N_\kappa : \in, I', a')\), we have that \(I' = (N_\kappa)^{V[h] \cap N_\kappa}\), but this is not relevant here; cf. footnote 9.
(i, τ) \sim_{\omega_1} (j, σ) \iff i \leq j \land \exists \rho \{ \gamma \check{\sigma}_{ij}(\tau) = \rho \land \gamma \check{N}_j \vDash \rho = \sigma \} \subset \bigcup g \\
\text{or } j \leq i \land \exists \rho \{ \gamma \check{\sigma}_{ji}(\sigma) = \rho \land \gamma \check{N}_j \vDash \rho = \tau \} \subset \bigcup g \\
[τ]_i = \{ σ : τ \sim_i σ \} \\
[(i, τ)] = \{ (j, σ) : (i, τ) \sim_{\omega_1} (j, σ) \} \\
M_i = \{ [τ]_i : τ \in \{ \check{n} : n < \omega \} \cup \omega_1 \land \gamma \check{N}_i \vDash \tau \in M_i \subset \bigcup g \} \\
M_{\omega_1} = (H_{\omega_2})^V \\
N_i = \{ [τ]_i : τ \in \{ \check{n} : n < \omega \} \cup \omega_1 \} \\
N_{\omega_1} = \{ [i, τ]_{\omega_1} : i < \omega_1 \land τ \in \check{M}_i \subset \bigcup g \} \\
[τ]_i, \check{c}_i [σ]_i \iff \gamma \check{N}_i \vDash τ \in σ \subset \bigcup g \\
[i, τ][\check{C}]_{\omega_1} [j, σ] \iff i \leq j \land \exists \rho \{ \gamma \check{\sigma}_{ij}(\tau) = \rho \land \gamma \check{N}_j \vDash \rho \in σ \} \subset \bigcup g \\
\text{or } j \leq i \land \exists \rho \{ \gamma \check{\sigma}_{ji}(\sigma) = \rho \land \gamma \check{N}_j \vDash \tau \in ρ \} \subset \bigcup g \\
[τ]_i \in I^{N_i} \iff \gamma \check{N}_i \vDash \tau \in \check{I} \subset \bigcup g \\
[i, τ] \in I^{N_{\omega_1}} \iff [τ]_i \in I^{N_i} \\
[τ]_i \in a^{N_i} \iff \gamma \check{N}_i \vDash τ \in \check{α} \subset \bigcup g \\
[i, τ] \in a^{N_{\omega_1}} \iff [τ]_i \in I^{N_i} \\
π_{ij}([τ]_i) = [σ]_j \iff \gamma \check{N}_j \vDash \check{π}_{ij}(τ) = σ \subset \bigcup g \\
π_{i\omega_1}([τ]_i) = x \iff \gamma \check{π}_{i\omega_1}(τ) = x \subset \bigcup g \\
π_{i\omega_1}(x) = x \iff x \in (H_{\omega_2})^V \\
σ_{ij}([τ]_i) = [σ]_j \iff \gamma \check{σ}_{ij}(τ) = σ \subset \bigcup g \\
σ_{i\omega_1}([τ]_i) = [i, τ] \\
(k, α) = (k^n, α^n) \iff \exists \check{u}, \check{α} \{ \gamma \check{u}(τ, \check{α}) \in \check{I} \subset \bigcup g \land k = \check{u}(n) \land α = \check{α}(n) \} \\
δ \in K^g \iff \exists \check{λ} \gamma \delta \Rightarrow \check{λ} \subset \bigcup g \\
\check{λ} = λ^g_δ \iff \delta \in K^g \land \gamma \delta \Rightarrow \check{λ} \subset \bigcup g \\
x \in X^g_δ \iff \delta \in K^g \land \gamma x \in X^g_δ \subset \bigcup g \\

We will first have that \check{C}_0 is wellfounded and that in fact (the transitive collapse of) the structure \check{N}_0 = (N_0; \check{C}_0, a^{N_0}, I^{N_0}) is an iterable \mathbb{P}_{\text{max}} condition. This is true
because straightforward density arguments give (C.2), i.e., that \( \langle (k_n, \alpha_n) : n < \omega \rangle \in [T] \) and \( \langle k_n : n < \omega \rangle \) will code the theory of \( N_0 \) à la (Σ.5).

Another set of easy density arguments will give that \( (N_i, \sigma_{ij} : i \leq j \leq \omega_1) \) is a generic iteration of \( N_0 \), were we identify \( N_i \) with the structure \( (N_i; \xi_i, a^{N_i}, I^{N_i}) \). To verify this, let us first show:

**Claim 3.4** For each \( i < \omega_1 \) and for each \( \xi \leq \omega_1^{N_i} \), \( \langle \xi \rangle_i \) represents \( \xi \) in (the transitive collapse of the well-founded part of) the term model for \( N_i \); moreover, \( a^{N_i} = A \cap \omega_1^{N_i} \). Hence \( a^{N_{\omega_1}} = A \).

**Proof** of Claim 3.4. Straightforward arguments using (Σ.1) show that \( \langle \xi \rangle_i \) must always represent \( \xi \) in \( N_i \) as given by any certificate. Claim 3.4 then follows by straightforward density arguments. \( \Box \) (Claim 3.4)

Similarly:

**Claim 3.5** Let \( i < \omega_1 \). \( N_{i+1} \) is generated from \( \text{ran}(\sigma_{ii+1}) \cup \{ \omega_i^{N_i} \} \) in the sense that for every \( x \in N_{i+1} \) there is some function \( f \in \omega_i^{N_i} (N_i) \cap N_i \) such that \( x = \sigma_{ii+1}(f)(\omega_i^{N_i}) \).

**Claim 3.6** Let \( i < \omega_1 \). \( \{ X \in \mathcal{P}(\omega_i^{N_i}) \cap N_i ; \omega_i^{N_i} \in \sigma_{ii+1}(X) \} \) is generic over \( N_i \) for the forcing given by the \( I^{N_i} \)-positive sets.

**Claim 3.7** Let \( i \leq \omega_1 \) be a limit ordinal. For every \( x \in N_i \) there is some \( j < i \) and some \( \bar{x} \in N_j \) such that \( x = \sigma_{ji}(\bar{x}) \).

\( (N_i, \sigma_{ij} : i \leq j \leq \omega_1) \) is then indeed a generic iteration of \( N_0 \). As \( N_0 \) is iterable, we may and shall identify \( N_i \) with its transitive collapse, so that (C.4) holds true.

Another round of density arguments will show that \( \mathcal{E} \) satisfies (C.1), (C.3), (C.5), (C.6), and (C.7), where we identify \( M_i \) with the structure \( (M_i; \in, (NS_{\omega_i^{M_i}}) M_i, A \cap \omega_i^{M_i}) \).

Let us now verify (C.8) and (C.9).

As for (C.8), \( X_\delta \cap \omega_1 = \delta \) for \( \delta \in K \) is easy. Let \( x_1, \ldots, x_k \in X_\delta, \delta \in K \). Suppose that

\[
(Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \exists v \varphi(v, x_1, \ldots, x_k).
\]

(13)

Let \( p \in g \) be such that \( \{ ^\gamma x_1 \in X_\delta \cap \gamma, \ldots, ^\gamma x_k \in X_\delta \cap \gamma, \gamma \delta \rightarrow \lambda_\delta \} \subset p \). Let \( q \leq p \), and let \( \Sigma \) be a syntactical certificate for \( q \) whose associated semantical certificate is

\[
\mathcal{E}' = (M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} : i \leq j \leq \omega_1), \langle (k_n', \alpha'_n) : n < \omega \rangle, \langle X'_\delta, X'_\delta : \delta \in K' \rangle.
\]
Then $\delta \in K'$ and 
\[
\{x_1, \ldots, x_k\} \subset X'_\delta \prec (Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}),
\]
so that by (13) we may choose some $x \in X'_\delta$ with 
\[
(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \vDash \varphi(x, x_1, \ldots, x_k).
\]
Let $r = q \cup \{\lceil x \in \dot{X}_\delta \rceil\}$.

By density, there is then some $y \in X_\delta$ such that 
\[
(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \vDash \varphi(y, x_1, \ldots, x_k).
\]

The proof of (C.9) is similar. Let again $\delta \in K$. Let $E \subset \mathbb{P}_{\lambda_\delta} \cap X^g_\delta$ be dense in $\mathbb{P}_{\lambda_\delta} \cap X_\delta$, and $r \in E$ iff $r \in \mathbb{P}_{\lambda_\delta} \cap X_\delta$ and 
\[
(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \vDash \varphi(r, x_1, \ldots, x_k).
\]
(14)

Let $p \in g$ be such that $\{\lceil x_1 \in \dot{X}_\delta \rceil, \ldots, \lceil x_k \in \dot{X}_\delta \rceil, \lceil \delta \mapsto \lambda_\delta \rceil\} \subset p$. Let $q \leq p$, and again let $\Sigma$ be a syntactical certificate for $q$ whose associated semantical certificate is 
\[
\mathcal{C}' = \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij}; i \leq j \leq \omega_1\rangle, \langle (k'_n, \alpha'_n); n < \omega\rangle, \langle \lambda_\delta, X'_\delta; \delta \in K'\rangle.
\]
Then $[[\Sigma]]^{\omega} \cap X'_\delta$ has an element, say $r$, such that (14) holds true. Let $s = q \cup r \cup \{\lceil r \in \dot{X}_\delta \rceil\}$.

By density, then, $g \cap X_\delta \cap E \neq \emptyset$. \hfill $\Box$

**Lemma 3.8** Let $g$ be $\mathbb{P}$-generic over $V$. Let

\[
\mathcal{C} = \langle M_i, \pi_{ij}, N_i, \sigma_{ij}; i \leq j \leq \omega_1\rangle, \langle (k_n, \alpha_n); n < \omega\rangle, \langle \lambda_\delta, X_\delta; \delta \in K\rangle
\]

be the semantic certificate associated with the syntactic certificate $\bigcup g$. Let 
\[
N_{\omega_1} = (N_{\omega_1}; \in, A, I^*),
\]
and let $T \in (\mathcal{P}(\omega_1) \cap N_{\omega_1}) \setminus I^*$. Then $T$ is stationary in $V[g]$.

If $\mathcal{C}$, $I^*$, and $T$ are as in the statement of Lemma 3.8, then by Lemma 3.3 and (C.3) and (C.4) we will have that $(\mathsf{NS}_{\omega_1})^V = I^* \cap V$, so that the conclusion of Lemma 3.8 also gives that $\mathbb{P}$ preserves the stationarity of $T$. In other words:

**Corollary 3.9** $\mathbb{P}$ preserves stationary subsets of $\omega_1$. 

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Proof of Lemma 3.8. Let $\hat{N}_{\omega_1} \in V^\mathbb{P}$ be a canonical name for $N_{\omega_1}$, and let $\hat{I}^* \in V^\mathbb{P}$ be a canonical name for $I^*$. Let $\bar{\rho} \in g, \hat{C}, \hat{S} \in V^\mathbb{P}$, and $i < \omega_1$ and $n < \omega$ be such that

(i) $T = \hat{S}^g$,
(ii) $\bar{\rho} \models \"\hat{C} \subset \omega_1 \text{ is club},\"$
(iii) $\bar{\rho} \models \"\hat{S} \in (\mathcal{P}(\omega_1) \cap \hat{N}_{\omega_1}) \setminus \hat{I}^*,\"$ and
(vi) $\bar{\rho} \models \"\hat{S} \text{ is represented by } [i,n] \text{ in the term model producing } \hat{N}_{\omega_1}.$

We may and shall also assume that

$\bar{\rho} \models \hat{n}$ is a subset of the first uncountable cardinal, yet $\hat{n} \notin \hat{I}^* \in \bar{\rho}$. (15)

Let $p \leq \bar{\rho}$ be arbitrary. We aim to produce some $q \leq p$ and some $\delta < \omega_1$ such that $q \models \hat{\delta} \in \hat{C} \cap \hat{S}$, see Claim 3.11 below.

For $\xi < \omega_1$, let

$$D_\xi = \{ q \leq p : \exists \eta \geq \xi (\eta < \omega_1 \land q \models \hat{\eta} \in \hat{C}) \},$$

so that $D_\xi$ is open dense below $p$. Let

$$E = \{(q, \eta) \in \mathbb{P} \times \omega_1 : q \models \hat{\eta} \in \hat{C}\}.$$

Let us write

$$\tau = ((D_\xi : \xi < \omega_1), E).$$

We may and shall identify $\tau$ with some subset of $H_\kappa$ which codes $\tau$.

By $(\Diamond(\mathbb{P}))$, we may pick some $\lambda \in C$ such that $p \in \mathbb{P}_\lambda$ and

$$(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\kappa; \in, \mathbb{P}, \tau).$$ (16)

Let $h$ be $\text{Col}(\omega, 2^{\aleph_2})$-generic over $V$, and let $g' \in V[h]$ be a filter on $\mathbb{P}_\lambda$ such that $p \in g'$ and $g'$ meets every dense set which is definable over $(N_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$ from parameters in $N_\lambda$. By Lemma 3.3, $\bigcup g'$ is a syntactic certificate for $p$, and we may let

$$\langle M'_i, \sigma'_{ij}, N'_i, \alpha'_n : i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) : n < \omega \rangle, \langle \lambda'_\delta, X'_\delta : \delta \in K' \rangle$$

be the associated semantic certificate. In particular, $K' \subset \lambda$. 16
Let $S$ denote the subset of $\omega_1$ which is represented by $[i, \dot{n}]$ in the term model giving $N'_{\omega_1}$, so that if $N'_{\omega_1} = \langle N'_{\omega_1}, \in, A, I' \rangle$, then by (15),

$$S \in (\mathcal{P}(\omega_1) \cap N'_{\omega_1}) \setminus I'.$$

(17)

Let us also write $\rho = \omega^V_1 = (2^{\aleph_2})^V$. Inside $V[h]$, we may extend $\langle N'_{i}, \sigma'_{ij}: i \leq j \leq \omega_1 \rangle$ to a generic iteration

$$\langle N'_{i}, \sigma'_{ij}: i \leq j \leq \rho \rangle$$

such that

$$\omega_1 \in \sigma'_{\omega_1, \omega_1+1}(S).$$

(18)

This is possible as $\omega^{N'_{\omega_1}}_1 = \sup \{ \omega^{N'_{i}}_1 : j < \omega_1 \} = \omega_1$ and by (17). Let

$$\langle M'_i, \pi'_{ij}: i \leq j \leq \rho \rangle = \sigma_{0, \rho}(\langle M'_i, \pi'_{ij}: i \leq j \leq \omega^{N'_{i}}_1 \rangle),$$

so that $\langle M'_i, \pi'_{ij}: i \leq j \leq \rho \rangle$ is an extension of $\langle M'_i, \pi'_{ij}: i \leq j \leq \omega_1 \rangle$.

Recalling (8), we may lift $\langle M'_i, \pi'_{ij}: \omega_1 \leq i \leq j \leq \rho \rangle$ to a generic iteration

$$\langle M'_i, \pi'_{ij}: \omega_1 \leq i \leq j \leq \rho \rangle$$

of $V$. Let us write $M = M'_\rho$ and $\pi = \pi'_{\omega_1, \rho}$.

The key point is now that $\langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij}: i \leq j \leq \rho \rangle$ may be used to extend $\pi'' \cup g'$ to a syntactic certificate

$$\Sigma \supset \pi'' \bigcup g'$$

(19)

for $\pi(p)$ in the following manner. Let $K^* = K' \cup \{ \omega_1 \}$. For $\delta \in K'$, let $\lambda^*_\delta = \pi(\lambda^*_\delta)$ and $X^*_\delta = \pi'' X^*_\delta$. Also, write $\lambda^*_{\omega_1} = \pi(\lambda)$ and $X^*_{\omega_1} = \pi'' Q_\lambda$. Let

$$\mathcal{C}^* = \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij}: i \leq j \leq \rho \rangle, \langle (k'_n, \pi(\alpha'_n)): n < \omega \rangle, \langle \lambda^*_{\delta}, X^*_\delta: \delta \in K^* \rangle.$$

It is then straightforward to verify that $\mathcal{C}^*$ is a semantic certificate for $\pi(p)$, and that in fact there is some syntactic certificate $\Sigma$ as in (19) such that $\mathcal{C}^*$ is certified by $\Sigma$.

Now let $[\dot{m}]_{\omega_1+1}$ represent $\sigma'_{\omega_1, \omega_1+1}(S)$ in the term model for $N'_{\omega_1+1}$ provided by $\Sigma$, so that\(^{13}\)

$$\{ \mathcal{G}\mathcal{G} \dot{\sigma}_{\omega_1+1}(\dot{\check{n}}) = \dot{m}^\gamma, \mathcal{G}\mathcal{G} \dot{N}_{\omega_1+1} \models \omega_1 \in \dot{m}^\gamma \} \subset \Sigma,$$

\(^{13}\)Here, $\dot{\sigma}_{\omega_1+1}$ and $\dot{N}_{\omega_1+1}$ are terms of the language associated with $\pi(P_\lambda)$ and $\mathcal{G}\mathcal{G} \dot{\sigma}_{\omega_1+1}(\dot{\check{n}}) = \dot{m}^\gamma$ and $\mathcal{G}\mathcal{G} \dot{N}_{\omega_1+1} \models \omega_1 \in \dot{m}^\gamma$ are formulae of that language.
in other words,

\[ \pi(p) \cup \{ \sigma_{i\omega_1+1}(\dot{n}) = \dot{m}, \dot{N}_{\omega_1+1} \models \omega_1 \in \dot{m} \} \text{ is certified by } \Sigma. \] (20)

Let us now define

\[ q^* = \pi(p) \cup \{ \sigma_{i\omega_1+1}(\dot{n}) = \dot{m}, \dot{N}_{\omega_1+1} \models \omega_1 \in \dot{m}, \omega_1 \mapsto \pi(\lambda) \}. \] (21)

We thus established the following.

**Claim 3.10** \( q^* \in \pi(\mathbb{P}) \), as being certified by \( \Sigma \).

The elementarity of \( \pi: V \to M^+_\rho \) then gives some \( \delta < \omega_1 \) and some \( \mu < \kappa \) such that

\[ q = p \cup \{ \sigma_{i\delta+1}(\dot{n}) = \dot{m}, \dot{N}_{\delta+1} \models \delta \in \dot{m}, \delta \mapsto \lambda \} \in \mathbb{P}. \] (22)

**Claim 3.11** \( q \models \bar{\delta} \in \dot{C} \cap \dot{S} \).

**Proof** of Claim 3.11. \( q \models \bar{\delta} \in \dot{S} \) readily follows from \( \{ \sigma_{i\delta+1}(\dot{n}) = \dot{m}, \dot{N}_{\delta+1} \models \delta \in \dot{m} \} \subset q \), the fact that \( \bar{p} \geq p \) forces that \( \dot{S} \) is represented by \([i, \dot{n}]\) in the term model giving \( \dot{N}_{\omega_1} \), and the fact that by Claim 3.4, \([\delta]_{\delta+1}\) represents \( \delta \) in the model \( \dot{N}_{\delta+1} \) of any semantic certificate for \( q \).

Let us now show that \( q \models \bar{\delta} \in \dot{C} \). We will in fact show that \( q \) forces that \( \bar{\delta} \) is a limit point of \( \dot{C} \). Otherwise there is some \( r \leq q \) and some \( \eta < \delta \) such that

\[ r \models \dot{C} \cap \bar{\delta} \subset \bar{\eta}. \] (23)

Let

\[ \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) : n < \omega \rangle, \langle \lambda'_\delta, \delta \in K' \rangle \] (24)

certify \( r \). We must have that

(a) \( \delta \in K' \),
(b) \( X'_\delta \prec (Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \),
(c) \( X'_\delta \cap \omega_1 = \delta \), and
(d) for some \( \Sigma \) such that the objects from (24) are certified by \( \Sigma \), \( [\Sigma]^{<\omega} \cap X'_\delta \cap E \neq \emptyset \) for every \( E \subset \mathbb{P}_\lambda \) which is dense in \( \mathbb{P}_\lambda \cap X'_\delta \) and definable over the structure

\[ (Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \]

from parameters in \( X'_\delta \).
Notice that $A_\lambda = \tau \cap Q_\lambda$, and hence $A_\lambda$ may be identified with $((D_\xi \cap Q_\lambda : \xi < \omega_1), E \cap Q_\lambda)$. As $\eta < \delta \subseteq X'_\delta$, $D_\eta$ is definable over the structure $(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$ from a parameter in $X'_\delta$. By (16), $D_\eta \cap Q_\lambda$ is dense in $\mathbb{P}_\lambda$. By (d) above, there is then some $s \in [\Sigma]^{<\omega} \cap X'_\delta \cap D_\eta \cap Q_\lambda$.

By (16) again, the unique smallest $\eta' \geq \eta$ with $s \Vdash \check{\eta}' \in \check{C}$ must be in $X'_\delta$, hence $\eta' < \delta$ by (c) above. By Lemma 3.1, $s$ is compatible with $r$. We have reached a contradiction with (23). □

References


[16] R. Schindler, *Woodin’s axiom (*), or Martin’s Maximum, or both?*, in: Foundations of mathematics, essays in honor of W. Hugh Woodin’s 60th birthday, Harvard University (Caicedo et al., eds.), pp. 177-204.


