2017/07/27 Gabriel Goldberg

<u>Thm.</u> Assume the Ultrapower Axiom (UA) + κ is supercompact. Then for all $\delta \geq \kappa$

$$2^{\delta} = \delta^+.$$

<u>Convention</u> A <u>uf</u> U is a uniform, countably complete ultrafilter on an ordinal α, denoted by $sp(U)$.

<u>Uf</u> denotes the class of ufs.

<u>Def.</u> Suppose $U_0, U_1 \in \mathrm{Uf}$.
A <u>comparison</u> of (U_0, U_1) is a pair (W_0, W_1)

s.t.

1) $W_i \in \mathrm{Uf}^{M_{U_i}}$, $i = 0, 1$
2) $M_{W_0}^{M_{U_0}} = M_{W_1}^{M_{U_1}}$
3) $j_{W_0}^{M_{U_0}} \circ j_{U_0} = j_{W_1}^{M_{U_1}} \circ j_{U_1}$.

<u>Def.</u> (UA)
Every pair of ufs admits a comparison.

Question. Suppose $\mathcal{U}$ is a normal ultrafilter on κ. Can $P(\kappa^+) \subseteq M_{\mathcal{U}}$?

Remark. If so, $2^\kappa = 2^{\kappa^+}$.

Proof. $2^{\kappa^+} = |P(\kappa^+)| \leq |j_{\mathcal{U}}(\kappa)| = 2^\kappa$. $\dashv$

Proposition. Suppose κ is supercompact. TFAE

1) $\exists\, \mathcal{U}$ s.t. $\mathcal{U}$ is normal on κ & $P(\kappa^+) \subseteq M_{\mathcal{U}}$,
2) $2^\kappa = 2^{\kappa^+}$.

Thm. (Solovay)
If κ is 2^{κ}-supercompact and $A \subseteq 2^\kappa$ then there is a normal uf $\mathcal{U}$ on κ s.t. $A \in M_{\mathcal{U}}$.

Proof of the proposition.
$|P(\kappa^+)| = 2^{\kappa^+}$ so $P(\kappa^+)$ is transitive, it's coded by $A \subseteq 2^{\kappa^+} = 2^\kappa$. Take $\mathcal{U}$ s.t. $A \in M_{\mathcal{U}}$, $\mathcal{U}$ normal.

<u>Def.</u> Suppose δ is a regular cardinal. We say δ is <u>M-commanded</u> if for all $A \subseteq 2^{\delta}$ $\exists$ uf U s.t. $sp(U) \leq \delta$ & $A \in M_U$.

<u>Proposition.</u> Every $\delta \geq \kappa$ is M-commanded.

<u>Proposition.</u> (UA) If δ is M-commanded and δ^+ carries a uf then $2^{\delta} < 2^{\delta^+}$.

~~Question~~. Suppose $U_0, U_1 \in Uf$ and $sp(U_0) \leq sp(U_1)$. Can $U_1 <_M U_0$?

~~Proposition. Suppose X is a regular cardinal. Let U_1~~

<u>Def.</u> Suppose $U_0, U_1 \in Uf$. Then

$$U_0 \leq_S U_1 :\Longleftrightarrow \exists (W_0, W_1) \text{ comparison of } (U_0, U_1) \text{ s.t. } j^{M_{U_0}}_{W_0}([id]_{U_0}) \leq j^{M_{U_1}}_{W_1}([id]_{U_1})$$

<u>Thm.</u> $\leq_S$ is a wellorder of Uf.

<u>Lemma.</u> If $U_0 \leq_S U_1$ then $sp(U_0) \leq sp(U_1)$.

<u>Proposition</u>. Suppose U_1 is the $\leq_S$ - least uf on a regular cardinal λ. Then for all U_0 s.t. $sp(U_0) \leq \lambda$:

$$U_1 \not\leq_M U_0 .$$

<u>Proof of weak GCH at δ.</u>

Let U_1 be the $\leq_S$ - least uniform uf on δ^+. By M-command at δ, there's an uf U_0 s.t. $sp(U_0) \leq \delta$ and $U_1 \in M_{U_0}$, since we may code U_1 as $A \subseteq 2^{\delta^+} = 2^{\delta}$. But this contradicts the proposition. $\dashv$

<u>Lemma</u>. Suppose $U_0, U_1 \in Uf$.

1) (U_0, U_1) admits a comparison (W_0, W_1) s.t.

$W_0 \leq_S^{M_{U_0}} j_{U_0}(U_1)$ &

$W_1 \leq_S^{M_{U_1}} j_{U_1}(U_0)$.

2) Suppose (U_0, U_1) admits a comparison of the form $(j_{U_0}(U_1), W_1)$. Then

$$j_{W_1}^{M_{U_1}} = j_{U_0} \restriction M_{U_1} .$$

Recall. $U_0 <_I U_1$ if $j_{U_0} \restriction M_{U_1}$ is an internal ultrapower embedding of M_{U_1}.

Lemma. Suppose λ is regular. Suppose U_1 is the $\leq_S$ - least uf on λ. Then for any U_0 s.t. $sp(U_0) \leq \lambda$:

$$U_0 <_I U_1 .$$

Proof. Let (W_0, W_1) be a comparison of (U_0, U_1) satisfying 1) of the previous lemma, i.e. $W_0 \leq_S^{M_{U_0}} j_{U_0}(U_1)$. It suffices to show

$$j_{U_0}(U_1) \leq_S^{M_{U_0}} W_0 .$$

Assume not, i.e. $W_0 <_S^{M_{U_0}} j_{U_0}(U_1)$.

W_0 is $<_S^{M_{U_0}}$ the $\leq_S$ - least uf ~~on λ~~ of M_{U_0} with space $j_{U_0}(\lambda)$.

So $sp(W_0) < j_{U_0}(\lambda)$.

$j_{W_0}^{M_{U_0}} \circ j_{U_0}$ is continuous at λ. But

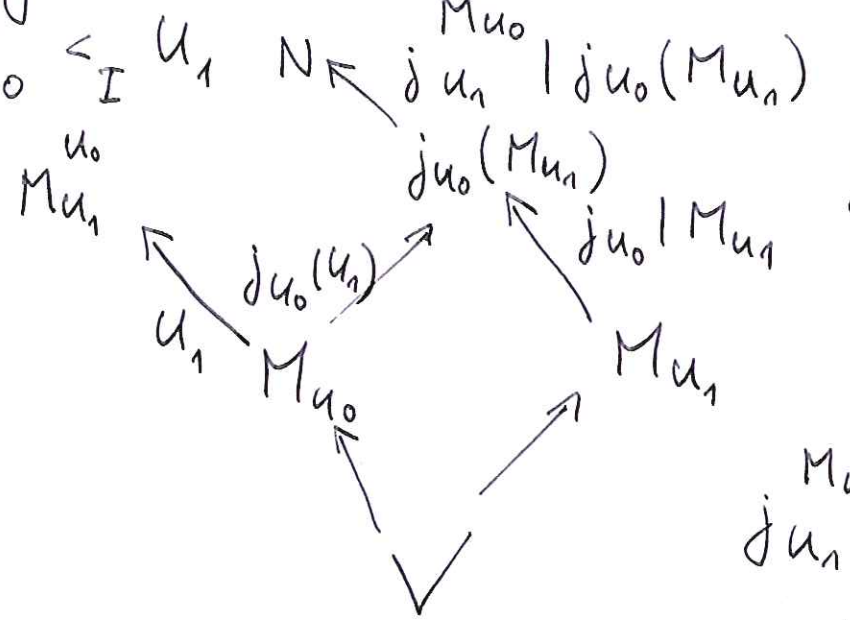
$j_{W_0}^{M_{U_0}} \circ j_{U_0} = j_{W_1}^{M_{U_1}} \circ j_{U_1}$ is discontinuous at λ. $\lightning$

$\dashv$

Proposition. Suppose U_1 is the $\leq_S$-least uf on λ regular. Suppose U_0 is an uf s.t. $sp(U_0) < \lambda$. Then $U_1 \not\leq_M U_0$.

Proof. Assume not. Fix U_0 on $sp(U_0) < \lambda$, say $sp(U_0) = \delta$, s.t. $U_1 <_M U_0$. We know

$U_0 <_I U_1$



$j_{U_1}^{M_{U_0}} \upharpoonright Ord$ is ~~amenable~~ amenable to M_{U_1}.

$j_{U_1}^{M_{U_0}} \upharpoonright H_{\lambda^+} = j_{U_1} \upharpoonright H_{\lambda^+}$ because $U_1 \in M_{U_0}$, so $P(\lambda) \in M_{U_0}$, so $H_{\lambda^+} \in M_{U_0}$, $H_{\lambda^+}^{\lambda} \subseteq H_{\lambda^+}$, so $Ult(H_{\lambda^+}, U_1)$ is computed correctly in M_{U_0}.

$j_{U_0}(\lambda) > \lambda$

$P(\lambda) \in M_{U_0}$

$j_{U_1} \upharpoonright \lambda^+ \in M_{U_1}$.

~~But U_1 has space λ.~~

~~$\sup j_{U_1}[\lambda^+] = j_{U_1}(\lambda^+)$.~~

~~$\Rightarrow$ cf $(\lambda^+) = \lambda^+$~~

$P(\lambda^+) \in M_{U_1}$ + $j_{U_0} \mid M_{U_1}$ "$\in$" M_{U_1}

$\Rightarrow U_0 \in M_{U_1}$. ↯ $\dashv$

<u>GCH from weak GCH:</u>

<u>Lemma</u>. Suppose $2^{<\delta} = \delta$ and δ is M-commanded, δ^+ is M-commanded, δ^+, δ^{++} carry ufs. Then $2^\delta = \delta^+$.

<u>Corollary</u>. (UA) If κ is supercompact then GCH holds above κ.

<u>Proposition</u>. Suppose $2^{<\delta} = \delta$ and δ carries 2^{2^δ} ufs. Then $2^{2^\delta} = (2^\delta)^+$.

<u>Sketch</u>. Prove if U is on δ then

$$|\{W \mid W <_S U\}| \leq \left(2^{2^{<\delta}}\right) = 2^\delta. \qquad \dashv$$

Proof of lemma. Assume $2^\delta \geq \delta^{++}$.

Then $2^\delta < 2^{(\delta^+)} < 2^{(\delta^{++})} \leq 2^{2^\delta} = (2^\delta)^+$ ↯ ⊣