Farmer Schlutzenberg 2

Thm. Let M be k-sound, (k, ω_1+1)-iterable. then M is $(k+1)$-solid, $(k+1)$-universal.

Solidity:

let $\mathcal{M}_k$ = set of all k-sound, cm. premice. for $N \in \mathcal{M}_k$, let $p^N = p^N_{k+1}$, $\rho^N = \rho_{k+1}(N)$.

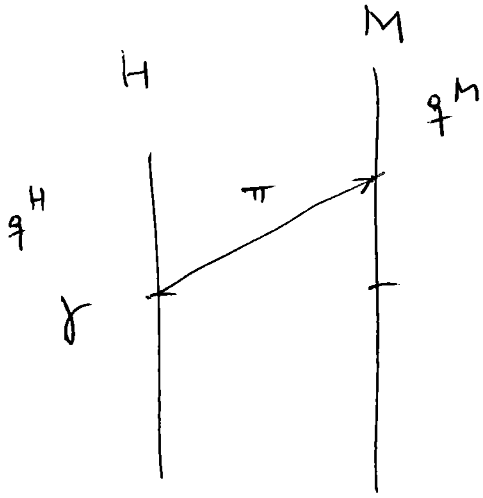
$\mathrm{Hull}^N = \mathrm{Hull}^N_{k+1}$

let $\gamma \in p^M$.

let $H = \mathrm{cHull}^M(q^M \cup \gamma)$,

collapsed

where $q^M = p^M \setminus (\gamma+1)$



$\pi : H \longrightarrow M$

uncollapse

$\mathrm{crit}(\pi) = \gamma$

$\pi(q^H) = q^M$

want: $H \in M$.

want to use bicephali to prove $H \in M$.

but H need not be $(k+1)$ sound.

$H = \mathrm{Hull}^H(q^H \cup \gamma)$.

if $q^H \trianglelefteq p^H$, and H is solid, then H is γ-sound.

i.e.,

$q^H = p^H \setminus \gamma$.

Somewhat like condensation proof

if $p^H < q^H$?

<u>case 1.</u> $q^H \trianglelefteq p^H$ <u>case 2.</u> $p^H < q^H$.

in case 2, will arrange an induction on a ~~the~~ mouse order $<^*$, which guarantees H is solid, universal, and <u>satisfies</u>

<u>Th.</u> (finitely generated mice)

let $N \in \mathcal{M}_k$, (k, ω_1+1)-iterable.

let $N = \mathrm{Hull}^N(p^N \cup \{x\})$,

some $x \in M$.

(again, $\mathrm{Hull}^N = \mathrm{Hull}^N_{k+1}$).

and N is $(k+1)$ solid, $(k+1)$ universal.

<u>then</u> N is a normal iterate of $\mathbb{C}_{k+1}(N)$ via a tree $\mathcal{I}$ of length $< \omega$.

applying this to H,

if $\mathbb{C}_{k+1}(H) \in M$, then $H \in M$.

also use sub induction on the length of p^M, ~~moves~~ moving downward ...

so may assume q^M is solid for

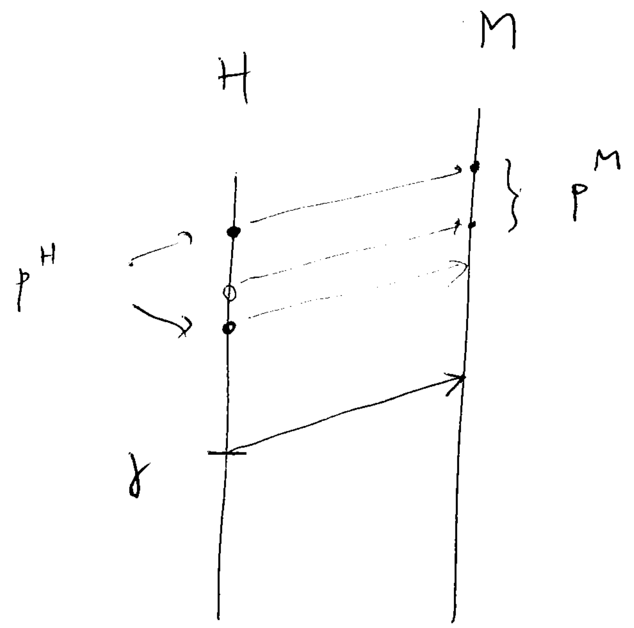
$M. \Rightarrow \mathbb{C}_{k+1}(H) \in M$.

$p^H < \varrho^H$.

get $t = Th^H(\rho^H \cup p^H)$ is

coupleted for one of the solidity

witness for ϱ^M in M.



t coupleted for

$Th^M(\ — \ )$.

mouse order $<^*$ on $\mathcal{M}_k$:

for $N, H \in \mathcal{M}_k$, say

$H <^* N$ iff there is a finite stack

$\vec{I}$, k-maximal, terminally non-dropping

finite length trees and

$\exists \pi : H \longrightarrow \mathcal{M}_\infty^{\vec{\mathcal{I}}}$ s.t.

π is a k-embedding

$\pi(p^H) < i^{\vec{\mathcal{I}}}(p^N)$.

<u>claim</u>. for $N \in \mathcal{M}_k$, if N is (k, ω_1+1)-iterable, then $<^*$ is well-founded below N.

<u>first</u>. why are models of $\vec{\mathcal{I}}$ well-founded.

<u>lem</u>. for $N \in \mathcal{M}_k$, let

$\mathcal{G}_k^N$ be the game where

(1) player I plays a stack $\vec{\mathcal{I}}$ as above

(2) players play the (k, ω_1+1) iterability game on $\mathcal{M}_\infty^{\vec{\mathcal{I}}}$

if N is (k, ω_1+1) iterable, then player II wins $\mathcal{G}_k^N$.

pf.: follows fr normalizati of trees

(no condensati of Σ_N required,
as trees are finite) $\uparrow$
strategy for N !

pf. of clai on p. 5 :
o.w. can construct a stack $\vec{\mathcal{I}}$,
$lh(\vec{\mathcal{I}}) = \omega$, each initial segment
$\vec{\mathcal{I}} \upharpoonright n$ is as in def. of $<^*$, and
$\mathcal{M}_\infty^{\vec{\mathcal{I}}}$ is ill-founded (use proof of
dodd-jensen).

then roughly, using the proof of
normalizati, $\mathcal{M}_\infty^{\vec{\mathcal{I}}}$ can be embedded
into the last model of a
normal tree $\mathcal{U}$ on N.

<u>lem.</u> let $N \in \mathcal{M}_k^{\text{ite}}$.

$$\mathcal{M}_k^{\text{ite}} = \{ \bar{N} \in \mathcal{M}_k : \bar{N} \ (k, \omega_1+1) \text{ iter} \}$$

supp. for all $H <^* N$, H is $(k+1)$ solid.

let $\mathcal{I}$ be a $\overbrace{\text{terminally non-dropping}}^{\text{non-dropping on main branch}}$ on N.

k-maximal

then $i^{\mathcal{I}}(p^N) = p^{M_\infty^{\mathcal{I}}}$.

<u>lem.</u> (p, ρ)-preservation.

let $N \in \mathcal{M}_k$, let E be semi-close to N (weakly amenable),

$\text{crit}(E) < \rho_k(N)$

$N | \kappa^{+N} = \text{ult}(N; E) | \kappa^{+\text{ult}}$, $\kappa = \text{crit}(E)$.

let $\rho' = \sup i_E{}'' \rho^N$,

$p' = i(p^N)$.

lu $N' = \mathrm{ult}_k(N; E)$, well-fdd.

then

(1) $\mathrm{Th}^{N'}_{k+1}(\rho' \cup p') \notin N'$.

(2) $\rho^{N'} \leq \rho'$.

(3) if $\rho^{N'} = \rho'$, then $p^{N'} \leq p'$.

(4) if $k < \rho^N$, then $\rho^{N'} = \rho'$

(5) if N' is solid, then N is solid

(6) if N is solid, and $\rho^{N'} = \rho'$,
then N' is solid and $i_E(p^N) = p' = p^{N'}$.

generalizes to $\langle E_\alpha \rangle_{\alpha < \lambda}$ sequence of semi-close extenders.

$$\begin{array}{ccccccc} N = N_0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & \cdots \\ & & \parallel & & & & \\ & & \mathrm{ult}(N_0, E_0) & & & & \end{array}$$

proof of 1st lemma on p. 7:

let $\mathcal{T}$ be a counterexample. say

$$M \xrightarrow[i^{\mathcal{T}}]{} M' \qquad p^{M'} < i^{\mathcal{T}}(p^M).$$

$$\boxed{\begin{array}{l}\text{must have}\\ p^{M'} \leq i^{\mathcal{T}}(p^M),\\ \text{cf. p. 9A}\end{array}}$$

\begin{tikzcd}
\bar{M}' \arrow[r, "\sigma"] & M' \\
M \arrow[u, squiggly, "\bar{\mathcal{T}}"] \arrow[ur, squiggly, "\mathcal{T}"'] &
\end{tikzcd}

$\exists$ finite $\bar{\mathcal{T}}$, k-max., non-dropping on main branch, on M

$\sigma : \bar{M}' \to M'$ a k-embedding

$p^{M'} \in \operatorname{ran}(\sigma)$, $\sigma \circ i^{\bar{\mathcal{T}}} = i^{\mathcal{T}}$.

say $\sigma(\bar{p}) = p^{M'}$.

claim. $\rho^{M'} = \sup i^{\mathcal{I}} {}'' \rho^M$.

pf. let $\beta \in b^{\mathcal{I}}$ (the main branch of $\mathcal{I}$) least

s.t. $\operatorname{crit}(i^{\mathcal{I}}_{\beta,\infty}) \geq \rho^* = \sup i^{\mathcal{I}}_{0,\beta} {}'' \rho^M$.

write $M^* = M^{\mathcal{I}}_\beta$.

then $\rho^* = \rho^{M^{\mathcal{I}}_\beta}$ by the lemma on p. 8 f.

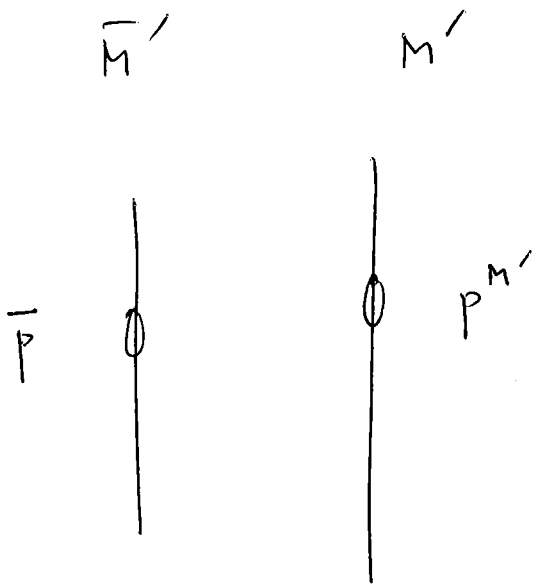
other extenders applied ~~to~~ along $[\beta, \infty)_{\mathcal{I}}$

are close to target model

(= model ext. appl. to)

So $\rho^{M^{\mathcal{I}}_\infty} = \rho^{M^{\mathcal{I}}_\beta} = \rho^*$. $\dashv$

therefore, $\rho^{M'} \leq i^{\mathcal{I}}(\rho^M)$.



here $i^{\bar{I}}(p^M) > \bar{p}$.

<u>case 1.</u> $p^{\bar{M}'} < i^{\bar{I}}(p^M)$.

then $\bar{M}' <^* M'$,

as $id : \bar{M}' \to \bar{M}'$

$id(\bar{p}^{M'}) < i^{\bar{I}}(p^M)$.

by induction $\bar{M}'$ is solid.

by lemma, M is solid.

$\Rightarrow$ $i^{\bar{I}}$ preserves parameters. $\lightning$

<u>Case 2.</u> $i^{\bar{J}}(p^M) = \bar{p}$.

we have $\rho^{\bar{M}'} = \sup i^{\bar{J}} {}'' \rho^M$

$$\rho^{M'} = \sup i^{J} {}'' \rho^{M} = \sup \sigma {}'' \rho^{\bar{M}'}$$

hence $p^{\bar{M}'} > \bar{p}$.

so $t = Th_{k+1}^{\bar{M}'}(\rho^{\bar{M}'} \cup \{\bar{p}\}) \in \bar{M}'$.

but σ is a k-embedding

(as $\sup \sigma {}'' \rho_k^{\bar{M}'} = \rho_k^{M'}$)

$\Rightarrow$ $\sigma(t)$ is a generalized witness for

$Th_{k+1}^{M'}(\sigma(\bar{p}) \cup \sigma(\rho^{\bar{M}'}))$

$\Rightarrow$ $Th_{k+1}^{M'}(p^{M'} \cup \rho^{M'}) \in M'$,

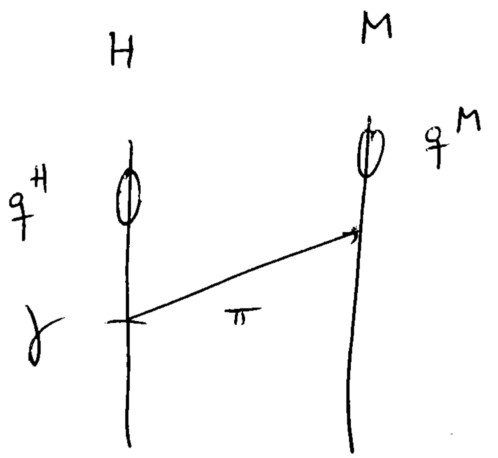
as the former thy computes the latter one. $\dashv$

<u>Situation</u>.

all $N <^* M$ has been analyzed.

($\Rightarrow$ N solid, universal, condensation)

also assume k - condensation for M.



$\gamma \in p^M$

$\pi(q^H) = q^M$.

want $H \in M$.

<u>Claim.</u> $p^H < q^H$.

$\Rightarrow$ $\pi(p^H) < q^M < p^M$

$\Rightarrow$ $H <^* M$.

$\Rightarrow$ H is solid etc.

+ H is a finite iterate of

$\mathbb{C}_{k+1}(H) \in M$

as q^M is solid for M.

so $H \in M$.

Case 2. $q^H \leq p^H$.

$\Rightarrow$ $q^H \trianglelefteq p^H$, and $p^H = q^H$ or
$p^H = q^H \cup \{\alpha_0 > \dots\}$,
$\alpha_0 < \gamma$,

because $H = \mathrm{Hull}^H(\gamma \cup q^H)$.

+ $H <^* M$, because

$\pi(p^H) < p^M$.

So H is solid etc.

$\boxed{\text{Case.}}$ assume: γ is a cardinal of M

$\gamma^{+H} < \gamma^{+M}$.

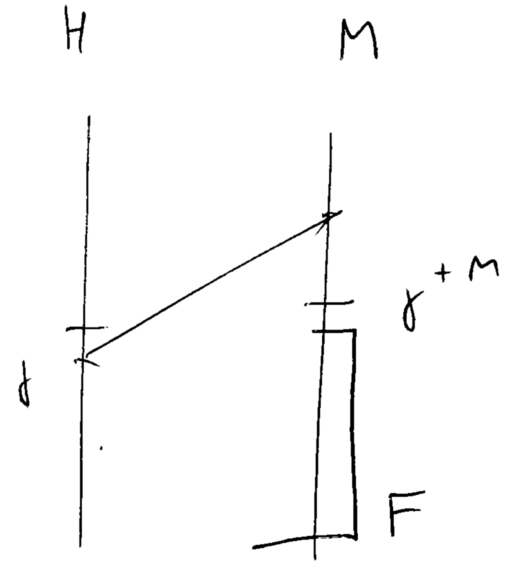
letting $\lambda = \gamma^{+M}$, $M|\gamma$ is active type 1 or 3.

$M = \mathrm{Hull}^M((\gamma+1) \cup p^M)$.

where $\delta = \gamma$.

bicephalus. H, M, δ, λ, F

$\lambda = \gamma^{+H}$



assume $H \ntriangleleft M$.

derive $\lightning$.

given E semi-close to B

$(\Leftrightarrow \ \mathrm{crit}(E) < \delta \ )$

define $B' = \mathrm{ult}(B, E)$

$= (H', M', \delta', \lambda', F')$ by:

$j : M \longrightarrow M' = \mathrm{ult}(M; E)$

$(\delta', \lambda', F') = j(\delta, \lambda, F)$.

$$H' = \mathrm{cHull}^{M'}(\delta' \cup q^{M'})$$

(rather than a ultrapower of H')

let $\pi' : H' \longrightarrow M'$ the uncoll. emb.

(we do this because maybe δ is regular in H, singular in M.

but want $i(\delta) = j(\delta)$, where i is the embedding.)

define $i : H \longrightarrow H'$ s.t.

$$i = \pi'^{-1} \circ j \circ \pi .$$

<u>claim.</u>

(1) $\mathrm{crit}(\pi') = \delta'$, as

$$\delta \notin \mathrm{Hull}^{M}(\delta \cup q^{M})$$

$$\Rightarrow \quad \delta' \notin \mathrm{Hull}^{M'}(\delta' \cup q^{M'}).$$

$$\lambda' = (\delta')^{+H'}$$

(2) $\lambda' = (\delta')^{+H'}$.

proof: e.g. $k = 0$.

consider usual stratification of Σ_1^M

(looking at where Σ_1 facts are verified).

$\dashv$

j is continuous at λ,

as $\mathrm{cf}^M(\lambda) = \kappa^{+M}$,

where $\kappa = \mathrm{crit}(F)$.

so $\mathrm{lh}(F') = \lambda' = (\delta')^{+H'}$.