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problem: prove square holds in hod mice,

i.e., supp. $P$ is a hod mouse.

prove that f.a. cardinals $\kappa$ of $P$,

$P \models$ "$\Box_\kappa \longleftrightarrow \kappa$ is not subcompact"

intended applications:

some determinacy theories.

- $AD^+$ / $AD^+ + \Theta > \theta_0$
- $AD_{\mathbb{R}}$
- $AD_{\mathbb{R}} + DC$
- $AD_{\mathbb{R}} + \Theta$ reg.
- $AD_{\mathbb{R}} + \Theta$ is measurable
- $AD_{\mathbb{R}} + \Theta$ is reg
- $AD^+ + \Theta = \theta_{\alpha+1} + \theta_\alpha$ is the largest Suslin cardinal $\Big\}$ LSA

**thm.** (Trang) supp. $\kappa$ is a cardinal s.t. $\kappa^{\aleph_0} = \kappa$. supp. $\forall \alpha \in [\kappa^+, (2^\kappa)^+]$, $\neg \square(\alpha)$. then there is a model $M$ s.t. $\mathbb{R} \cup OR \subset M$,

$M \models AD_{\mathbb{R}} + \Theta$ reg. (in fact, [illegible] or even reg)

**Cor.** assume PFA

(or $\omega$-closed $\omega_2$-generic models or $\exists$ strongly compact cardinal)

then there is an $M$ as above.

**Sample problem:** prove $\mathrm{con}(PFA) \to \mathrm{con}(LSA)$.

**lsa-small hod premouse**

$P$

$\delta_{\alpha+1}$

$\delta_\alpha$ — lim of woodins, reg up to $\delta_{\alpha+1}$

$\vdots$

$\delta_1$

$\delta_0$

outline of M. of $\square_\kappa$ in $L[E]$.

- $P$ is $L[E]$-model, i.h.
- $\kappa$ is a cardinal of $P$, not subcompact
- $S = \{ \tau < \kappa^+ : P|\tau \prec P|\kappa^+,$ $P|\tau$ is passive $\}$

def. a coll. structure for $\tau \in S$ is a sound pm $Q \in P$ s.t.

$Q|\tau = P|\tau$,

$Q \cap OR \geq \tau$,

$Q \models \tau = \kappa^+$, $\rho_\omega(Q) = \kappa$.

def. fix a coll. structure $Q$ for $\tau$.

say $n(Q) =$ the $n$ s.t.

$\rho_{n+1}(Q) = \kappa < \rho_n(Q)$.

Say $(\mu, q)$ is a <u>divisor of</u> $Q$

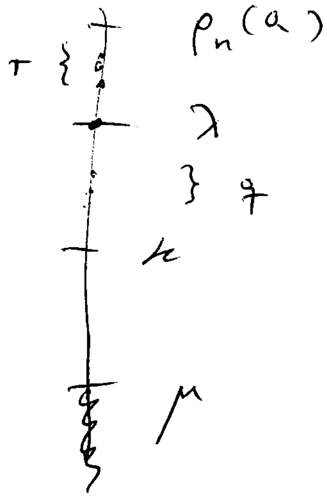
if $\exists \lambda$ s.t.

- $\mu \le \kappa < \lambda < \rho_n(Q)$
- $q = P_Q \cap \lambda$
- $\tilde{h}^{n+1}_Q(\mu \cup \{r\}) \cap \rho_n(Q)$ is

$$\|$$

$$P_Q - q$$

cofinal in $\rho_n(Q)$

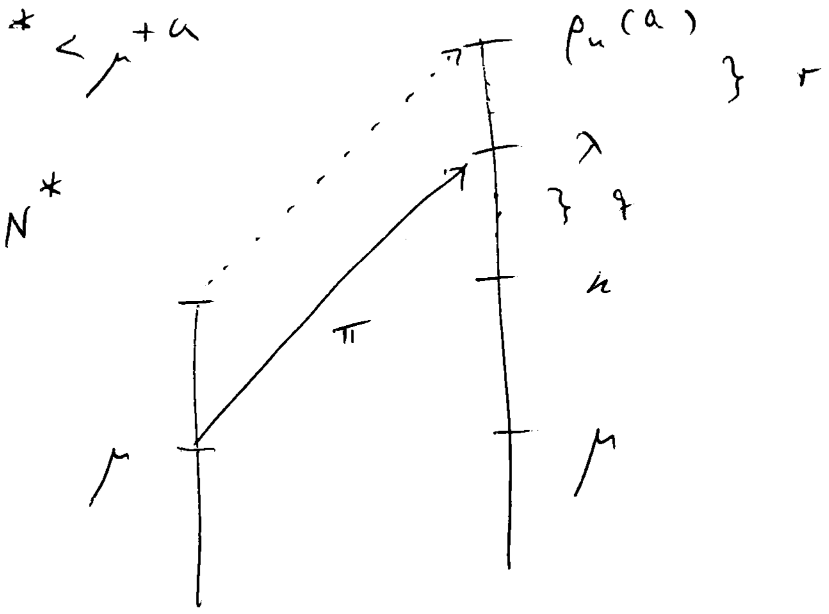



- $\lambda = \min(\mathrm{OR} \cap \tilde{h}^{n+1}_Q(\mu \cup \{r\}) - \mu)$

let $N^* =$ tr. coll. of $\tilde{h}^{n+1}_Q(\mu \cup \{r\})$

$N^* \triangleleft Q$

let $\vartheta = \mu^{+N^*} < \mu^{+Q}$



let $F = E_\pi \restriction \mathcal{P}(\mu) \cap N^*$.

then $(Q \mid \sup \pi''\vartheta, F)$ is a protomouse, call it $Q(\mu, q)$

<u>step 1.</u> choose the coll. structure $N_\tau$ for $\tau \in S$.

$N_\tau \trianglelefteq Q^P$ is the level of $Q^P$ s.t.

$\rho_\omega(N_\tau) = \kappa$ + $N_\tau \models \tau = \kappa^+$.

<u>step 2.</u> for each $\tau \in S$ choose a "canonical divisor" $(\mu_\tau, q_\tau)$ for $N_\tau$ if it ex.

$S_0 = \{\tau \in S : N_\tau$ doesn't have a can. divisor $(\mu_\tau, q_\tau)\}$

$S_1 = S \setminus S_0$.

$\forall \tau \in S_0$, let $B_\tau = \{ \bar{\tau} < \tau : \bar{\tau} \in S_0 \wedge$

$\exists \sigma_{\bar{\tau},\tau} : N_{\bar{\tau}} \longrightarrow N_\tau \quad \Sigma_0^{(n(N_{\bar{\tau}}))}$-pres.

$n(N_{\bar{\tau}}) = n(N_\tau)$.

$\sigma_{\bar{\tau},\tau}(\bar{\tau}) = \tau$, $\sigma_{\bar{\tau},\tau}(p_{N_{\bar{\tau}}}) = p_{N_\tau}$.

$\forall \alpha \in p_{W_\tau}$ $\exists$ gen. solidity witness for $(N_\tau, p_{W_\tau})$ w.r.t. $\alpha$ in the range of $\sigma_{\bar{\tau},\tau}$ $\}$.

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now let $\tau \in S_1$.

$B_\tau = \{ \bar{\tau} < \tau :$

$\exists \sigma_{\bar{\tau},\tau} :$ $\cancel{N_{\bar{\tau}} \to N_{\bar{\tau}}}$

$$N_{\bar{\tau}}(\mu_{\bar{\tau}}, q_{\bar{\tau}}) \longrightarrow N_\tau(\mu_\tau, q_\tau)$$

$\Sigma_0$ preserving,

$$(\mu_{\bar\tau}, |q_{\bar\tau}|) = (\mu_\tau, |q_\tau|)$$

$\mathrm{crit}(\sigma_{\bar\tau,\tau}) = \bar\tau$, $\sigma_{\bar\tau,\tau}(\bar\tau) = \tau$,

$\sigma_{\bar\tau,\tau}(q_{\bar\tau}) = q_\tau$ $\wedge$ Solidity when in the range }

$q_{\bar\tau}$: st. para of protomouse

Can prove:

- $\forall \tau \in S \; \forall \bar\tau \in B_\tau$

$$B_\tau \cap \bar\tau = B_{\bar\tau} \setminus \min(B_\tau)$$

- $\forall \tau \in S$ of unctbl. cofinality $B_\tau$ is closed + unbounded on a tail, i.e., $\exists \bar\tau < \tau$ s.t. $B_\tau - \bar\tau$ is club in $\tau$.

want to see:

$\tau \in S$, $cf(\tau) > \omega$

$\Longrightarrow$ $\mathcal{B}_\tau$ is unbounded

1) $\tau \in S_0$: fix $\tau' < \tau$.

let $\sigma : \bar{N} \longrightarrow N_\tau$ be fully elementary, $\bar{N}$ cttbl, $\operatorname{ran}(\sigma) \ni \tau'$.

let $\tilde{\tau} = \sup \sigma''\bar{\tau}$,

where $\bar{\tau} = \sigma^{-1}(\tau)$.

then $\tau' < \tilde{\tau} < \tau$.

let $\tilde{N} = \operatorname{ult}(\bar{N}, F)$, where

$F = (\operatorname{crit}(\sigma), \tilde{\tau})$ extender derived from $\sigma$.

$$\begin{tikzcd} & N_\tau \\ \bar{N} \arrow[ur, "\sigma"] \arrow[r, "\tilde{\sigma}"'] & \tilde{N} \arrow[u] \end{tikzcd}$$

can prove $\tilde{N} = N_{\tilde{\tau}}$,

$\tilde{\tau} \in \mathcal{B}_\tau$.

2) $\tau \in S_1$.

fix $\tau' < \tau$.

let $\sigma : \bar{N} \longrightarrow N_\tau (\mu_\tau, \rho_\tau)$

could be $= \ \ldots$

$N_\tau$

let $\tilde{\tau} =$ as before.

$\hat{\tilde{N}} = \mathrm{ult}(\bar{N}, F)$ as before.

$\tilde{N}$ is not a premouse, just a protomouse, in general.

$\hat{\tilde{N}} = (\hat{\tilde{N}}^-, \tilde{F})$ is a protomouse.

$\tilde{F}$ doesn't measure all of $\mathrm{crit}(\tilde{F}) \cap \hat{\tilde{N}}^-$.

let $R \trianglelefteq \hat{\tilde{N}}^-$ be longest

s.t. $\rho_\omega(R) = \mu$, $P(\mu) \cap R$

is measured by $\tilde{F}$.

$$\begin{tikzcd}
R \arrow[r, "\pi_0"] \arrow[d, phantom, "\Delta"] \arrow[dr, "\pi_1(k)"'] & \tilde{R} = \mathrm{ut}(K, \tilde{F}) \arrow[d, "\pi_2"] \\
N_\tau \arrow[r, "\pi_1"'] & \mathrm{ut}(N_\tau, E_\tau)
\end{tikzcd}$$

$$\pi_2(\pi_0(f)(a)) = \pi_1(f)(\sigma'(a)).$$

can prove $\tilde{R} = N_{\tilde{\tau}}$.

$\longrightarrow$ $\tilde{N} = N_{\tilde{\tau}}(\mu_{\tilde{\tau}}, q_{\tilde{\tau}})$.

$\longrightarrow$ $\tilde{\tau} \in B_\tau$.