Hugh Woodin.

<u>thm.</u> Suppose δ is supercompact.

then there is an alternating chain

$(E_i : i < \omega)$ on V s.t.

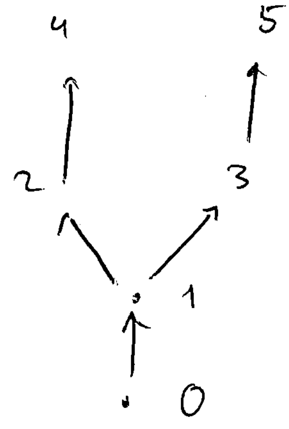
(1) $\forall i$ $\cancel{\mathrm{crit}(E_i)}$) E_i is a short extender

(2) $\forall i > 0$ $\rho(E_i) = \mathrm{length}(E_i)$ is strongly inaccessible in the model M_i

(3) if $i <_T j$, then $\mathrm{crit}(E_j) > j_{E_i}(\mathrm{crit}(E_i))$

(4) both branches are well-founded

tree order:



all that is needed is

$$j : V \longrightarrow M, \quad V_{j(\kappa)+2} \subset M.$$

assuming $L[E]$ ex., in $L[E]$ thm',

no conver. for $V_{j(\kappa)+1} \subset M$

if ext. are for the seq.

<u>prf.</u>: fix $j : V \longrightarrow M$, $\mathrm{crit}(j) = \delta_0$,

$\delta = j(\delta_0)$, $V_{\delta+1} \subset M$.

fix $G \subset \mathrm{Col}(\omega, V_{\delta_0+2})$, V-generic.

view $G : \omega \xrightarrow[\text{onto}]{} V_{\delta_0+2}$. work in $V[G]$.

supp. X, Y are sets of ultrafilters s.t.

each ultrafilter in $X \cup Y$ concentrates

on finite sequences.

<u>def.</u> a bijection $\pi : X \longleftrightarrow Y$ is

a tower isomorphism if f.a.

$(u_i : i<\omega)$ from X,

$(u_i : i<\omega)$ is a tower iff

$(\pi(u_i) : i<\omega)$ is a tower,

and matching preserves well-foundedness.

<u>lem.</u> Suppose $\pi : X \longrightarrow Y$ is a tower isomorphism, $\bar{\bar{X}} = \bar{\bar{Y}} < \kappa$, and all $u \in X \cup Y$ are κ-complete. ~~suppose κ is a limit of Woodin cardinals~~. then $\forall \mathbb{P} \in V_\kappa$, if $g \subset \mathbb{P}$ is V-generic, π is a tower isomorphism.

define $R_G : V_\delta \longrightarrow V_{\delta_0+2}$ as follows (R_G is the "reduction" map given by G).

fix $\mathcal{U} \in V_\delta$, $\mathcal{U}$ concentrates on

i tuples.

$\mathcal{U}$ is δ_0^+ — complete.

supp. $A \in \mathcal{U}$, $(b_k : k < i) \in A$.

recall $G : \omega \longrightarrow V_{\delta_0+2}$

consider $(\underbrace{G(0), b_0, G(1), b_1, \ldots}_{\text{of type } i}),$ call it s.

~~la~~ let $\{B \subset V_{\delta_0} : s \in j(B)\} \sim W_s$.

$\in V_{\delta_0+2}$

$\mathcal{U}$ is δ_0^+ complete, so

there is some $A \in \mathcal{U}$ s.t.

$\forall \vec{b} \in A$, W_s is the same.

this W_s is $R_G(\mathcal{U})$.

key observation.

lem. let $X_G = \{ j(f)(G(i)) : \begin{array}{l} f \in V \\ i < \omega \end{array} \}$

$= \{ j(f)(a) : \begin{array}{l} f \in V, \\ a \in V_{\delta_0 + 2} \end{array} \}$

let $N =$ coll. of X_G.

Supp. $(\mathcal{U}_i : i < \omega)$ is a tower of δ_0^+-complete ultrafilters such that $\mathcal{U}_i \in X_G \cap V_\delta$ $\forall i$

then the tower

$(R_G(\mathcal{U}_i) : i < \omega)$

is well-founded (i.e., the direct lim of the V-ultrapowers is well-founded)

iff $\lim_i \operatorname{ult}(N, \mathcal{U}_i^N)$ is well-founded, and $\mathcal{U}_i^N$ is the preimage of $\mathcal{U}_i$ under the coll. map.

let X be the set of δ_0 - complete ultrafilters on V_{δ_0} in V.

~~(1)~~ let $\delta_0 < \kappa < \delta$ be least s.t.

(1) $V_\kappa \prec V_\delta$ (hence a lim of woodin)

(2) κ is measurable

(3) $\exists Y \subset V_{\kappa+2}$ of κ complete ultrafilters on V_κ s.t. Y is tower isom. to X.

in V, fix $\pi : X \longleftrightarrow Y$,

$Y, \pi \in X_G$.

$V_\kappa[G] \models \kappa$ is a lim of woodins.

so in $V[G]$, if $A \subset \mathbb{R}^{V[G]}$ is $(<\kappa)$ - weakly hom. Souslin, then any M in $L(\mathbb{R}^{V[G]}, A)$

is also $(<\kappa)$ - weakly hom. Souslin.

for each $F : V_{\delta_1+2} \longrightarrow V_{\kappa+2}$,

$F \in V[G]$.

let A^G_F = the set of $x \in {}^\omega\omega$

s.t. $(F \circ G(x(i)) : i < \omega)$

is a countably complete tower of

κ - complete ultrafilters.

$A^G_\pi \in X_G[G]$.

choose $F : V_{\delta_0+2} \longrightarrow V_{\kappa+2}$ s.t.

$F'' V_{\delta_0+2} \subset$ κ-complete ultrafilters on V_κ

s.t. A^G_F is wadge above $\left(\left(A^G_\pi\right)^\#\right)^\#$

in $V[G]$.

may choose $F \in X_G$.

$Y_F = \{$ ~~of~~ all κ-complete U in $F'' V_{\delta_0+2} \}$.

$Y_F \in X_G$, $|Y_F|^V \geq |V_{\delta_0+2}|$.

let Z be a κ-complementing set of ultrafilters on V_κ,

$e : Y_F \longrightarrow Z$ witness this

(taking w.fdd. towers to ill-founded ones and vice versa),

$Z, e \in X_G$.

<u>key claim.</u>

$\exists$ tower $(U_i : i<\omega)$ in $V[G]$ from Y_F s.t. both the towers

$$\langle\, j(R_G(U_i)) : i < \omega \,\rangle \quad \text{and}$$

$$\langle\, j(R_G(e(U_i))) : i < \omega \,\rangle$$

are well-founded.