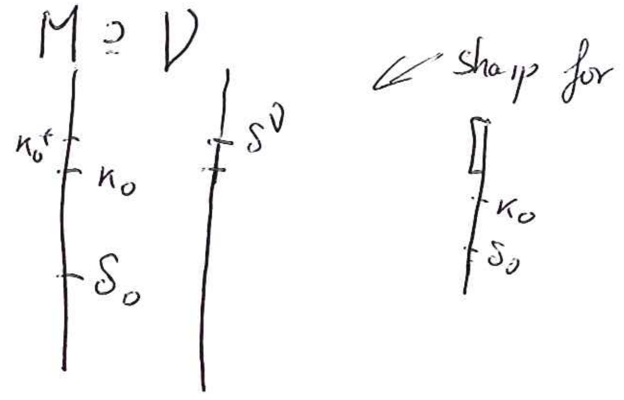
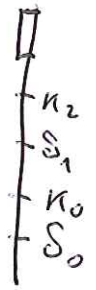
2017/07/20 R. Schindler Varsovian Models II

joint work with Grigor



Assume $M^{\#}_{swsw}$ exists



δ_i Woodin

κ_i strong

$M = M_{swsw}$

Fix the iteration strategy Σ for M induced by the iteration strategy for $M^{\#}_{swsw}$

<u>Question.</u> Which grounds does M have?

• $\mathcal{M}^{\circ}_{\infty}$ - system:

consists of $\mathcal{P}$'s s.t.

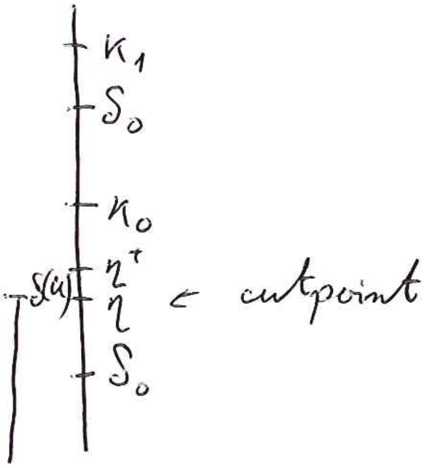
$\exists$ cutpoint $\eta < \kappa_0$ $\mathcal{M}(\mathcal{U})$

$\mathcal{U}$ lives on $M|\delta_0$ $\mathcal{U}$

$\delta(\mathcal{U}) = \eta^{+M}$

$\mathcal{U}$ is guided by $\mathcal{P}$-constructions

$M = \mathcal{P}^M(\mathcal{M}(\mathcal{U}))[M|\eta^+]$



(Actually $P^M(\mathcal{M}(\mathcal{U})) \cong \mathcal{M}_\infty^{\mathcal{U}}$.)

System is

$(P, \pi_{P,P'} \mid P$ is of the form $P^M(\mathcal{M}(\mathcal{U}))$ as above, $\pi_{P,P'}$ the canonical iteration map if $\pi_{P,P'}\downarrow$ $)$

The model M contains enough fragments of the maps $\pi_{P,P'}$ so that inside M we can define a system with the same direct limit.

Let $(\mathcal{M}_\infty^0, \pi_{P,\infty} \mid P \in \text{system})$ be the direct limit. This system is a definable class in M.

M an inner model.

Suppose $(P_i, \pi_{ij} \mid i \preceq j \in I)$, $0 \in I$, $P_0 = M$, producing $(\mathcal{M}_\infty, \pi_{i\infty} \mid i \in I)$.

Collection of the P_i is definable in M.

Each P_i is a ground for M "in a uniform way".

Let $\xi^* = \min\{\pi_{i,\infty}(\xi) \mid i \in I\}$, $\xi \in OR$

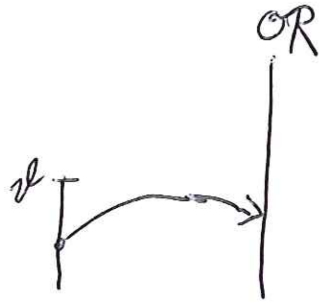
Suppose $\xi \mapsto \xi^*$ is in M.

Then $L[\mathcal{M}_\infty, \xi \mapsto \xi^*]$ is a ground.

<u>Why?</u> Let $f : \vartheta \to OR$, $f \in M$

Fix λ large enough.

Suffices to ~~prove~~ get



$$f^* : \vartheta \to \mathcal{P}(OR) \quad \text{s.t.}$$

$$f(\xi) \in f^*(\xi) \quad \forall \xi < \vartheta$$

$$\overline{\overline{f^*(\xi)}} < \lambda$$

$$f^* \in L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$$

(In our case $\lambda = \kappa_0^+$.)

<u>Proof</u>. Each P_i has some such f^*, call it f_i, $i \in I$.

Can arrange that $\vec{f} = (f_i \mid i \in I)$ is in M.

[True in our case.]

Consider $\pi_{0,\infty}(\vec{f})$.

$$h(\xi) = \bigcup \{ g(\xi) \mid g \in \pi_{0,\infty}(\vec{f}) \}$$

$$f^*(\xi) = (h(\xi^*))^{*^{-1}}$$

Say $f(\xi) = \xi'$.

$$f(\xi) \in f_i(\xi) \text{ for all } i.$$

Take $i \in I$ s.t. $\pi_{i,\infty}(f(\xi), \xi) = \underbrace{f(\xi)^*}_{= \xi'^*}, \xi^*$

$$\xi'^* \in h(\xi^*)$$

$\Rightarrow \xi' \in f^*(\xi)$.

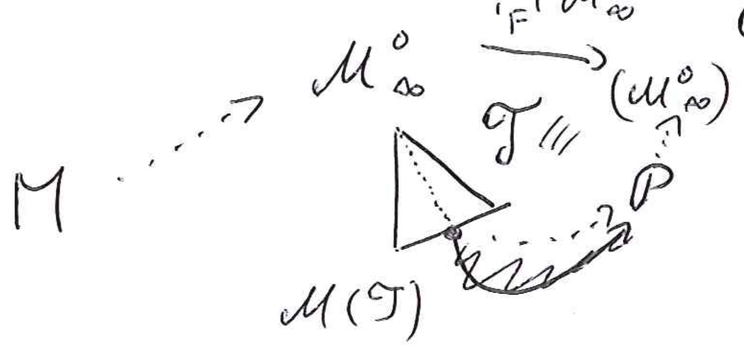
$M = M_{swsw}$

$\rightsquigarrow \mathcal{M}^0_\infty$

$L[\mathcal{M}^0_\infty, g \mapsto g^*]$ is a ground contained in all $\mathcal{P}$'s from the $\mathcal{M}^0_\infty$-system

- $\mathcal{M}^0_\infty | \underset{\text{least Woodin of } \mathcal{M}^0_\infty}{\overset{\parallel}{\delta_0^{\mathcal{M}^0_\infty}}}$ is fully iterable in M.

<u>Proof</u>.



$\mathrm{crit}(F) = \kappa_0$

F long enough

$\dashv$

- Let $Q = \mathcal{M}^0_\infty | \underbrace{\kappa_0^{+\mathcal{M}^0_\infty}}$

the ordinal α s.t. $\mathcal{M}^0_\infty \models$ "α is the cardinal succ. of the least strong"

Let $i : \mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty} \to \mathcal{M}^*$ arise from making Q generic.

κ_0^{++M}

$\kappa_0^{+M} = \delta^{\mathcal{M}^0_\infty}$

κ_0

δ_0

$\mathcal{U}$ on $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$ of length $\kappa_0^{+\mathcal{M}^0_\infty} + 1$ s.t. $\mathcal{U} \upharpoonright \kappa_0^{+\mathcal{M}^0_\infty}$ is definable on Q

Q $\tau = \kappa_0^{+\mathcal{M}^0_\infty}$

$\kappa_0^{\mathcal{M}^0_\infty}$

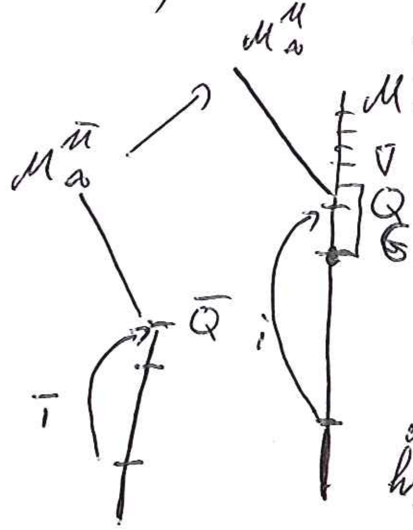
$\delta_0^{\mathcal{M}^0_\infty} = \kappa_0^{+M}$

$i = \pi^{\mathcal{U}}_{0\tau}$

Claim. $Q = \mathrm{Hull}^Q(i)$. (Soundness)

Proof.

Let $\bar{Q} \overset{\sigma}{\cong} \mathrm{Hull}^Q(i)$



$\mathcal{U}$ construed as a tree on $\mathcal{M}_\infty^0$

$$\rho^{\mathrm{Ult}(\mathcal{M}_\infty^0; G)}(\mathcal{M}(\mathcal{U})) \overset{!?}{=} \rho^{\mathcal{M}_\infty^0}(\mathcal{M}(\mathcal{U})) = \mathcal{M}_\infty^{\mathcal{U}}$$

$$\mathcal{M}_\infty^{\mathcal{U}}[Q] = \mathrm{Ult}(\mathcal{M}_\infty^0; G)$$

Will get

$$\mathcal{M}_\infty^{\bar{\mathcal{U}}}[\bar{Q}] \to \underset{W}{\mathrm{Ult}(\mathcal{M}_\infty^0; G)}$$

Corollary. $\bar{\bar{Q}} = \delta_0^{\mathcal{M}_\infty^0}$ in $L[\mathcal{M}_\infty^0, g \mapsto g^*]$.

$W = \mathrm{Ult}(\mathcal{M}_\infty^0; G)$ indisc.

$$\mathcal{M}_\infty^0 = \mathrm{Hull}^{\mathcal{M}_\infty^0}(\delta^{\mathcal{M}_\infty^0} \cup I)$$

$$\mathrm{Ult}(\mathcal{M}_\infty^0; G) = \mathrm{Hull}^W(\delta_0^{\mathcal{M}_\infty^0} \cup \{\kappa_0^{\mathcal{M}_\infty^0}\} \cup I)$$

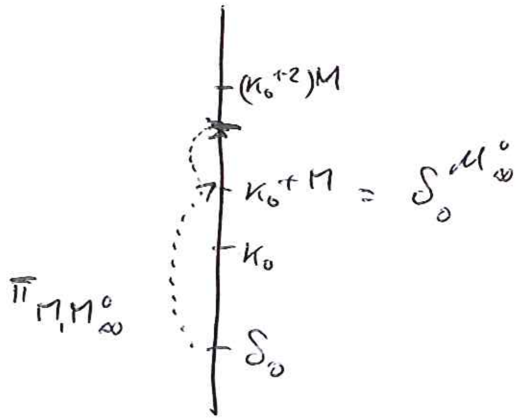
$$\mathcal{M}_\infty^{\bar{\mathcal{U}}}[\bar{Q}] \longrightarrow \mathcal{M}_\infty^{\mathcal{U}}[Q]$$

$$\tau^{\bar{Q}} \longmapsto \sigma(\tau)^Q$$

$$\| \quad \mathrm{Ult}(\mathcal{M}_\infty^0; G)$$

$M = M_{swsw}$

$L[\mathcal{M}^\circ_\infty, g \mapsto g^*] =: V_0$



$V_0[M \mid \kappa_0^{+2}]$

$\uparrow$

V_0 generic for $\mathbb{B} \in V_0$

$\mathbb{B} \subseteq \kappa_0^{+2}$ has the $\delta_0^{\mathcal{M}^\circ_\infty}$-c.c. in V_0

$\cap$

$V_0 | \kappa_0^{+3}$

Can show: $V_0 = \mathrm{HOD}^{M^{Col(\omega, <\kappa_0)}}$

$$H^{V_0}_{\delta_0} = \mathcal{M}^\circ_\infty \mid \delta_0^{\mathcal{M}^\circ_\infty}$$

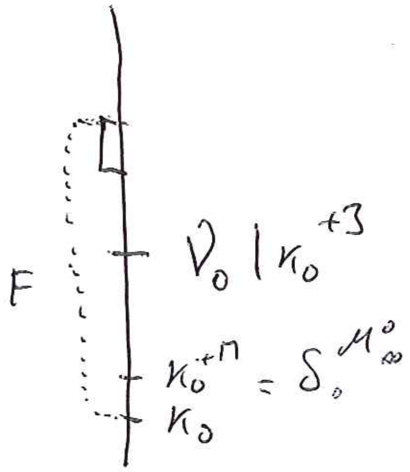
Also, $\delta_0^{\mathcal{M}^\circ_\infty}$ is Woodin in V_0.

To get more grounds (and more information on V_0) let's reorganize it.

$$\mathcal{P}^{\bar{\Sigma}}(V_0 \mid \kappa_0^{+3})^M$$

$$\cup$$

$$\mathbb{B}$$



Notation: $\bar{\Sigma} = \Sigma_{\mathcal{M}^\circ_\infty \mid \delta_0^{\mathcal{M}^\circ_\infty}}$

$$i^M_F \restriction \mathcal{M}^\circ_\infty : \mathcal{M}^\circ_\infty \to (\mathcal{M}^\circ_\infty)^{Ult(M;F)}$$

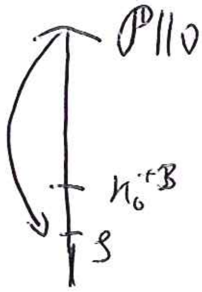
Claim: $\underbrace{\mathcal{P}^{\Sigma}(V_0|\kappa_0^{+3})^M}_{\mathcal{P}} = V_0$.

Proof. $\mathcal{P}$ never projects across κ_0^{+3}.

$V_0[g] = M$

$\uparrow$

$\mathbb{B}$-gen. $/M$



$\tau \in V_0|\kappa_0^{+3}$

$\tau^g = \{\xi<\varrho \mid \mathcal{P}\|\nu \models \varphi(\xi,\vec{p})\}$

$p \Vdash \tau = \{\xi<\varrho \mid \mathcal{P}\|\nu \models \varphi(\xi,\vec{p})\}$

$\cap$

g

if g, g' are mutually generic $\mathcal{P}\|\nu$ $\mathcal{P}\|\nu$

$\Rightarrow a = \tau^g = \tau^{g'} \in \mathcal{P}[g] \cap \mathcal{P}[g']$

$= \mathcal{P}$

$V_0|\kappa_0^{+3}[g] = M|\kappa_0^{+3}$

$V_0|\kappa_0^{+3} = \mathcal{P}|\kappa_0^{+3}$

M is $\mathcal{P}$-generic via $\mathbb{B}$

M is $\cancel{\mathcal{P}}\ V_0$-generic via $\mathbb{B}$

Woodin & Laver $\Rightarrow$ $V_0 = \mathcal{P}$.

- V_0 is iterable with respect to trees which live on $(0, \delta_0^{\mathcal{M}_\infty^0}) \cup (\kappa_0^{+3}, \infty)$

Proof.

$L[\mathcal{M}_\infty^0, \varrho \mapsto \varrho^*]$

$M = \mathrm{Hull}^{\mathcal{M}_\infty^0}(\mathrm{ran}(\pi_{M\mathcal{M}_\infty^0}))$

trivially

$$L[M, \pi_{M,\mathcal{M}^0_\infty}] \cong \mathrm{Hull}^{L[\mathcal{M}^0_\infty,\, \rho \mapsto \rho^*]}(\mathrm{ran}(\pi_{M,\mathcal{M}^0_\infty}))$$

$$L[M, \pi_{M,\mathcal{M}^0_\infty}] \xrightarrow{\tilde{\pi}_{M,\mathcal{M}^0_\infty}} L[\mathcal{M}^0_\infty,\, \rho \mapsto \rho^*].$$

Suffices to prove: $L[M, \pi_{M,\mathcal{M}^0_\infty}]$ is iterable in V with respect to extenders from $\underset{\substack{\| \\ V_0^M}}{(0,\delta_0)} \cup (\kappa_0^{+3}, \infty)$.

$$L[M, \pi_{M,\mathcal{M}^0_\infty}] \longrightarrow L[\mathcal{M}^0_\infty,\, \rho \mapsto \rho^*] \subseteq M$$

as a tree on M

living on $\mathcal{T}$ $M|\delta_0$ $\quad \mathcal{M}^{\mathcal{T}}_\alpha \longrightarrow V_0^{\mathcal{M}^{\mathcal{T}}_\alpha} \subseteq \mathcal{M}^{\mathcal{T}}_\alpha \qquad \mathcal{M}^{\mathcal{T}}_\alpha$

$$L[M^*, \pi^*] \xrightarrow{\sigma} V_0^{M^*} \subseteq M^*$$

$$V_0^{M^*} = \mathcal{P}^{\Sigma}(V_0^{M^*}|(\kappa_0^{+3})^{M^*})^{M^*}$$

$\mathcal{T}'$ $\qquad \sigma\mathcal{T}'$

We can lift $\sigma\mathcal{T}'$ to M^* by fattening the extenders. Since M^* – as an iterate of M – is iterable, we get a branch for $\sigma\mathcal{T}'$ and hence $\mathcal{T}'$.

V_0

κ_0^{+3}

$?$

$\delta_0^{V_0} = \kappa_0^{+M}$

$W = L^{\bar{\Sigma}}[E](\mathcal{M}^0_\infty \mid \delta_0^{\mathcal{M}^0_\infty})$ $\quad \bar{\Sigma} = \Sigma_{\mathcal{M}^0_\infty \mid \delta_0^{\mathcal{M}^0_\infty}}$

$\uparrow$

certified by total

M-ext. / V_0-ext.

with critical point $> \kappa_0$

$W \models \delta_1$ is a Woodin cardinal

$M \mid \delta_1$ is generic over W.

$$\Rightarrow R := \underbrace{P^{\bar{\Sigma}}(W \mid \delta_1)}_{\text{is iterable in } V}[M \mid \delta_1] = M.$$

Claim: $\neg \exists \delta \in (\delta_0^{\mathcal{M}^0_\infty}, \delta_1)$

$R \models$ "δ is Woodin".

o.w. $R[M \mid \delta] \models$ "δ is Woodin" $\quad$ δ_1, δ, $\delta_0^{\mathcal{M}^0_\infty}$

...

R: κ_1^+, κ_1, η^+, η, δ_1^R, $\delta_0^R = \delta_0^{\mathcal{M}^0_\infty}$

$R \dashrightarrow \mathcal{M}^1_\infty$

$L[\mathcal{M}^1_\infty, \xi \mapsto \xi^{**}]$

?

$V_1 = V^M$