Hugh Woodin : long extender models.

probl. : assume UBH. can one build a weak extender model for δ is supercompact assuming δ is extendible ?

background :

def. N is a weak extender model for δ is supercompact iff f.a. $\lambda > \delta$ $\exists$ normal fine, δ-complete measure $\mathcal{U}$ on $P_\delta(\lambda)$ s.t.

① $N \cap P_\delta(\lambda) \in \mathcal{U}$

② $N \cap \mathcal{U} \in \mathcal{U}$.

$\boxed{\omega\text{- barrier}}$

def. (foreman, magidor)

weak^2 $\square$ holds at κ iff

$\exists\ (C_\alpha : \alpha < \kappa^+)$ s.t.

① C_α is closed cof. in α, $\forall \alpha$ lim

② $\operatorname{otp}(C_\alpha) \leq \kappa$ $\forall \alpha$

if $\operatorname{cf}(\alpha) < \kappa$, then $|C_\alpha| < \kappa$

③ $\forall \alpha\ \forall \zeta < \alpha\ \exists \beta < \alpha$

$$C_\beta \cap \zeta = C_\alpha \cap \zeta$$

__lem.__ (foreman - magidor) TFAE.

① weak^2 $\square$ holds at κ

② $\forall X \prec H_{\kappa^{++}}$, $\overline{\overline{X}} = \kappa$, $\kappa \subset X$,

there is $C \subset X \cap \kappa^+$ of

order type κ if $\operatorname{cf}(X \cap \kappa^+) = \kappa$

and $|C| < \kappa$ if $\operatorname{cf}(X \cap \kappa^+) < \kappa$

$C \cap \zeta \in X$ $\forall \zeta < X \cap \kappa^+$.

<u>lem</u> (F-M) (GCH)

if κ is $\kappa^{+\omega}$-supercompact, then weak^2 $\square$ fails at $\kappa^{+\omega}$

<u>lem.</u> (GCH) assume global choice.

TFAE.

① weak^2 $\square$ holds at all $\kappa > \omega$

② there is $\mathbb{P}$ s.t. $V = J(\mathbb{P})$ and all the levels of $J(\mathbb{P})$ are ω-sound and acceptable.

<u>Theorem.</u> Suppose $V = HOD$, and there is a proper class of extendible cardinals. then there exists a generalized Martin-Steel extender sequence $\mathbb{E}$ s.t.

① $\mathbb{E}$ is Σ_2-definable

② if E is on $\mathbb{E}$, then $lh(E) \leq j_E(\kappa_E) + 1$

③ $V = L[\mathbb{E}]$.

<u>def.</u> supp. $M \models ZFC$ is transitive, $\delta < \lambda < M \cap OR$. supp. $\mathbb{E} \in M$. then δ is witnessed by $\mathbb{E}$ to be λ-supercompact iff

$\exists \alpha \in \mathrm{dom}(\mathbb{E})$ s.t. letting $E = \mathbb{E}_\alpha$,

① E is a M-extender

② $\kappa_E = \delta$ and $\lambda < \mathrm{spt}(E) = i_E$

③ $\cancel{j_E(\lambda)}$

$j_E{}''\lambda \in M_E = \mathrm{ult}(M; E)$

notation:

if E is an extender $j_E : V \to \mathrm{ult}(V; E) = M_E$

① $\kappa_E = \mathrm{crit}(E)$

② $\nu_E = \sup\{\xi + 1 : \xi$ is a generator$\}$

③ $i_E = \sup\{\lambda : \exists$ generator ξ of E s.t. $\sup j_E{}''\lambda < \rho < j_E(\lambda)\}$

(4) $\mathrm{spt}^*(\mathbb{E})$ is the set of all $\lambda \subseteq i_E$ s.t.

$$\mathbb{E} \mid \hat{j}_E(\lambda) \neq E \mid \sup \hat{j}_E{}''\lambda .$$

Consider structures $(M, \mathbb{E})$ s.t.

M is transitive,

$(M; \mathbb{E}) \models ZFC$

$\mathbb{E} \cap M_\alpha \in M_{\cancel{\alpha}}$ $\forall \alpha \in OR$.

$\mathbb{E} \subset OR^M \times M$

- Supp. $\alpha \in \mathrm{dom}(\mathbb{E})$, $\mathbb{E}_\alpha$ is an M-extender. Set $E = \mathbb{E}_\alpha$.

$\cancel{\hat{j}_E(M)}$. $\hat{j}_E : M \longrightarrow \mathrm{ult}(M; E) = M_E$,

~~$i \in \mathrm{spt}(E)$.~~

$i = i_E$, $\gamma = (i^+)^M$.

(1) Supp. $\cancel{\hat{j}_E(i)}$ $\hat{j}_E{}''i \notin M_E$ and let $\delta \leq i$ be least s.t. $\hat{j}_E{}''\delta \notin M_E$, supp. $\delta < i$.

then

(a) $cf(\delta)^M < \kappa_E$

(b) either $i = \delta^{+M}$ or $\mathbb{E}$ does not witness in M that κ_E is j-supercompact

(2) Supp. $i = \lambda^{+M}$, $cf^M(\lambda) \geq \kappa_E$, and $\mathbb{E}$ witness in M that κ_E is λ-supercompact.

then $j_E{}''i \in M_E$ and

(a) $\nu_E < j_E(i)$ and $\nu_E = \zeta + 1$, for ζ.

(b) $E|\eta \in M_E \quad \forall \eta < \zeta$.

(c) if $E|\zeta \notin M_E$, then $cf^{M_E}(\zeta) < j_E(\kappa_E)$.

(3) $$M|\beta = M_E|\beta$$ where
$$\beta = \sup \hat{j}_E{}''\gamma = j_E(\gamma).$$

def. supp. $(M; \mathbb{E}) \models ZFC$ etc.
then $(M; \mathbb{E})$ is finitely generated iff $\exists a \in M$ s.t. every $b \in M$ is def.ble in $(M; \mathbb{E})$ from a ~~ord ordinal~~ parameters.

def. Suppose $M, N \models ZFC$.
$\pi : M \longrightarrow N$ is a elementary embedding with $\pi''M$ cof. in N.
then π is close to M if for each $z \in M$ and each $x \in \pi(z)$
$$\{A \subset z : A \in M, x \in \pi(A)\} \in M.$$

<u>lem.</u> Supp. $(M; \mathbb{E})$, $(N, \mathbb{F})$ are finitely generated. Supp.

$$\pi_0 : (M; \mathbb{E}) \longrightarrow (N, \mathbb{F})$$
$$\pi_1 : (M, \mathbb{E}) \longrightarrow (N, \mathbb{F})$$

close to M.

pf.: let ξ be least s.t.

$$(M; \mathbb{E}) = \mathrm{Hull}_\omega^{(M; \mathbb{E})}(\{\xi\}).$$

assume $\underset{\overset{\shortparallel}{\xi_0}}{\pi_0(\xi)} > \underset{\overset{\shortparallel}{\xi_1}}{\pi_1(\xi)}$

$\xi_0 > \xi_1$

$$\begin{tikzcd}
 & & (N; \mathbb{F}) \ni \xi_1 \\
(M; \mathbb{E}) \arrow[urr, "\pi_0"] \arrow[rr, "\pi_{U_0}"] & & (\bar{N}; \bar{\mathbb{F}}) \arrow[u, "c"]
\end{tikzcd}$$

$\hat{\xi}_0 > \hat{\xi}_1$

$\mathrm{ult}(M; U_0)$

U_0 generated by ξ_1

$$\mathrm{Hull}_\omega^{(N;\mathbb{F})}(\{\xi_1\}) \cong (M;\mathbb{E})$$

so $$\mathrm{Hull}_\omega^{(\bar N;\bar F)}(\{\bar\xi_1\}) \cong (M;E)$$

<u>defn.</u> $(M;\mathbb{E}) \models \mathrm{ZFC}$

a semi-iteration of $(M;\mathbb{E})$ is continuous (linearly) directed system

$$((N_\alpha, \mathbb{F}_\alpha), \pi_{\alpha\beta}, E_\alpha : \alpha \le \beta \le \eta)$$

s.t. the following hold where

E_α is the N_α-ext, giving $\pi_{\alpha,\alpha+1}$

① $(N_0, \mathbb{F}_0) = (M, \mathbb{E})$

② no $\delta < \kappa_{E_\alpha}$ is witnessed to be $(<\kappa_\alpha)$-supercompact in N_α by $\mathbb{F}_\alpha$

$\pi_{\alpha\alpha+1}(\mathrm{crit}(E_\alpha))$
$\quad \|$
$\cdot\lambda_{E_\alpha}$
$\quad \|$

(3) $\kappa^*_{(E_\alpha)} \leq \kappa_{E_\beta} \quad \forall \alpha < \beta$

(4) supp $\pi_{\alpha,\alpha+1}{}'' i_{E_\alpha} \notin N_{\alpha+1}$

and let δ be least st.

$\pi_{\alpha,\alpha+1}{}''\delta \notin N_{\alpha+1}$, supp; $\delta < i(E_\alpha)$ then

a) $\mathrm{cf}(\delta)^{N_\alpha} < \kappa_{E_\alpha}$, and

b) either $i_{E_\alpha} = \delta^{+N_\alpha}$ or $\mathbb{F}_\alpha$ does not when in N_α then

κ_{E_α} is δ^{+N_α} – supercompact.

(5) if $i_{E_\alpha} = \lambda^{+N_\alpha}$,

$\mathrm{cf}(\lambda)^{N_\alpha} \geq \kappa_{E_\alpha}$, the blablabla

(6) for all $a \in [\mathrm{lh}(E_\alpha)]^{<\omega}$,

$(E_\alpha)_a \in N_\alpha$ (closedness)

<u>def.</u> supp. $(M; \mathbb{E}) \models ZFC$ etc.

then $(M; \mathbb{E})$ satisfies <u>comparison</u>

if $X, Y \prec (M; \mathbb{E})$, $X \neq Y$

$X \cap \mathbb{R} = Y \cap \mathbb{R}$,

both finitely generated.

let $(M_X, \mathbb{E}_X)$ be the tr. coll. of X

$(M_Y, \mathbb{E}_Y)$ — " — Y.

then there ex. semi-iterations

$$((N^X_\alpha, \mathbb{F}^X_\alpha), \pi^X_{\alpha,\beta}, E^X_\alpha : \alpha < \beta \leq \eta_X)$$

$$((N^Y_\alpha, \mathbb{F}^Y_\alpha), \pi^Y_{\alpha,\beta}, E^Y_\alpha : \alpha < \beta \leq \eta_Y)$$

of $(M_X, \mathbb{E}_X)$, $(M_Y, \mathbb{E}_Y)$ respectively, s.t.

① $(N^X_{\eta_X}, \mathbb{F}_{\eta_X}) = (N^Y_{\eta_Y}, \mathbb{F}_{\eta_Y})$

② $E^X_0 \neq E^Y_0$

③ supp. $(\mathrm{spt}(E_0^X) \cup \mathrm{spt}(E_0^Y)) \subset \lambda$

$$P(\lambda) \cap M_X = P(\lambda) \cap M_Y .$$

then $\pi^X_{0\eta_X} \restriction P(\lambda) \neq \pi^Y_{0\eta_Y} \restriction P(\lambda)$.

__def.__ $(M_0, \mathbb{E}_0)$, $(M_1, \mathbb{E}_1) \models$ ZFC etc.

let κ be a reg. cardinal in each,

$\kappa^{+M_0} = \kappa^{+M_1}$

$(M_0, E_0) \restriction \kappa^{+M_0} = (M_1, E_1) \restriction \kappa^{+M_1}$.

__coherent pair__ at κ

__def.__ suppose $(M_0, \mathbb{E}_0)$, $(M_1, \mathbb{E}_1)$

is a coher. pair at κ.

a semi-iteration is a continuous

(lin.) directed system

$$((N_\alpha, \mathbb{F}_\alpha), \pi_{\alpha,\beta}, E_\alpha ; \alpha < \beta \leq \eta)$$

s.t.

① $(N_0, \mathbb{F}_0) \in \{(M_0, \mathbb{E}_0), (M_1, \mathbb{E}_1)\}$ and (—) is a semi-iterate of $(N_0, \mathbb{F}_0)$.

② if $N_0 = M_1$, then $\kappa < i$ for some $i \in \operatorname{spt}^*(E_0)$.

defn. supp. $(M; \mathbb{E}) \models ZFC$ etc., and

κ is a measurable cardinal in V,

U is a normal measure on κ,

$U \cap M \in M$. Let

$$(M_U, E_U) = \operatorname{ult}_0((M; \mathbb{E}), U \cap M)$$

Supp. $(M, \mathbb{E}), (M_U, \mathbb{E}_U)$ is a coherent pair at κ.

then $(M_U, \mathbb{E}_U)$ satisfies comparison backed up by $(M; \mathbb{E})$ if the following holds.

supp. $X \prec (M; \mathbb{E})$ is finitely generated and $u \cap M \in X$.

let (M_X, E_X) be the collapse of X.

let α_X = image of α under collapse.

$(M_u^X; \mathbb{E}_u^X)$ = the image of $(M_u \cap X, \mathbb{E}_u \cap X)$ under the collapse.

then the ex. seri iterats

$$((N_\alpha^0, \mathbb{F}_\alpha^0)\ \pi^0_{\alpha\beta},\ E^0_\alpha,\ \alpha < \beta < \eta_0)$$

of $(M_X, \mathbb{E}_X)$

$$(((N_\alpha^1, \mathbb{F}_\alpha^1),\ \pi^1_{\alpha,\beta},\ E^1_\alpha,\ \alpha < \beta < \eta_1)$$

s.t.

(1) $(N^0_{\eta_0}, \mathbb{F}^0_{\eta_0}) = (N^1_{\eta_1}, \mathbb{F}^1_{\eta_1})$

② $E^0_0 \neq E^1_0$

③ supp. $\operatorname{spt}(E^0_0) \cup \operatorname{spt}(E^1_0) \subset \lambda$

$$P(\lambda) \cap N^0_0 = P(\lambda) \cap N^1_0,$$

then $\pi^0_{0,\eta_0} \mid P(\lambda) \neq \pi^1_{0,\eta_1} \mid P(\lambda)$.

<u>Th.</u> Supp. δ is supercompact,

$\Omega > \delta$, Ω is strongly inaccessible

then there is <u>no</u> weakly backgrounded

$(M; \mathbb{E}) \models ZFC$ s.t.

① $\Omega = M \cap OR$, δ is witnessed by $\mathbb{E}$ to be supercompact in M

② $\exists$ measurable cardinal $\delta < \kappa < \Omega$

and a normal measure U on κ

such that the following hold, where

$$(M_U, \mathbb{E}_U) = \operatorname{ult}_0(M; U \cap M)$$

a) $((M, \mathbb{E}), (M_u, \mathbb{E}_u))$

is a coherent pair at κ

b) $U \cap M \in M$ (~~not~~ must hold anyway by backgroundedness)

c) $(M_u, \mathbb{E}_u)$ satisfies compatibility backed up by $(M, \mathbb{E})$ at κ.