Yizheng I

The new derived model theorem (Woodin)

Let λ be a lim of Woodins. let G be $\mathrm{Col}(\omega, <\lambda)$-generic.

let $\mathbb{R}^*_G = \bigcup_{\alpha<\lambda} \mathbb{R} \cap V[G \restriction \alpha]$.

$$\mathrm{Hom}^*_G = \{ A^* : A \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]}, \text{ for } \alpha < \lambda \}$$

recall: the old derived model is

$$L(\mathbb{R}^*_G, \mathrm{Hom}^*_G) \models AD^+.$$

let $$A_G = \{ B \subset \mathbb{R}^*_G : B \in \overset{\mathrm{HOD}^{V[G]}_{V\cup\mathbb{R}^*_G\cup\{\mathbb{R}^*_G\}}}{\overset{\parallel}{V(\mathbb{R}^*_G)}}, \; L(B, \mathbb{R}^*_G) \models AD^+ \}.$$

then (1) for $B, C \in A_G$, either

$$L(B, \mathbb{R}^*_G) \subset L(C, \mathbb{R}^*_G) \quad \text{or}$$

$$L(C, \mathbb{R}^*_G) \subset L(B, \mathbb{R}^*_G)$$

(2) $L(A_G, \mathbb{R}^*_G) \models AD^+$.

some facts about AD^+.

assume AD^+.

μ = martin measure on $\mathcal{D}$ = turing degrees.

$\mu(A) = 1 \longleftrightarrow \exists d \, \forall e \geq_T d \quad e \in A$.

black box 1. $\prod_{\mathcal{D}} \mathrm{Ord}/\mu$ is well-fdd.

assume further there is a largest suslin cardinal,

e.g. $V = L(B, \mathbb{R})$, some B.

κ = largest suslin cardinal.

$S(\kappa)$ = ~~t~~ pointclass of κ-suslin sets.

Ch T = a tree on $\omega \times \kappa$ s.t.

$p[T]$ = a universal set for κ-suslin sets.

black box. $\forall A \subset \mathbb{R}$, for μ-a.e. x,

$A \cap L[T,x] \in L[T,x]$.

$\mathbb{R} \cap L[T,x] = C_\Gamma(x)$, Γ = "lightface vrsn of $S(\kappa)$"

$= OD(T,x)$.

pointed Sacks forcing.

$\mathbb{P} = \{ U : U$ is a pointed perfect tree on $\omega \}$

pointed : every $x \in [U]$ $U \leq_T x$.

$U \leq_{\mathbb{P}} U'$ iff $U \subset U'$.

let g be $\mathbb{P}$-gen. then $\forall x \in \mathbb{R}^V$ $x \leq_T x_g$.

theorem (martin) let $A \subset \mathbb{R}$. if

$\forall x \in \mathbb{R}\ \exists y \geq_T x \ (y \in A)$.

then $\exists$ pointed perfect U $([U] \subset A)$

the proof is a variant of "μ is an ultrafilter".

g will induce a V-ultrafilter $\mathcal{U}_{x_g}$.

for $A \subset \mathbb{R}$, $A \in V$,

$$\left(A \in \mathcal{U}_{x_g} \longleftrightarrow \exists T \in g \ \ [T]^V \subset A \right)$$

"$\mathcal{U}_{x_g}$ is an ultrafilter" uses martin.

$\forall A \subset \mathbb{R} \quad A \in V$

$\{ u : [u] \subset A$ or $[u] \cap A = \emptyset \}$

is dense in $\mathbb{P}$.

take $\prod_{x \in \mathbb{R}^V} L[T, x] / \mathcal{U}_{x_g}$, using fcts in V.

$[id]_{\mathcal{U}_{x_g}} = x_g$.

if $f : \mathbb{R}^V \longrightarrow OR$, $f \in V$, then

$\{ u : f \restriction [u]$ is turing invariant$\}$ is dense.

then f is turing invariant mod $\mathcal{U}_{x_g}$.

$[f]_{\mathcal{U}_{x_g}} = [f']_\mu$, for some $f' : \mathcal{D} \longrightarrow OR$,

$f' \in V$.

so $[\mathrm{const}_T]_{\mathcal{U}_{x_g}} = [\mathrm{const}_T]_\mu = T^*$.

by Łoś, $\prod_{x \in \mathbb{R}^V} L[T,x] / \mathcal{U}_{x_g} = L[T^*, x_g]$.

<u>theorem</u>. $V = L(T^*, \mathbb{R}^V)$, assuming $V = L(\mathcal{P}(\mathbb{R}))$.

pf.: say $A \subset \mathbb{R}$.

let $A^* = [x \longmapsto \underbrace{A \cap L[T,x]}_{\in}]_{\mathcal{U}_{x_g}}$

$L[T,x)$ for a cone of x.

So $A = A^* \cap \mathbb{R}^V$.

$A^* \in L[T^*, x_g)$ by Łoś.

So $A \in L(T^*, \mathbb{R}^V)[x_g]$;

here $T^* \in V$ by $T^* = [\mathrm{const}_T]_\mu$,

so it can be computed in V;

also $\mathbb{P}$ def.ble on HC, so x_g is also $L(T^*, \mathbb{R}^V)$-generic for $\mathbb{P}$.

$\ddagger \quad A \in L(T^*, \mathbb{R}^V)[x_g]$

for all g, so

$A \in L(T^*, \mathbb{R}^V)$. $\dashv$

proof of the compatibility of $\mathcal{A}_G =$
$\{A \subset \mathbb{R}^*_G : A \in V(\mathbb{R}^*_G),\ L(A, \mathbb{R}^*_G) \models AD^+\}$.

for all $B, C \in \mathcal{A}_G$ $\quad B \leq_W C$ or $C \leq_W \neg B$,
via a $\ddagger$ continuous fct. coded in $\mathbb{R}^*_G$.

supp. not. let B, C be a counterexample.

let $T^*_B =$ the T^* ~~from~~ above as being computed in $L(B, \mathbb{R}^*_G)$

$T^*_C =$ — T^* — $L(C, \mathbb{R}^*_G)$.

assume w.l.o.g. that $T^*_B, T^*_C \in V$
$(\in V[G \upharpoonright \alpha]$, some $\alpha < \lambda)$.

T_B = the T for above as being defined in $L(B, \mathbb{R}^*_G)$

[so $T^*_B = \left(\prod_\alpha T_B / \mu\right)^{L(B, \mathbb{R}^*_G)}$.]

T_C = — T — $L(C, \mathbb{R}^*_G)$

T_B certifies a can. good w.o. of $\mathbb{R} \cap L[T_B, x]$, uniformly in x.

fix $k \in \omega$. $(x, y, k) \in p[T_B] \longleftrightarrow$ $x, y \in \mathbb{R}$, y codes an initial segment of $<^{T_B}_x$.

to get this, use AD^+ in $L(B, \mathbb{R}^*_G)$.

then T^*_B certifies a can. good w.o. of $\mathbb{R} \cap L[T^*_B, x_g]$, where x_g is a pointed Sacks generic over $L(B, \mathbb{R}^*_G)$.

for any $B_0 \in L(B, \mathbb{R}^*_G)$, there is $z \in L[T^*_B, x_g]$ which codes B_0 relative to $\mathbb{R}^*_G$:

$\forall \ell \in \omega$ if $\{\ell\}^{x_g} \in \mathbb{R}^*_G$, then

$$\{\ell\}^{x_g} \in B_0 \longleftrightarrow \ell \in z .$$

here, $\{\ell\}^{x_g}$ is the ℓth real rec. in x_g.

pick β_0, γ_0 so that the β_0th real in $<^{T^*_B}_{x_g}$ codes B_0, the γ_0th real in $<^{T^*_C}_{x_g}$ codes C_0.

and $B_0 \not\leq_W C_0 \wedge C_0 \not\leq_W \neg B_0$ in $\mathbb{R}^*_G$.

minimize (β_0, γ_0) in gödel pairing, fix $p \in \mathbb{P}$ = pt. Sachs forcing,

$V(\mathbb{R}^*_G) \models p \Vdash$ "(β_0, γ_0) defines a minimum counterexample certified by

T^*_B, T^*_C.

assume $p \in V$ (w.l.o.g.).

Yizheng II

T^*_β T^*_C

let $\phi_1(\cancel{B}, \cancel{C}, p, x)$ be :

$\exists \beta, \gamma$ s.t. $p \Vdash_{\mathbb{P}}$ "(β, γ) defines a min. counterexample certified by T^*_β, T^*_C $\wedge$ if B, C are the counterexample sets, then $x \in B$".

so for $x \in \mathbb{R}^*_G$, $x \in B_0 \iff$

$$V(\mathbb{R}^*_G) \models \cancel{p}\ \phi_1(p, T^*_\beta, T^*_C, x)$$

tree production lemma.

let δ be woodin, $\varphi(x,v)$ a fmla, a a set. suppose

(1) (generic absoluteness) for G $<\delta$-generic and H $\leq \delta$-generic over $V[G]$ and $x \in \mathbb{R}^{V[G]}$

$$V[G] \models \varphi(x,a) \longleftrightarrow V[G,H] \models \varphi(x,a).$$

(2) (stationary tower correctness) for G $\mathbb{Q}_{<\delta}$-generic, $\sigma : V \longrightarrow M = \text{ult}(V;G)$, $x \in \mathbb{R}^{V[G]}$,

$$V[G] \models \varphi(x,a) \longleftrightarrow M \models \varphi(x,\sigma(a)).$$

then there are $<\delta$-absolute complementing T, U s.t. $p[T] = \{x : \varphi(x,a)\}$ in all $<\delta$-gen. extensions.

let G be $\mathbb{Q}_{<\lambda}$- gen.

$$j^* : V \longrightarrow M \quad \text{gen. ultrapow}$$

$\beta_0, \gamma_0 \in wfp(M)$, $\mathbb{R}^M = \mathbb{R}^*_G$.

for $\delta < \lambda$ woodin

$$j_\delta : V \longrightarrow M_\delta = ult(V; H \cap V_\delta)$$
$$\subset V[H \cap V_\delta].$$

$$j^*_\delta : M_\delta \longrightarrow M \quad \text{tail of} \quad j^*.$$

M_δ is well-fdd.

lem. $M \models p \Vdash (\beta_0, \gamma_0)$ is the min. counterexample certified by $j^*(T^*_\beta)$, $j^*(T^*_C)$.

Sublem. let g be pointed sacks gen. / $M \cup V(\mathbb{R}^*_G)$.

supp. $z \in \mathbb{R}^{L[T^*_B, x_g]}$, z is the α^{th} real in $<^{T^*_B}_{x_g}$, $\alpha \leq \max(\beta_0, \gamma_0)$, z codes $E \subset \mathbb{R}^*_G$ relativ to x_g.

then: let z' be the α^{th} real in $<^{j^*(T^*_B)}_{x_g}$, z' codes $E \subset \mathbb{R}^*_G$ relativ to x_g.

and vice versa. no proof.

prf. of the sublem: assume M is well-fdd. then $T^*_B \longmapsto j^*(T^*_B)$, and $<^{T^*_B}_{x_g}$ is an initial sgmt of $<^{j^*(T^*_B)}_{x_g}$.

$V(\mathbb{R}^*_G) \models p \Vdash (\beta_0, \gamma_0)$ is the min. counterexample certified by T^*_B, T^*_C.

$$\mathbb{R} \cap V(\mathbb{R}^*_G) = \mathbb{R}^*_G = \mathbb{R} \cap M .$$

$\dashv$ (lem, asng M w.fdd.)

for any $\delta < \lambda$, δ woodin, verify the assmpts of the tree product lem to get $<\delta$-complementing trees to ϕ_1, ϕ_2.

stat. tower correctness

K $\mathbb{Q}_{<\delta}$-gen., $x \in V[K]$.

$$V[K] \models \phi_1(p, T^*_B, T^*_C, x)$$

$$\overset{?}{\Longleftrightarrow} \quad ult(V;K) \models \Phi_1(p, j_K(T^*_B), j_K(T^*_C), x)$$

for this, we may use $k \in V[G]$,

and $ult(V;K) \xrightarrow{k} M$, $j^* = k \circ j_K$.

no need

[by the lem,

$ult(V;K) \models$ "$pH(k^{-1}(\beta_0), k^{-1}(\gamma_0))$ is the min. codn.graph certified by $j_K(T^*_B)$, $j_K(T^*_C)$."

$$\begin{tikzcd} \mathrm{ult}(V;\kappa) \models \phi_1(p, j_\kappa(T_B^*), j_\kappa(T_C^*), x) \\ \iff x \in B_0 . \\ M \models \phi_1(p, j^*(T_B^*), j^*(T_C^*), x) \arrow[uu, leftrightarrow] \end{tikzcd}$$

Sublem: $x \in B_0 \longleftrightarrow V[\kappa] \models \phi_1(p, T_B^*, T_C^*, x)$

pf.: we may construct $V[\kappa] \xrightarrow{j'} M'$

with $\mathbb{R} \cap M' = \mathbb{R}^*_G$

as the dir. lim. of stat. tower ultrapowers

apply the lem w/ M', j' in place of M, j^*.

$$\begin{tikzcd} M' \models \phi_1(p, j'(T_B^*), j'(T_C^*), x) \\ \iff x \in B_0 . \\ V[\kappa] \models \phi_1(p, T_B^*, T_C^*, x) . \arrow[uu, leftrightarrow] \end{tikzcd}$$

⊣ (sublem)

(stat. tower correction)

gen. absoluteness follows for the same sublemma.

So by free product lemma:

$$B_0, C_0 \in \mathrm{Hom}^*,$$

but any two sets in Hom^* are wadge compatible. $\dashv$ $\mathcal{A}_G$ - compatibility.