trevor I

plan:

1. suslin sets + scales
2. martin's closure operator
3. kechris - woodin transfer thm.
4. woodin's thm. on $AD_{\mathbb{R}}$ in the intersection of divergent models of AD^+.

work in ZF.

pointclasses

baire space $\mathcal{N} = {}^\omega\omega$.

a <u>product space</u>: $X_1 \times \ldots \times X_n$, each $X_i \in \{\mathcal{N}, \omega\}$.

a <u>pointset</u> is a subset of a product space.

a <u>pointclass</u> is a set of pointsets.

a tree on X is a $T \subset {}^{<\omega}X$, closed under init'l segments.

$[T]$ = set of all infinite branches.

$[T]$ is closed in ${}^\omega X$ + every closed subset of ${}^\omega X$ arises that way.

every ill-fdd. tree on OR has a branch: the leftmost branch (lex-least)

projection for a set A, pA = projection onto the 1^{st} coordinate.

for a tree T on $\omega \times OR$:

$$p[T] = \{ x \in {}^\omega\omega : \exists f \in {}^\omega OR \ \forall n \ (x\restriction n, f\restriction n) \in T \}.$$

$A \subset {}^\omega\omega$ Suslin iff $A = p[T]$ for some T on $\omega \times OR$.

$A \subset {}^\omega\omega$ κ-Suslin iff $A = p[T]$ for an T on $\omega \times \kappa$.

for $x \in {}^\omega\omega$, define section T_x of T:

$$T_x = \{ s \in {}^{<\omega}OR : (x\restriction \mathrm{lh}(s), s) \in T \}.$$

note : ① $x \mapsto T_x$ is continuous.

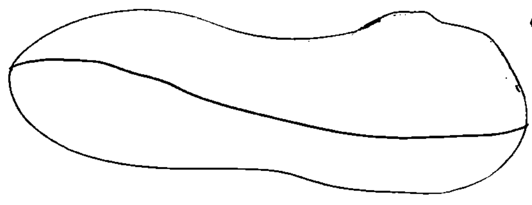
② $x \in p[T] \longleftrightarrow T_x$ is ill-fdd.

Suslin $\Longleftrightarrow$ continuously reducible to ill-foundedness of trees on ordinals.

<u>example</u> : $A \subset \mathcal{N}$ is ω-Suslin $\Leftrightarrow$ A is analytic ($\underset{\sim}{\Sigma}^1_1$).

<u>theorem</u>: let $A \subset \mathcal{N} \times \mathcal{N}$ be Suslin. then can be uniformized.

A



pf : say $A = p[T]$, T on $\omega \times \omega \times OR$.

for $x \in pA = \mathrm{dom}(A)$, let $(y_x, f_x) =$ the leftmost branch of T_x. i.e.,

$(y_x(0), f_x(0), y_x(1), f_x(1), \ldots)$ is lex-least.

$x \mapsto y_x$ is a uniformizati.

Scales. Let A be a pt. set. A norm on A is $\varphi : A \to \mathrm{OR}$.

for a seq. $\vec{\varphi} = (\varphi_n : n < \omega)$ of norms on A, write

$$x_k \to y \mod \vec{\varphi}$$

for: $x_k \to y$ in the usual sense, and $(\varphi_n(x_k) : k < \omega)$ is eventually constant f.a. n.

$\vec{\varphi}$ is a semi scale on A if

whenever $\{x_i : i < \omega\} \subset A$ and $x_i \to y \mod \vec{\varphi}$, then $y \in A$. $\quad (*)$

call $(*)$ the semi-scale property. (i.e., closure in a weaker sense.)

$\vec{\varphi}$ is a scale on A if it is a semiscale + for x_i, y as above,

$$\underbrace{\varphi_n(y) \leq \lim_{i<\omega} \varphi_n(x_i)}$$ for all n.

lower semi continuity property

<u>theorem</u>. ($ZF + CC_{\mathbb{R}}$) for $A \subset \mathcal{N}$, TFAE.

cth. choice for reals

(1) A is suslin.

(2) A has a scale.

(3) A has a semi scale.

proof: (1) ⟹ (2): Say $A = p[T]$, T on $\omega \times OR$.

for $x \in A$, let l_x = leftmost branch of T_x.

define $\varphi_n(x) = \| l_x \upharpoonright n \|_{lex}$ (lex in ${}^n OR$).

"leftmost norm"

if $\underset{\underset{A}{\cap}}{x_k} \longrightarrow y$ mod $\vec{\varphi}$. let $f \in {}^\omega OR$,

$$f(n) = \lim_{k \to \infty} \varphi_n(x_k).$$

then $(y, f) \in [T]$, so $y \in A$.

have $f \in [T_y]$, so $\ell_y \leq_{lex} f$,

so lower semicont. also follows.

③ $\Rightarrow$ ① Let $\vec{\varphi}$ be a semiscale on A.

$T_{\vec{\varphi}} = \{ (x \upharpoonright n, (\varphi_0(x), \dots, \varphi_{n-1}(x))) : x \in A \}$.

clearly, $A \subset p[T_{\vec{\varphi}}]$. Let $y \in p[T_{\vec{\varphi}}]$,

say $(y, f) \in [T_{\vec{\varphi}}]$.

for all $n < \omega$, choose $x_n \in A$ s.t.

$x_n \upharpoonright n = y \upharpoonright n$, $\varphi_i(x_n) = \varphi_i(y)$.

then $x_n \longrightarrow y \bmod \vec{\varphi}$. so $y \in A$.

showed $p[T_{\vec{\varphi}}] \subset A$. $\dashv$

in Cohen's model in which there is A

with no ctble subset of A:

trivially, A has a scale (as converge

is a trivial concept). but A is not suslin, as o.w. ~~its~~ one of its elts. would ~~be~~ be definable from others. $\lessgtr$

So need $CC_{\mathbb{R}}$. [observed by Gabe Goldberg.]

example. let A be $\underset{\sim}{\Pi}^1_1$, say $A = {}^\omega\omega \setminus p[T]$.

for $s \in {}^{<\omega}\omega$, define

$$\varphi_s : A \longrightarrow \omega_1$$

$$\varphi_s(x) = \begin{cases} \mathrm{rank}_{T_x}(s) & \text{if } s \in T_x \\ 0 & \text{if } s \notin T_x \end{cases}$$

then $\{\varphi_s : s \in {}^{<\omega}\omega\}$ forms a scale on A (in any enumeration).

proof: let $\underset{A}{\underset{\cap}{x_n}} \longrightarrow y \bmod \vec{\varphi}$.

define $\rho : T_y \longrightarrow \mathrm{OR}$:

$$\rho(s) = \lim_{n\to\infty} \varphi_s(x_n) = \lim_{n\to\infty} \mathrm{rank}_{T_{x_n}}(s)$$

$\rho(t) < \rho(s)$ for $s \subsetneqq t$, i.e., ρ is a rank fcn. So $y \in A$.

lower semi-continuous, because $\vec{\varphi}$ was defined using the least rank fctn.

<u>note</u>. this is a $\underset{\sim}{\Pi}^1_1$ scale.
(Π^1_1 scale if A Π^1_1).

<u>defin</u>. A Γ-scale is a scale $\vec{\varphi}$ on $A \in \Gamma$ s.t.

$$\{(x,y,n) : \varphi_n(x) \leq \varphi_n(y)\} \in \Gamma$$

$$\{(x,y,n) : \varphi_n(x) < \varphi_n(y)\} \in \Gamma$$

add here "$x \in A \wedge (y \in A$ or ...$)$"

note : for reasonably closed pt. classes,

$$\text{scale}(\Gamma) \implies \text{unif}(\Gamma).$$

<u>theorem.</u> (Kondo – Addison)

unif(Π^1_1).

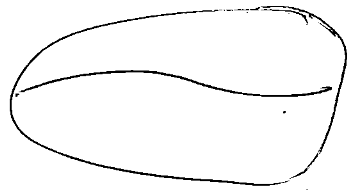
scales compute uniformizations locally.

<u>theorem</u> ($ZF + CC_{\mathbb{R}}$) let $A \subset W \times W$ +

let $\vec{\varphi}$ be a scale on A.

then A has uniformization.

$f : pA \longrightarrow W$



such that

for all $n < \omega$, the function

$x \longmapsto f(x) \upharpoonright n$ is definable from

$\varphi_0, \ldots, \varphi_{n-1}$.