dominik 2

defin. M a ctm sharp premouse, etc.

let $\kappa < \lambda$ be inacc.

(Q, Σ) is captured at κ iff

- (Q, Σ) is a κ-HOD pair ($\overline{\overline{Q}} \leq \kappa$, Σ is a (κ^+, κ^+) strategy w/ hull condensation)
- Σ uniquely extends to a $< \lambda$-u.b. strategy on $Col(\omega, <\lambda)$.
- Σ and its extension have branch condensation + $\mathcal{D}(M, \lambda)$-fullness preservation.

defn. let λ be a lim of inacc, $\alpha \in OR$.

$\kappa < \lambda$ is $(\alpha, <\lambda)$-wg iff κ is inacc. + $\forall \beta \cancel{<} \alpha$ $\forall \gamma < \lambda$ $\exists E \in V_\lambda$

$M = \mathrm{ult}(V, E)$ is w.fdd.

$V_\gamma \subset M$, $M \models \kappa$ is $(\beta, < i_E(\lambda))$-wg lem. let λ be a lim of meas., λ non-measurable cof.

$\exists \alpha < \lambda^+$ s.t. no $\kappa < \lambda$ is $(\alpha, <\lambda)$ strg.

Pf: $C_E = \{\alpha < \lambda^+ : i_E(\alpha) = \alpha\}$.

C_E contains an ω-club.

$C = \bigcap_{E \in V_\lambda} C_E$, has otp λ^+.

$\forall \alpha < \beta$ in C, the least $(\alpha, <\lambda)$ strg is $<$ the least $(\beta, <\lambda)$ strg. $\dashv$

lem. let λ be a lim of measurables.

Supp. $x^\#$ ex. let $x^\#_\alpha$ be the α^{th} it. of $x^\#$.

defn $\leq^\lambda_x$: domain consists of $(\tau, \vec{\alpha}, \vec{y})$ s.t.

τ is an Σ_1-term in the language of $x^\#$, with $lh(\vec{\alpha}) = lh(\vec{y}) + 1$ free variables

$\vec{\alpha}$ finite set of x-indiscernibles $< \lambda$,

$\vec{y} \subset x$.

$(\tau, \vec{\alpha}, \vec{y}) \leq^\lambda_x (\sigma, \vec{\beta}, \vec{z})$ iff

$\cancel{\tau^{L[x^\#_{\lambda+1}]}(\vec{y}, \vec{\alpha}, \lambda) \leq \sigma}$

$\tau^{x^\#_{\lambda+1}}(\vec{y}, \vec{\alpha}, \lambda) \leq \sigma^{x^\#_{\lambda+1}}(\vec{z}, \vec{\beta}, \lambda)$

$\in OR$.

$\delta^\lambda_x(\xi) = \cancel{\ldots} \; \|\xi\|_{\leq^\lambda_x}$

$\Delta^\lambda_x = \sup_\xi \delta^\lambda_x(\xi)$.

<u>Lem</u>. Let λ be a limit of inacc.

$cf(\lambda)$ nonmeasurable. Let $x \in V_\lambda$, $x^\#$ ex. Let $E \in V_\lambda$, $crit(E) > rk(x)$.

$$s \le^{\lambda}_{x} t \iff s\, i_E(\le^{\lambda}_{x})\, t .$$

$$i_E(\delta^{x}_{\lambda}) = \delta^{x}_{\lambda} .$$

__Lem.__ Let λ be a Woodin cardinal, then there is $\kappa < \lambda$ s.t. $\forall x \in V_\kappa$ κ is $(\Delta^{\lambda}_{x}, <\lambda)$ strong.

pf.: let $\kappa < \lambda$ reflect $A = \{ (\eta, s, x) :$ η is $(\delta^{\lambda}_{x}(s), <\lambda)$ - strong $\}$.
fix $x \in V_\kappa$,

Say κ is already $(\delta^{\lambda}_{x}(s), <\lambda)$ strong, then let $E \in V_\gamma$, $\mathrm{crit}(E) = \kappa$, $V_\gamma \subset \mathrm{ult}(V; E)$, and $i_E(A) \cap V_\gamma =$ $\cancel{i_E(A)}$ $A \cap V_\gamma$. then

$(\kappa, s, x) \in i_E(A)$. ⊣

lem. let λ be a lim of $<\lambda$ strongs. let $\kappa < \lambda$ reflect the set of $<\lambda$ strongs. then $\forall x \in V_\kappa$ κ is $(\Delta^\lambda_x, <\lambda)$-strg.

pf.: exercise!

def. let $M^\#_{\Theta_{\omega_2}}$ = the least active sound premouse $(M; \in, \vec{E}, F)$ with a good ω_1+1 iteration strategy s.t.

$M \| \mathrm{crit}(F) \models$ "$\exists \lambda$ lim of woodins s.t.
$\forall x \in V_\lambda \ \exists \kappa > rk(x)$
κ is $(\Delta^\lambda_x, <\lambda)$-strong."

lemma. let λ be $M^\#_{\Theta_{\omega_2}}$'s lim of woodins. then $M^\#_{\Theta_{\omega_2}}$ rebuilds itself below λ.

<u>thm</u>. $\mathcal{D}(M^\#_{\theta_{\omega_2}}; \lambda) \models$ "$\theta \geq \theta_{\omega_2}$"

pf.: fix a $<\lambda$-reg κ.

let $(\mathcal{P}, \Sigma^{\mathcal{P}})$ the local hod limit at κ, i.e., the direct lim of all (Q, Σ) captured at some $\alpha < \kappa$ under comparison. $\Sigma^{\mathcal{P}} = \bigoplus_{\alpha < \lambda^{\mathcal{P}}} \Sigma^{\mathcal{P}}_\alpha$.

let $\mathcal{P}^+ =$ the least $\mathcal{M} \lhd L_p^{\bigoplus_{\alpha<\lambda^{\mathcal{P}}} \Sigma^{\mathcal{P}}_\alpha, \mathcal{D}(M,\lambda)}(\mathcal{P})$, $M = M^\#_{\theta_{\omega_2}}$, s.t. $\rho_\omega(\mathcal{M}) < OR \cap \mathcal{P}$, if ex., o.w. $= L_p \ldots$.

let $\mathcal{T}$ be a normal tree on $\mathcal{P}$ with branch embedding $\pi : \mathcal{P}(\alpha) \longrightarrow Q(\alpha)$. let E be an extender with $\mathrm{crit}(E) = \kappa$, and $\mathcal{T} \in \mathrm{ult}(M; E) = N$.

<u>clai.</u> $\exists \sigma^+ : Q^+ \cong j(P^+)$

s.t. $j \restriction P^+ = \sigma^+ \circ \pi^+$

$Q^+ = \mathrm{ult}(P^+, \pi)$.

pf. of cl.: have $\sigma : Q(\alpha) \longrightarrow j(P(\alpha))$

s.t. $j \restriction P(\alpha) = \sigma \circ \pi$.

let $\sigma^+(\pi(f)(a)) = j(f)(\sigma(a))$.

now arg. for a contradiction that

$\rho_\omega(P^+) < P \cap OR$.

let $\alpha < \lambda$ be min. s.t. $\rho_\omega(P^+) < \delta_\alpha^P$.

let $P^* \cong \mathrm{Hull}_{n+1}^{P^+}(P(\alpha) \cup \{p_{n+1}^{P^*}\})$.

$D = D(M, \lambda) = D(N, \lambda)$.

$\mathcal{P}^*$ has a unique strategy modulo $\Sigma^{\mathcal{P}}_\alpha$.

let a be the new set, then

$$a \in OD^{D}_{\Sigma^{\mathcal{P}}_\alpha} \xRightarrow{\text{mouse capturing}} a \in \mathcal{P}(\alpha+1) \quad \not\ni .$$

$\dashv$

can now build

$$\mathcal{P}^+ \lhd \mathcal{P}^{++} = \mathbb{L}^{M, \Sigma^{\mathcal{P}}}_{k}(\mathcal{P}^+) .$$

let I, π, Q be as before

$$(Q^{++}, \Sigma^{Q}) = m(\mathcal{P}^{++}, \pi)$$

$$\sigma^{++} : Q^{++} \longrightarrow j(\mathcal{P}^{++})$$

<u>claim.</u> $\forall \alpha < \lambda_Q$ $\quad Q \models$ "Σ^{Q}_α extends to a fullness preserving strategy"

<u>proof of claim</u>.

$$D = D(\mathcal{P}^{++}, \lambda) = D(Q^{++}, \lambda) = D(j(\mathcal{P}^{++}), \lambda) .$$

$P^{++} \models \forall \alpha < \lambda^P \quad \Sigma_\alpha^P$ extends.

$\Rightarrow \quad Q^{++} \models \forall \alpha < \lambda^Q \quad \Sigma_\alpha^Q$ extends.

$\Rightarrow \quad D \models$ —"—

let $\mathcal{U}$ be a normal tree on Q^+ on the interval $(\delta_\alpha^{Q^+}, \delta_{\alpha+1}^{Q^+})$

$\alpha = \sup \pi^{+}{}'' \lambda^P$