2017/07/20 M. Zeman - Tutorial III, part 2

Recall the definition of B_α

$\bar\alpha \in B_\alpha$ iff

i) $n_{\bar\alpha} = n_\alpha$

ii) there is a map $\sigma : N_{\bar\alpha} \to N_\alpha$ s.t.

a) $\mathrm{crit}(\sigma) = \bar\alpha$ and $\sigma(\bar\alpha) = \alpha$

b) $\sigma(p_{\bar\alpha}) = p_\alpha$

c) σ is $\Sigma_0^{(n_\alpha)}$ - preserving

We saw that σ is unique and denote it by $\sigma_{\bar\alpha}$.

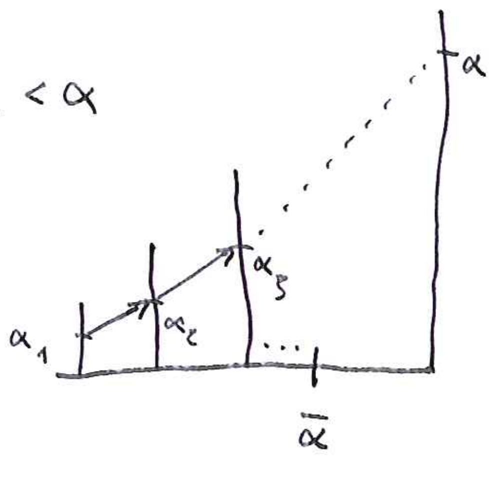
- if $\alpha^* < \bar\alpha$ are in B_α, then $\mathrm{rng}(\sigma_{\alpha^*}) \subseteq \mathrm{rng}(\sigma_{\bar\alpha})$ and we have a map $\sigma_{\alpha^*\bar\alpha} := \sigma_{\bar\alpha}^{-1} \circ \sigma_{\alpha^*}$

  which satisfies i) and ii).

We work with $n = n_\alpha = 0$, so $\Sigma_0^{(n_\alpha)}$ is Σ_0.

<u>Lemma B</u> B_α is a closed subset of α.

<u>Proof</u>. Assume $\bar\alpha \in \lim(B_\alpha)$, $\bar\alpha < \alpha$



We let $\bar N$ = the transitivized direct limit of the diagram

$(N_{\alpha^*}, \sigma_{\alpha'\alpha^*} \mid \alpha' < \alpha^* < \bar\alpha)$

$\bar{\sigma}_{\alpha^*} : N_{\alpha^*} \longrightarrow \bar{N}$ the direct limit map

$\sigma : \bar{N} \longrightarrow N_\alpha$ the natural embedding

We have

1) All maps above are Σ_0-preserving

2) $\mathrm{crit}(\sigma) = \bar{\alpha}$ and $\sigma(\bar{\alpha}) = \alpha$.

Let $\bar{p} := \bar{\sigma}_{\alpha^*}(p_{\alpha^*})$.

3) $\sigma(\bar{p}) = p_\alpha$.

We show $\bar{N} = N_{\bar{\alpha}}$ and $\sigma = \sigma_{\bar{\alpha}}$.

This will show: $\bar{\alpha} \in B_\alpha$.

$\rightarrow$ $\bar{N}$ is Σ_1-generated from $\bar{p}$ and ordinals below κ.

This says $\rho^1_{\bar{N}} = \kappa$ and $p_{\bar{N}} \leq^* \bar{p}$.

Here $\leq^*$ is the canonical w.o. of finite sets of ordinals.

<u>Why</u>: Pick $x \in \bar{N}$. Find $\alpha^* \in B_\alpha \cap \bar{\alpha}$ large enough s.t. $x \in \mathrm{rng}(\bar{\sigma}_{\alpha^*})$. Let $\bar{x} := \sigma^{-1}_{\alpha^*}(x)$.

Because $\alpha^* \in B_\alpha$:

$$\bar{x} = h_{\alpha^*}(i, (\xi, p_{\alpha^*})) \quad \text{some } i < \omega, \xi < \kappa$$

Apply $\bar{\sigma}_{\alpha^*}$:

$$x = h_{\bar{N}}(i, (\xi, \bar{p}))$$

$\longrightarrow p_{\bar{N}} \not<^* \bar{p}$.

Why: If $p_{\bar{N}} <^* \bar{p}$ then there are $i \in \omega$ and $\xi < \kappa$ s.t.

$$\bar{p} = h_{\bar{N}}(i, (\xi, p_{\bar{N}})).$$

Apply σ: We get

$$p_\alpha = h_\alpha(i, (\xi, \sigma(p_{\bar{N}})))$$

Because $p_{\bar{N}} <^* \bar{p}$:

$$\sigma(p_{\bar{N}}) <^* p_\alpha$$

Contradiction to the minimality of p_α. $\dashv$ Lemma B

So $\bar{\sigma}_{\alpha^*} = \sigma_{\alpha^* \bar{\alpha}}$ and $\sigma = \sigma_{\bar{\alpha}}$.

So we have:

a) $f(\alpha) > \omega \Rightarrow B_\alpha \subseteq \alpha$ is unbounded

b) B_α is a closed subset of α.

We prove:

c) if $\bar{\alpha} \in B_\alpha$ then $B_{\bar{\alpha}} = B_\alpha \cap \bar{\alpha}$.

<u>Proof</u>. If $\alpha^* \in B_{\bar{\alpha}}$ then we have a map

$$\sigma_{\alpha^* \bar{\alpha}} : N_{\alpha^*} \to N_{\bar{\alpha}} .$$

But we also have

$$\sigma_{\bar{\alpha}} : N_{\bar{\alpha}} \to N_\alpha .$$

Also notice $n_{\alpha^*} = n_{\bar{\alpha}}$ and $n_{\bar{\alpha}} = n_\alpha$

So $\sigma_{\alpha^*} = \sigma_{\bar{\alpha}} \circ \sigma_{\alpha^* \bar{\alpha}}$ witnesses $\alpha^* \in B_\alpha$.

If $\alpha^* \in B_\alpha \cap \bar{\alpha}$ we have $\sigma_{\alpha^*} : N_{\alpha^*} \to N_\alpha$.

We also have $\sigma_{\bar{\alpha}} : N_{\bar{\alpha}} \to N_\alpha$.

By Fact 3 (last talk) $\operatorname{rng}(\sigma_{\alpha^*}) \subseteq \operatorname{rng}(\sigma_{\bar{\alpha}})$

and it's easy to see that

$$\sigma_{\alpha^*\bar{\alpha}} = \sigma_{\bar{\alpha}}^{-1} \circ \sigma_{\alpha^*} .$$

This witnesses $\alpha^* \in B_{\bar{\alpha}}$.

<u>Remark</u>. The sequence $(B_\alpha \mid \alpha \in (\kappa, \kappa^+) \cap \mathrm{lim})$ cannot be threaded in <u>any</u> outer universe $W \supseteq V$ s.t. $\mathrm{cf}(\kappa^{+L})^W > \omega$.

<u>Why</u>: If B is a thread, we have a diagram

$$(N_\alpha, \sigma_{\bar{\alpha}\alpha} \mid \bar{\alpha} < \alpha \text{ are in } B).$$

Let N be the direct limit. If $\mathrm{cf}(\kappa^{+L}) > \omega$ then N is well-founded, so w.l.o.g. N is transitive. Then $N = J_\delta$ for some $\delta \in \mathrm{Ord}$.

Also: if $\tilde{\sigma}_\alpha : N_\alpha \to N$ are the direct limit maps then $\tilde{\sigma}_\alpha(\nu) = \kappa^{+L}$. As before we then show that J_δ is a collapsing structure for κ^{+L}. ↯

Getting a $\square_\kappa$-sequence

We thin out the sets B_α:

Given B_α, define sequences α_i, ξ_i where $\alpha_i \in B_\alpha$, $\xi_i < \kappa$ are as follows

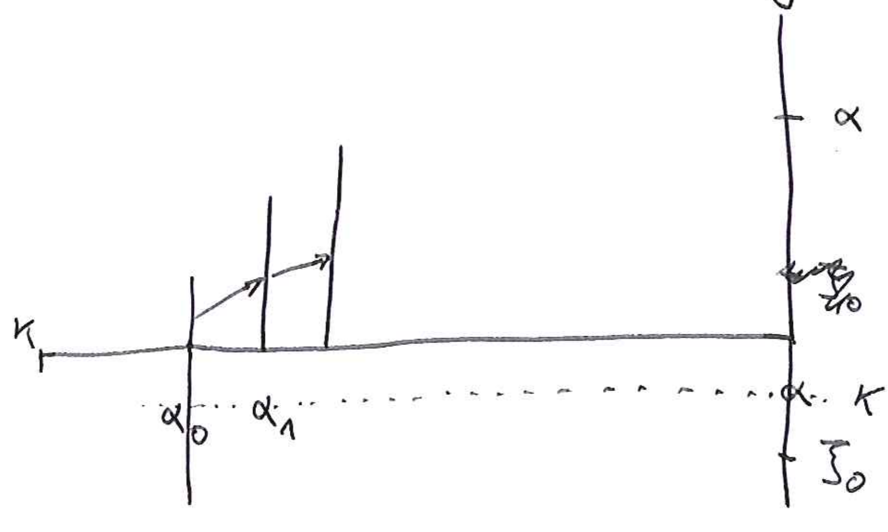
- $\alpha_0 = \min(B_\alpha)$
- if i is a limit then $\alpha_i = \sup_{j<i} \alpha_j$
- if we have α_i let

  $\xi_i =$ the least $\xi < \kappa$ s.t.

  $X_\alpha^\xi := h_\alpha(\{\xi\} \cup \{p_\alpha\}) \not\subseteq rng(\sigma_{\alpha_i})$
- if we have ξ_i:

  $\alpha_{i+1} :=$ the least $\bar{\alpha} \in B_\alpha$ s.t.

  $h_\alpha(\{\xi_i\} \cup \{p_\alpha\}) \subseteq rng(\sigma_{\bar{\alpha}})$.



- The sequences (α_i), (ξ_i) are increasing and (α_i) is continuous.

~~The monotonicity of (ξ_i) requires and explanation. What is used here is this:~~

$$\cancel{X_\alpha^\xi \subseteq \mathrm{rng}(\sigma_{\bar\alpha}) \text{ then}}$$

$$\cancel{X_\alpha^\xi = \sigma_{\bar\alpha}[X_{\bar\alpha}^\xi].}$$

The monotonicity of (ξ_i) follows from the fact that $\underset{\text{in } B_\alpha}{\alpha^* < \bar\alpha} \Rightarrow \mathrm{rng}(\sigma_{\alpha^*}) \subseteq \mathrm{rng}(\sigma_{\bar\alpha})$.

Define

$$C_\alpha = \{\alpha_i \mid i < \vartheta_\alpha\}$$

when ϑ_α is the length of (α_i). We get

a) if $f(\alpha) > \omega$ then C_α is unbounded in α

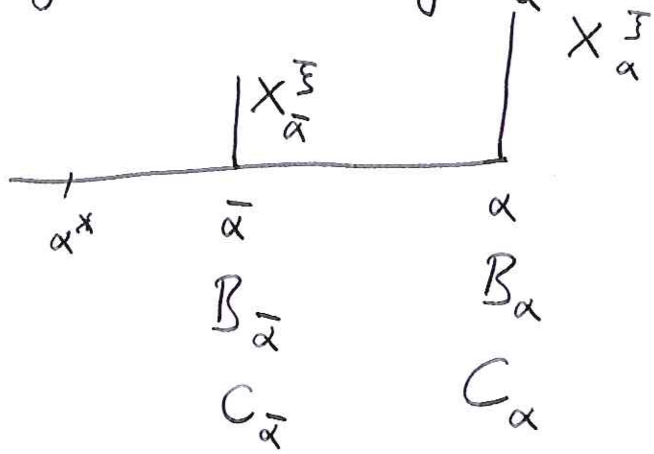
b) $C_\alpha \subseteq \alpha$ is closed

c) $\bar\alpha \in C_\alpha \Rightarrow C_{\bar\alpha} = C_\alpha \cap \bar\alpha$

d) $\mathrm{otp}(C_\alpha) \leq \kappa$.

d) follows as the map $\alpha_i \mapsto \xi_i$ is monotonic.

c) follows from coherence of B_α



And from the fact that if $\alpha_i < \bar\alpha$ then

$$X_\alpha^{\xi_i} = \sigma_{\bar\alpha\alpha}[X_{\bar\alpha}^{\xi_i}].$$

Doing things in $L[E]$: Focus is on constructing B_α.

~~Some Fine Structural Facts~~

~~A) Solidity witnesses:~~

We work in models with λ-indexing. Premice are coherent structures with specific properties.

A coherent structure has the form (J_ν^E, F) s.t. letting $\kappa = \mathrm{crit}(F)$ and $\alpha = \kappa^{+J_\nu^E}$

$$i_F : J_\alpha^E \xrightarrow[F]{} J_\nu^E$$

and $F = i_F \restriction (P(\kappa) \cap J_\alpha^E)$.

We write $\lambda_F = i_F(\kappa)$.

A cutpoint of F is an ordinal $\bar{\lambda} < \lambda$ s.t. $i_{F|\bar{\lambda}}(\kappa) = \bar{\lambda}$. The set of cutpoints is closed.

A coherent structure is of type

A if there are no cutpoints,

B if there is a largest cutpoint,

C if cutpoints are unbounded in λ.

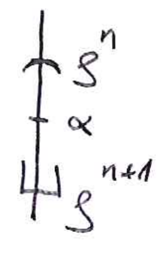
Some Fine Structural Facts

Let M be an amenable J-structure.
+ acceptable

Let $\alpha \in OR \cap M$ and $p \in [OR]^{<\omega} \cap M$. The standard solidity witness for α w.r.t. p in M is

$$W_M^{\alpha,p} = \text{the transitive collapse of } \underbrace{\tilde{h}_M^{n+1}(\alpha \cup \{p \setminus (\alpha+1)\})}_{(*)}$$

where $(*)$ is the $\Sigma_1^{(n)}$-hull of $\alpha \cup \{p \setminus (\alpha+1)\}$ in M and $n < \omega$ is s.t. $\rho_M^{n+1} \leq \alpha < \rho_M^n$.



- ~~If~~ $\cancel{W_M^{\alpha,p} \in}$
- If p is $(n+1)$-good, i.e. there is a $\Sigma_1^{(n)}(M)$ in p subset $a \subseteq \mathcal{J}_M^{n+1}$ s.t. $a \in M$ and $W_M^{\alpha,p} \in M$ for all $\alpha \in p$ then $p = P_M^{n+1}$.
- M is solid $:\Leftrightarrow W_M^{\alpha, p_M^n} \in M$ for all $n < \omega$, all $\alpha \in p_M^n$.

<u>Remark</u>. $W_M^{\alpha,p}$ has "high" complexity in terms of definability.

For this reason we consider a generalized version:

Given α, p as above a generalized solidity witness for α w.r.t. p in M is any structure (Q, r) s.t.

- Q is an acceptable J-structure,
- $r \in [OR]^{|p \setminus (\alpha+1)|} \cap M$,
- if $\varphi(\vec{u}, v)$ is a $\Sigma_1^{(n)}$-formula, n as above, and $\vec{\xi} \in [\alpha]^{<\omega}$

$$M \models \varphi(\vec{\xi}, p \setminus (\alpha+1)) \Rightarrow Q \models \varphi(\vec{\xi}, r)$$

If $(Q, r) \in M$ then $(*)$ can be expressed over M as a $\Pi_1^{(n)}$-statement.

Important point: If $\sigma : \bar{M} \underset{\Sigma_0^{(n)}}{\longrightarrow} M$ and $\sigma(\bar{p}) = p$, $\sigma(\bar{\alpha}) = \alpha$
and there is some generalized solidity witness (Q, r) for α w.r.t. p in M s.t. $(Q,r) \in \operatorname{rng}(\sigma)$ then $\sigma^{-1}((Q,r))$ is a generalized solidity witness for $\bar{\alpha}$ w.r.t $\bar{p}$ in $\bar{M}$.

<u>Fact</u>. $W_M^{\alpha, p} \in M$ iff there is some generalized solidity witness (Q,r) for α w.r.t. p in M s.t. $(Q,r) \in M$.

Condensation Lemma

Assume $\bar{M}, M$ are premice and M is countably iterable. Assume

$$\sigma : \bar{M} \underset{\Sigma_0^{(n)}}{\longrightarrow} M \quad \text{and}$$

$$\sigma \upharpoonright \varrho_{\bar{M}}^{n+1} = \mathrm{id}$$

Then

1) $\bar{M}$ is countably iterable

is generated by $p_{\bar{M}} \cup \nu$

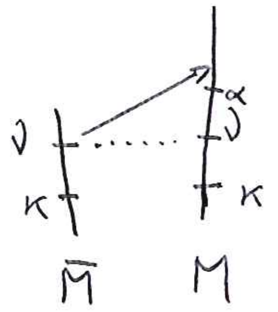
2) if $\bar{M}$ is sound above $\nu = \operatorname{crit}(\sigma)$ then one of the following holds

a) $\bar{M} = \mathrm{core}_\nu(M)$ and σ is the core map,

b) $\bar{M} \lhd M$,

c) $\bar{M} = Ult^*(M\|\beta, E_\alpha^{M\|\beta})$ where

- $\nu = \kappa^{+M\|\beta}$ where $M\|\beta$ is the collapsing level of M for ν and κ is the cardinal predecessor of ν in $M\|\beta$



- $\mathrm{crit}(E_\alpha^{M\|\beta}) = \kappa$
- κ is the only generator of $E_\alpha^{M\|\beta}$.

d) $\bar{M} \lhd Ult(M, E_\nu^M)$

For us a weaker form

d') $\bar{M} \lhd Ult^*(M, E_\nu^M)$

will suffice.

Construction of B_α

We let, for $\alpha \in (\kappa, \kappa^{+L[E]})$

$N_\alpha :=$ collapsing level of $L[E]$ for α.

Assuming w.l.o.g. $L[E]\|\alpha \prec L[E]\|\kappa^+$.

We would like to let $\bar{\alpha} \in B_\alpha$ iff

i) $n_{\bar{\alpha}} = n_\alpha$

ii) there is a map $\sigma : N_{\bar{\alpha}} \to N_\alpha$ s.t.

a) $\bar{\alpha} = \operatorname{crit}(\sigma)$ and $\sigma(\bar{\alpha}) = \alpha$

b) $\sigma(p_{\bar{\alpha}}) = p_\alpha$

c) σ is $\Sigma_0^{(n_\alpha)}$ - preserving.

Work with $n_\alpha = 0$.

Now try to prove that B_α is closed.

We have the diagram

$$(N_{\bar{\alpha}}, \sigma_{\alpha^* \bar{\alpha}} \mid \alpha^* < \bar{\alpha} \text{ in } B_\alpha)$$

$\bar{N}$ = the transitivized direct limit

$$\bar{\sigma}_{\alpha^*} : N_{\alpha^*} \to \bar{N}$$

$$\sigma : \bar{N} \to N$$

Want to use the condensation lemma to conclude $\bar{N} \triangleleft N_\alpha$. As before we show $n_{\bar{N}} = n_\alpha$ and $h_{\bar{N}}(\kappa \cup \{\bar{p}\}) = \bar{N}$ where $\bar{p} = \bar{\sigma}_{\alpha^*}(p_{\alpha^*})$.

This tells us that $\bar{p}$ is a good parameter, so $p_{\bar{N}} \leq^* \bar{p}$. This would follow if we had solidity witnesses for $\bar{p}$.

We add another clause to the definition of B_α:

ii) d) For each $\beta \in p_\alpha$ there is some generalized solidity witness ~~Q~~ (Q, r) for β w.r.t. p_α in N_α s.t. $(Q, r) \in \mathrm{rng}(\sigma_{\bar{\alpha}})$.

With this we get $\bar{p} = p_{\bar{N}}$. Hence $\bar{N}$ is sound. Also know $\bar{N}$ is a collapsing structure for $\bar{\alpha}$.

To see $\bar{N} = N_{\bar{\alpha}}$, check the clauses of the condensation lemma:

a) False, as N_α is sound, so σ would be Σ_1.

c) False as otherwise $\bar{N}$ would be sound above κ.

d) False: We only define B_α for α as above s.t. $E_\alpha^{L[E]} = \emptyset$. We say more about this.