omer III

recap:

let $\mathcal{U} = (\mathcal{U}_{\alpha,\tau} : \alpha \leq \kappa, \tau < o^{\mathcal{U}}(\alpha))$

be a coh. seq. in V.

$\mathbb{R}(\mathcal{U})$ adds a gen. club C to κ.

properties of $\mathcal{U}$	prop. of κ in $V[C]$
$o^{\mathcal{U}}(\kappa) = \tau < \kappa$, τ reg.	$cf^{V[C]}(\kappa) = \tau$.
$o^{\mathcal{U}}(\kappa) = \kappa$	$cf^{V[C]}(\kappa) = \omega$
$\tau \leq \kappa^+$, $o^{\mathcal{U}}(\kappa) = (\kappa^+)^{1+\tau}$	κ is τ-mahlo in $V[C]$
$o^{\mathcal{U}}(\kappa) = \kappa^{++}$	κ reflects stat. sets; moreover, $\forall (S_i : i < \kappa)$ stat. $\exists$ unboundedly many $\delta < \kappa$ $(S_i \cap \delta : i < \delta)$ stat. in δ

~~WRCR~~

WRP	κ is weakly compact
LRP	?
RP	κ is measurable

remark all 'length' properties (i.e., $o(\kappa) = \kappa^{++}$) are ~~[illegible]~~ consistently compatible w/ $o^{\mathcal{U}}(\kappa) < 2^{\kappa}$.

LRP (and hence RP) is incompatible w/ $o^{\mathcal{U}}(\kappa) \leq 2^{\kappa}$.

theorem (Woodin) SPS $2^{\kappa} > o^{\mathcal{U}}(\kappa)$ and $cf(o^{\mathcal{U}}(\kappa)) \geq \kappa^{+}$, then $\Diamond_{\kappa}$ fails in $V[C]$.

recall: $(\mathcal{R}(\mathcal{U}), \leq, \leq^{*})$ is a prikry forcing notion. conditions are of the form

$p = d_0 {}^\frown d_1 {}^\frown \dots {}^\frown d_{\ell-1} {}^\frown d_{\ell}$,

$d_i = (\kappa_i, a_i)$

$\kappa_0 < \kappa_1 < \dots < \kappa_{\ell} = \kappa$. for each $i \leq \ell$, $a_i \in \mathcal{E}_{\kappa_i}$.

$$\mathcal{E}_{\alpha} = \begin{cases} \bigcap_{\tau < o(\alpha)} \mathcal{U}_{\alpha,\tau} & \text{if } o^{\mathcal{U}}(\alpha) > 0 \\ \{\emptyset\} & \text{o.w.} \end{cases}$$

for every $m < \ell$,

$$\mathcal{R}(\mathcal{U})/_{p} \cong \overbrace{\mathcal{R}(\mathcal{U}\restriction \kappa_m+1)/_{d_0{}^\frown\cdots{}^\frown d_m}}^{\kappa_m^+\text{-c.c.}} \times \underbrace{\mathcal{R}(\mathcal{U}\setminus \kappa_m+1)/d_{m+1}{}^\frown\cdots{}^\frown d_\ell}_{(2^{\kappa_m})^+\text{-closed}}$$

<u>proof</u> of woodin's thm.

Supp. $p \in \mathcal{R}(\mathcal{U})$, $\sigma = (\sigma_\alpha : \alpha < \kappa)$
a sq. of names s.t. $p \Vdash \sigma_\alpha \subset \check{\alpha}$
f.a. $\alpha < \kappa$.

we plan to find $X \subset \kappa$ in V,
$\bar{p} \le p$ s.t. $\bar{p} \Vdash \check{X} \cap \alpha \neq \sigma_\alpha$ for
club many α.

since $X \cap \alpha \in V$ for all α then,
may assume σ_α is a $\mathcal{R}(\mathcal{U})$-name for a
V-set.

step 1. reduce the $\mathcal{R}(\mathcal{U})$ names σ_α with an $\mathcal{R}(\mathcal{U}\restriction\alpha+1)$ name of V-set.

step 2. counting argument using 2^κ-~~[illegible]~~ large.

step 1: write $p = \underbrace{d_0{}^\frown \cdots}_{\vec{d}}{}^\frown d\ell = \vec{d}\,{}^\frown(\kappa, A)$

for each $\alpha \in A$, consider

$$p^\frown\langle\alpha\rangle = \vec{d}\,{}^\frown(\alpha, A\cap\alpha)^\frown(\kappa, A\setminus\alpha).$$

Since $\mathcal{R}(\mathcal{U}\setminus(\alpha+1))$ has $\leq^*$ which is $(2^\alpha)^+$ closed, there is $A_\alpha \in \mathcal{F}_\kappa$, $A_\alpha \subset A$ and S_α, an $\mathcal{R}(\mathcal{U}\restriction\alpha+1)$-name s.t. $\vec{d}\,{}^\frown(\alpha, A\cap\alpha)^\frown(\kappa, A_\alpha) \Vdash \sigma_\alpha = S_\alpha$.

we thus obtain $(A_\alpha : \alpha<\kappa) \subset \mathcal{F}_\kappa$ and $(S_\alpha : \alpha<\kappa)$ where S_α is an $\mathcal{R}(\mathcal{U}\restriction\alpha+1)$-name. let $A^* = \underset{\alpha<\kappa}{\Delta} A_\alpha \in \mathcal{F}_\kappa$, and $p^* = \vec{d} \frown (\kappa, A^*)$.

$p^* \leq^* p$ and $\forall \alpha \in A^*$,

$p^* \frown \langle \alpha \rangle \Vdash \sigma_\alpha = S_\alpha$.

<u>step 2</u>: fix $\tau < o^{\mathcal{U}}(\kappa)$, and let

$j_{\kappa,\tau} : V \longrightarrow M_{\kappa,\tau} \cong \mathrm{ult}(V; \mathcal{U}_{\kappa,\tau})$, we

have $j_{\kappa,\tau}(p^*) \frown \langle \kappa \rangle \Vdash j_{\kappa,\tau}(\sigma)_\kappa = j_{\kappa,\tau}(S)_\kappa$,

where $j_{\kappa,\tau}(S)_\kappa$ is a $\mathbb{R}(j_{j_{\kappa,\tau}}(\mathcal{U}) \restriction \kappa+1) =$

$\mathbb{R}(\mathcal{U} \restriction (\kappa,\tau))$ name of a V-subset of κ.

the fact that $\mathbb{R}(\mathcal{U} \restriction (\kappa,\tau))$ satisfies

the κ^+-c.c. implies there are only

κ-many options for the V-set

$j_{\kappa,\tau}(S)_\kappa$, for each $\tau < o^{\mathcal{U}}(\kappa)$.

since $o^{\mathcal{U}}(\kappa) < 2^\kappa$, there must be $X \in \mathcal{P}(\kappa) \cap V$

s.t. $\mathbb{1} \Vdash j_{\kappa,\tau}(S)_\kappa \neq \check{X}$ for each $\tau < o^{\mathcal{U}}(\kappa)$.

back in V, let $B = \{\alpha < \kappa : p^* \frown \langle\alpha\rangle$

$\Vdash S_\alpha \neq X \cap \check{\alpha}\}$.

then $\kappa \in j_{\kappa,\tau}(B)$

$\Rightarrow$ $B \in \mathcal{U}_{\kappa,\tau}$ for each $\tau < o^{\mathcal{U}}(\kappa)$.

$\Rightarrow$ $B \in \mathcal{F}_\kappa$.

let $\bar{p} \leq^* p^*$, $\bar{p} = \vec{d} \frown (\kappa, A^* \cap B)$.

then $\forall \alpha \in A^* \cap B$

$p^* \frown \langle\alpha\rangle \Vdash \sigma_\alpha = S_\alpha \neq \check{X} \cap \alpha$.

since $p^* \Vdash \underset{\sim}{C} \setminus \max(\vec{d}) \subset A^* \cap B$,

then $p^* \Vdash \forall \alpha \in \underset{\sim}{C} \setminus \max(\vec{d}),\ \sigma_\alpha \neq \check{X} \cap \alpha$.

hence $p^* \Vdash (\sigma_\alpha : \alpha < \kappa)$ is not a

$\diamondsuit_\kappa$ - seq. $\dashv$