John Steel 2.

given $\mathcal{I}$ on M, fine str., say,

F on the M^{S}_{α}-seq. (S also on M)

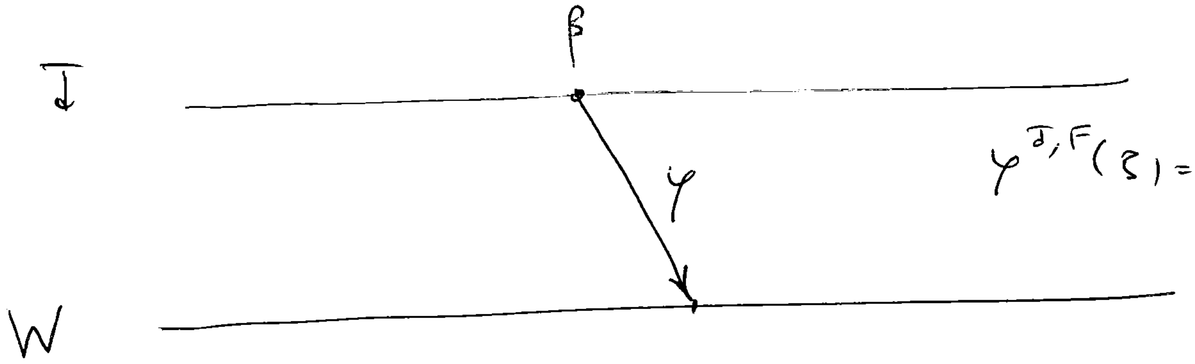
(α = least s.t. this is true)

fix β least s.t. $\mu = \mathrm{crit}(F) < \lambda_{E^S_\beta}$

$\mathcal{I} \restriction \beta+1 = S \restriction \beta+1$

then $W(\mathcal{I}, F) =$

$$S \restriction \alpha+1 {}^\frown \langle F \rangle {}^\frown i_F \text{''} \mathcal{I}^{>\mathrm{crit}(F)}$$



$$\varphi^{\mathcal{I},F}(\xi) = \begin{cases} \xi & \xi < \beta \\ (\alpha+1)+(\xi-\beta) & \xi \geq \beta \end{cases}$$

in the "coarse case",

$$M_{\varphi(\xi)} = \mathrm{ult}(M_\xi, F)$$

then $\pi_\tau : M^{\mathcal{T}}_\tau \longrightarrow M^{W}_{\varphi(\tau)}$

if F doesn't move all lhs in $M^S_\beta \restriction lh\ E^S_\beta$, then $dom(\varphi) = \beta+1$

define then $W(\mathcal{T}, \mathcal{U})$ for $\mathcal{T}$ on M, $\mathcal{U}$ a last model of $\mathcal{T}$.

define $W_\gamma = W(\mathcal{T}, \mathcal{U} \restriction \gamma+1)$ by ind on γ.

then $\sigma_\gamma : M^{\mathcal{U}}_\gamma \longrightarrow R_\gamma =$ last model of W_γ.

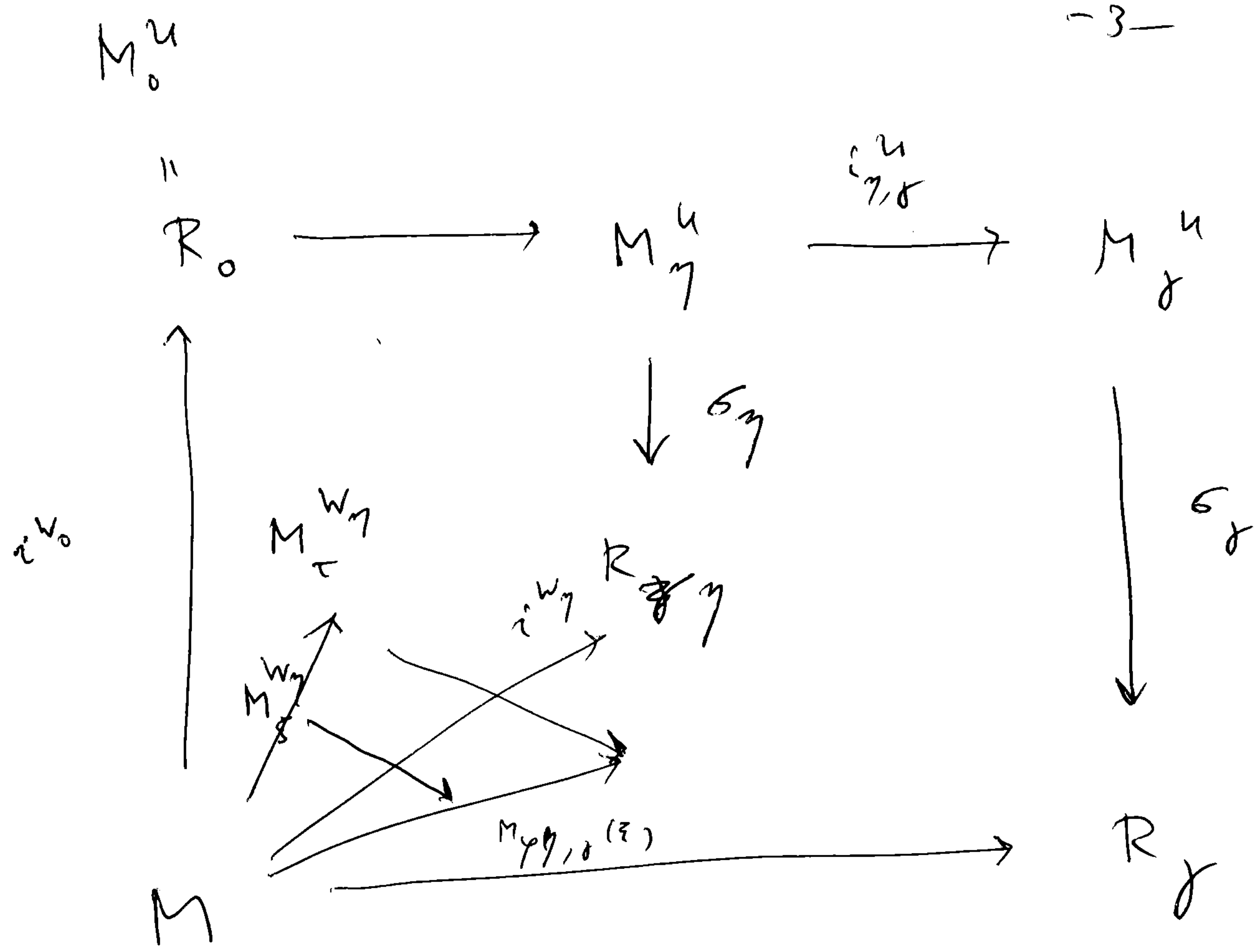
if $\nu = \mathcal{U}\text{-pred}(\gamma+1)$

$W_{\gamma+1} = W(W_\nu, F_\gamma)$

where $F_\gamma = \sigma_\gamma(E^{\mathcal{U}}_\gamma)$

then $\varphi_{\nu,\gamma+1} : lh(W_\nu) \longrightarrow lh(W_{\gamma+1})$

$\pi^{\nu,\gamma+1}_\tau : M^{W_\nu}_\tau \longrightarrow M^{W_{\gamma+1}}_{\varphi_{\nu,\gamma+1}(\tau)}$.



$$W(\mathcal{I}, u) = \bigcup_{\xi<\lambda} W_\xi \restriction \alpha_\xi + 1$$

if $lh(u) = \lambda$ lim

for $\lambda + 1 = lh(u)$.

then $\delta = lh\ W(\mathcal{I}, u \restriction \lambda) = \sup_{\xi<\lambda} \alpha_\xi$

$= \sup\{crit(\varphi_{\eta,\gamma}) : \eta < \gamma \in [0,\lambda]_u\}$

let b = a cof. branch of $\mathcal{U}$

<u>case 1.</u> b does not drop.

let $\tau = \tau_0$ = least $\alpha < lh(\mathcal{T})$

s.t. $\forall \gamma <_{\mathcal{U}} \xi$ w/ $\xi \in b$,

$$\varphi_{0\gamma}(\alpha) \geq \operatorname{crit}(\varphi_{0,\xi}) .$$

set $\varphi_{0,b}(\tau) = \delta$.

$$lh(W_b) = \delta + (lh(\mathcal{T}) - \tau)$$

$$\varphi_{0,b} = \delta + (\xi - \tau) \quad \text{for } \xi \geq \tau$$

here $\pi_\xi^{\eta,b} : M_\xi^{W_\eta} \longrightarrow M_{\varphi_{0,b}(\xi)}^{W_b}$

this also depends what ~~M^{W_b}~~ $M^{W_b}_{\varphi_{0,b}(\xi)}$ is.

$$E^{W_b}_{\varphi_{0,b}(\xi)} = \pi_\xi^{\eta,b}(E_\xi^{W_\eta}) .$$

Σ 2-normalizes well iff

whenever

$\langle \mathcal{I}, \mathcal{U} \rangle$ is by Σ, then

$W(\mathcal{I}, \mathcal{U})$ is by Σ.

for any it. tree W,

$$\mathrm{Ext}(W) = \{ E^W_\alpha : \alpha+1 < lh(W) \}.$$

for $\alpha < lh(W)$, s^W_α = incr. enumeration of the E^W_β used in $[0, \alpha)_W$.

$$W^{ext} = \{ s^W_\alpha : \alpha < lh(W) \}$$

for $\nu < \gamma$ in $\mathcal{U}$, we have

$$\psi^{\nu,\gamma}(E^{W_\nu}_\tau) = E^{W_\gamma}_{\varphi_{\nu,\gamma}(\tau)}.$$

then can define

$$\hat{\psi} : W_\nu^{ext} \longrightarrow W_\gamma^{ext}$$

$\hat{\psi}(s)$ = downward closure of
$\{\psi(s(i)) : i \in \mathrm{dom}(s)\}$.

$\hat{\psi}$ preserves $\subseteq$, $\perp$.

<u>strong hull condensation.</u>

notation: for any it. tree $\mathcal{U}$, $\alpha <_{\mathcal{U}} \beta$

$\hat{\imath}^{\mathcal{U}}_{\alpha\beta} : M^{\mathcal{U}}_{\alpha} \longrightarrow M^{\mathcal{U}}_{\beta}$ is the
embedding given by $(\alpha, \beta]_{\mathcal{U}}$
acting on the longest initial
segment of $M^{\mathcal{U}}_{\alpha}$ possible.

<u>def.</u> Let $\mathcal{T}$, $\mathcal{U}$ be normal iteration
trees on M. A pseudo hull
emb. for $\mathcal{T}$ into $\mathcal{U}$ is a system
$(u, \langle t^0_\beta : \beta < \mathrm{lh}(\mathcal{T})\rangle, \langle t^1_\beta : \beta + 1
< \mathrm{lh}(\mathcal{T})\rangle, p)$

s.t.

(a) $u : \{\alpha : \alpha+1 < lh(\mathcal{T})\} \longrightarrow \{\alpha : \alpha+1 < lh(\mathcal{U})\}$

$\alpha < \beta \to u(\alpha) < u(\beta)$
λ lim iff $u(\lambda)$ lim

(b) $p : Ext(\mathcal{T}) \longrightarrow Ext(\mathcal{U})$
so that if E is used before F on a branch of $\mathcal{T}$,
then $p(E)$ is used before $p(F)$ on a branch of $\mathcal{U}$

(so have $\hat{p} : \mathcal{T}^{ext} \longrightarrow \cancel{\mathcal{E}}\ \mathcal{U}^{ext}$)

(c) let $v : lh(\mathcal{T}) \to lh(\mathcal{U})$ be given by $s^{\mathcal{U}}_{v(\beta)} = \hat{p}(s^{\mathcal{T}}_{\beta})$
($\neq s^{\mathcal{U}}_{u(\beta)}$ in general)

then

(d) (1)

also if $\alpha+1 = \mathrm{lh}(\mathcal{I})$, then

$\mathcal{U}$ has a last model $M_\xi^{\mathcal{U}}$,

and $\nu(\alpha) \leq_{\mathcal{U}} \xi$.

(d) (2)

if $\alpha+1 < \mathrm{lh}(\mathcal{J})$, then

$\nu(\alpha) \leq_{\mathcal{U}} u(\alpha)$ and

$$t'_\alpha = \hat{\imath}^{\mathcal{U}}_{\nu(\alpha), u(\alpha)} \circ t^0_\alpha$$

and

$$p(E^{\mathcal{J}}_\alpha) = t'_\alpha(E^{\mathcal{J}}_\alpha) = E^{\mathcal{U}}_{u(\alpha)}$$

(e) if $\beta = T\text{-pred}(\alpha+1)$, then

$\mathcal{U}\text{-pred}(u(\alpha)+1) \in [\nu(\beta), u(\beta)]_{\mathcal{U}}$

and setting $\beta^* = \mathcal{U}\text{-pred}(u(\alpha)+1)$,

$$t^0_{\alpha+1}\left([a,f]^{P}_{E^{\mathcal{J}}_\alpha}\right) = \left[t^1_\alpha(a),\ \hat{\imath}^{\,\mathcal{U}}_{\nu(\beta)},\ \beta^*\right]^{P^*}_{E^{\mathcal{U}}_{u(\alpha)}}$$

as $P \trianglelefteq M^{\mathcal{J}}_\beta$ [illegible] then $E^{\mathcal{J}}_\alpha$ gets appl. to

$P^* \trianglelefteq M^{\mathcal{U}}_{\beta^*}$ and by then $E^{\mathcal{U}}_{u(\alpha)}$ gets appl. to.

diagram:

$$\begin{tikzcd}
M^{\mathcal{J}}_\beta \arrow[rr, "E^{\mathcal{J}}_\alpha"] \arrow[dd, "t^0_\beta"'] \arrow[ddr, "\rho"] & & M^{\mathcal{J}}_{\alpha+1} \arrow[d, "t^0_{\alpha+1}"] \\
 & & M^{\mathcal{U}}_{u(\alpha)+1} = M^{\mathcal{U}}_{u(\alpha+1)} \\
M^{\mathcal{U}}_{\nu(\beta)} \arrow[r] & M^{\mathcal{U}}_{\beta^*} \arrow[ur, "E^{\mathcal{U}}_{u(\alpha)}"] \arrow[r] & M^{\mathcal{U}}_{u(\beta)}
\end{tikzcd}$$

def. Σ has strong hull condensation iff whenever $\mathcal{U}$ is by Σ and $\mathcal{T}$ is a pseudo hull of $\mathcal{U}$, then $\mathcal{T}$ is by Σ.

L[E] — constructions.

Let W be a w.o. of V_δ and $\kappa < \delta$. A W-construction above κ is a sequence

$$\mathbb{C} = \langle M_{\nu,k}, F_\nu : \langle \nu, k\rangle <_{lex} \langle \nu_0, k_0\rangle \rangle,$$

where

$$M_{0,0} = (V_\omega; \in, -)$$

$$\cancel{M_{0}}\quad M_{\nu,k+1} = \operatorname{core}(M_{\nu,k}) = \mathbb{C}_{k+1}(M_{\nu,k})$$

$$M_{\nu,\omega} = \text{eventual value of } M_{\nu,k}, \text{ as } k \to \omega$$

$M_{\nu+1,0}$ = mod clos of

$M_{\nu,\omega} \cup \{M_{\nu,\omega}\}$, ~~if~~

arranged as a passive pm

if ν is a lim,

$M^{<\nu}_{\cancel{\nu,0}}$ = unique passive P s.t.

$\forall$ pm N

$N \lhd P$ iff

$N \lhd M_{\alpha,0}$ $\forall$ s.l. large $\alpha < \nu$

$M_{\nu,0}$:

can 1. there is an F s.t. $(M^{<\nu}, F)$

is a λ-pm, and F is

certifiable.

(as in "epimorphisms with subcompacts" neeman - steel)

F unique by bicephalus arg

$M_{\nu,0} = (M^{<\nu}, F)$

Case 2. o.w. $M_{\nu,0} = M^{<\nu}$.

a certificate for F is a short extender F^* with $\kappa_F = \mathrm{crit}(F)$,

$\lambda_F = i_F(\kappa_F)$.

F^* must have strength = length some inacc. cardinal, call it η, $\eta > \lambda_F$.

$F^* \restriction \lambda_F = F \restriction \lambda_F$

also: (i) $\forall \tau < \nu:\ lh(F^{\mathbb{C}}_\tau) < \eta$

(ii) $i_{F^*}(W) \cap V_\eta = W \cap V_\eta$

(iii) $F^* \in V_\delta$, $\mathrm{crit}(F^*) > \kappa$.

we then choose $F^{\mathbb{C}}_\nu$ to be the unique certificate F^* for F s.t.

(*) F^* is a certificate for F, minimal in Mitchell order among all certificates and W-least among those.

<u>rmk.</u> So $i_{F^*}(\mathbb{C}) \restriction \nu+1 =$

$\mathbb{C} \restriction \nu \frown$ "passive."

associated to $\mathbb{C}$ we have resurrection maps $R_{\nu,k}$, $\sigma_{\nu,k}$ for $(\nu,k) <_{lex}$ length$(\mathbb{C})$. for each $\mathcal{N} \trianglelefteq M_{\nu,k}$,

$$R_{\nu,k}[\mathcal{N}] = \langle \eta, \underset{\sigma}{l} \rangle \leq_{lex} \langle \nu, k \rangle$$

$$\sigma_{\nu,k}[\mathcal{N}] : \mathcal{N} \longrightarrow M_{\langle \eta, l \rangle}.$$

e.g. $\sigma_{\nu,k+1}[\mathcal{M}_{\nu,k+1}]$:

$$\mathrm{core}(M_{\nu,k}) \longrightarrow M_{\nu,k}$$

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define for $\mathcal{I}$ on $M_{\nu,k}$ a <u>conversion system</u>

lift$(\mathcal{I}, M_{\nu,k}, \mathbb{C})$.

this system consists of

(i) an iteration tree $\mathcal{I}^*$ on V,

(ii) indices $\langle \eta_\xi, l_\xi \rangle$ for $\xi < lh(\mathcal{I})$

(iii) π_ξ for $\xi < lh(\mathcal{I})$ s.t.

using P_ξ, $i_{\xi,\nu}$, F_ξ, P^*_ξ, $i^*_{\xi,\nu}$, F^*_ξ for the models, embeddings, extensions of $\mathcal{I}, \mathcal{I}^*$.

0. $\mathcal{I}, \mathcal{I}^*$ have the same tree order

1. $\pi_\xi : P_\xi \longrightarrow M_{\eta_\xi, l_\xi}$ P^*_ξ $\leftarrow$ in $i^{\mathcal{I}^*}_{0\xi}(\mathbb{C})$

2. if $\xi <_T \nu$, and $(\xi, \nu]_T$ doesn't drop in model or degree, then

$\langle \eta_\nu, l_\nu \rangle = i^*_{\xi,\nu}(\langle \eta_\xi, l_\xi \rangle)$ and

$\pi_\nu \circ i_{\xi\nu} = i^*_{\xi,\nu} \circ \pi_\xi$.

3. if $\xi = T\text{-pred}(\nu+1)$ and this is ~~not~~ a drop to $\bar{P} \triangleleft P_\xi$,

then $\langle \eta_{\nu+1}, \ell_{\nu+1} \rangle = i^*_{\xi,\nu+1}\left(\mathrm{Res}^{P^*_\xi}_{\eta_\xi,\ell_\xi}[\pi_\xi(\bar{p})]\right)$

4. let $\lambda_\xi = \lambda_{F_\xi}$, and $\alpha_\xi = lh(F_\xi)$ = the index of F_ξ in P_ξ.

$$\sigma_\xi = \sigma^{P^*_\xi}_{\eta_\xi,\ell_\xi}[\pi_\xi(P_\xi \| \langle \alpha_\xi, 0 \rangle)].$$

for $\xi < \nu$,

$$\pi_\nu \restriction \lambda_\xi = \sigma_\xi \circ \pi_\xi \restriction \lambda_\xi, \text{ and}$$

$$P^*_\xi \restriction \sup \sigma_\xi \circ \pi_\xi{}'' \lambda_\xi =$$

$$P^*_\nu \restriction \sigma_\xi \circ \pi_\xi{}'' \lambda_\xi.$$

lift $(\mathcal{T}, M_{\nu,h}, \mathbb{C})$ is determined by this and

(a) let $\xi = T\text{-pred}(\nu+1)$, and $\alpha_\nu = lh(F_\nu)$

(so F_ν = the last ext. of

$P_\nu \mid \langle \alpha_\nu, 0 \rangle)$.

let G = the least extender of

$$\mathrm{Res}^{P_\nu^*}_{\eta_\nu, \ell_\nu} [\pi_\nu(P_\nu \| \langle \alpha_\nu, 0 \rangle)]\ .$$

let G^* = the background for G in $i^*_{0,\nu}(\mathbb{C})$.

then $G^* = F_\nu^*$.

+ a couple of more conditions ☺

def. given a partial iteration strategy Σ^* for V, we define Σ for $M^{\mathbb{C}}_{\nu,k}$ by $\mathcal{T}$ is by Σ iff

$\mathrm{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C})$ is by Σ^* (i.e., $\mathcal{T}$ is)

def. Ω^{UBH} is the partial strategy for V defined on normal trees on V by $\Omega^{UBH}(\mathcal{T})$ = the unique

cofinal branch b of $\mathcal{T}$ s.t.
$M_b^{\mathcal{T}}$ is w. fdd,

for $\mathcal{T}$ played by Ω^{UBH}.

rmk. UBH above κ supercompact

$\Longrightarrow$ Ω^{UBH} is total on all "nice" iterati trees

a theorem of woodin!

all extender have lengt = strength a inacc. in the model when they are taken from.

rmk. what's Ω^{UBH} for stacks ?

given $(\mathcal{T}, \mathcal{U})$, $\Omega^{UBH}(\langle\mathcal{T}, \mathcal{U}\rangle) =$ the unique b s.t. $W(\mathcal{T}, \mathcal{U}^\frown b)$ has well-founded models.

given $\mathbb{C}$, let $\Omega_{\nu,k}$ = the partial strategy for $M_{\nu,k}$ induced by Ω^{UBH}.

thm. assume Ω^{UBH} is total on trees above some κ (e.g. a supercompact). then $\Omega^{\mathbb{C}}_{\nu,k}$ is total (for $\mathbb{C}$ a W-construction above κ).

moreover, $\Omega^{\mathbb{C}}_{\nu,k}$ normalizes well and has strong hull condensation, and is universally baire.

thm: let (P, Σ) be s.t. Σ normalizes well, has strong hull condensation, and is UB, w/ $P \in HC$.

let $\mathbb{C}$ be a W-construction above κ.

then for any $(\nu,k) <_{lex} \text{length}(\mathbb{C})$, either (a) there is a Σ-iterate (via normal $\mathcal{T}$) $Q \trianglerighteq P$

s.t. $M_{\nu,k} \trianglelefteq Q$ and

$$\Omega^{\mathbb{C}}_{\nu,k} = \Sigma_{\mathcal{I}, M_{\nu,k}},$$

or

(b) some $M_{\eta,j}$, $\langle \eta, j\rangle \leq_{lex} \langle \nu, k\rangle$ satisfies (a) w/ $Q = M_{\eta,j}$.

<u>Cor.</u> assume UBH for nice trees. Let κ be superstrong, and suppose $\exists$ ~~superstrong~~ 1-extendible $\lambda > \kappa$. then there is a canonical inner model

$M \models$ "there is a superstrong cardinal and I am iterable"

<u>potential applications.</u>

① κ supercompact, λ a Woodin limit of Woodins $> \kappa$.

(Neeman) $\Rightarrow \exists M \models$ "$\exists$ Woodin limit of Woodins + I am iterable"

(2) $\mathrm{PFA} \Longrightarrow \exists M \models$

$AD_{\mathbb{R}}$-hypo +

i am iterable.