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Recall: Always work in $ZF + DC$, unless stated o.w.

def. ω_1 is X-supercompact if there is $\mu \subseteq \mathcal{P}(\mathcal{P}_{\omega_1}(X))$, a ultrafilter

i) countably complete
ii) fine
iii) μ is normal,
i.e., for any $f : \mathcal{P}_{\omega_1}(X) \to \mathcal{P}_{\omega_1}(X)$
s.t. $\forall^*_\mu \sigma \; f(\sigma) \subset \sigma, \; f(\sigma) \neq \emptyset$,
$\exists x \; \forall^*_\mu \; x \in f(\sigma)$.

thm. (Solovay) $ZF + AD_{\mathbb{R}}$ implies:

if C is the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$
$\Rightarrow$ C is an $\mathbb{R}$-supercompact measure.

<u>thm.</u> (woodin) ZF + $AD_{\mathbb{R}}$ then

$\mathcal{C}$ is the unique

$\mathbb{R}$ - supercompactness measure for ω_1.

<u>question.</u> assume AD.

is it the case that there is at most

one model of the form $L(\mathbb{R}, \mu)$

$\exists$ "ω_1 is $\mathbb{R}$ - supercompact as being witnessed by μ",

$\mu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$.

<u>thm.</u> (rodriguez - trang) assume $M^{\#}_{\omega^2}$ exists and is

$\omega_1 + 1$ iterable. then

1) $L(\mathbb{R}, \mathcal{C}) \models$ "AD^+ + $\mathcal{C}$ is a $\mathbb{R}$-s.c. measure"

$\mathcal{C}$ is the club filter

on $\mathcal{P}_{\omega_1}(\mathbb{R})$

2) if $\mu < \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ and

$\forall A \in \mu \; (A \text{ is stat})^V$ and

$L(\mathbb{R}, \mu) \models AD^+ + \mu$ is an $\mathbb{R}$-s.c. measure

$$\Rightarrow \quad L(\mathbb{R}, C) = L(\mathbb{R}, \mu).$$

<u>Th.</u> (Woodin)

Supp. $M \models ZFC$ and M has ω^2 Woodins, $(\delta_i^M : i < \omega^2)$,

$$\lambda_\alpha^M = \sup_{\beta < \alpha} \delta_\beta^M .$$

let $G \subset \text{Col}(\omega, < \lambda^M_{\omega^2})$ be M-generic.

$$\mathbb{R}^* = \bigcup_{i < \omega^2} \mathbb{R}^{M[G \restriction i]}$$

$$\sigma_i = \bigcup_{\alpha < \omega \cdot i} \mathbb{R}^{M[G \restriction \alpha]}$$

$$\bigcup_{i < \omega} \sigma_i = \mathbb{R}^*$$

defin $\mathcal{F}^*_G$ = "the tail filter"

$A \in \mathcal{F}^*_G \iff$

$\exists n \ \forall m \geq n \quad \sigma_m \in A$.

$L(\mathbb{R}^*, \mathcal{F}^*_G) \models$ "AD^+ +

$\mathcal{F}^*_G$ is $\mathbb{R}$-s.c. measure"

<u>lem</u>. supp. γ is big enough.

and $X_0 \prec X_1 \prec H_\gamma$

Σ, $M^\#_{\omega^2} \in X_0$

$\mathbb{R} \cap X_0 \in X_1$.

where Σ is the it. strategy of $M^\#_{\omega^2}$ w.r.t. trees of length $\leq \omega_1 + 1$.

then there is $\mathcal{I} \in X_1$ on $M^\#_{\omega_2}$
with last model P,
$M^\#_{\omega^2} \longrightarrow P$ and $g \subset \mathrm{Col}(\omega, < \lambda^P_\omega)$
s.t. $\mathbb{R} \cap \sigma_0 = \mathbb{R}^*_g$.

pf.: in X_1, enumerate
$\sigma_0 = (x_i : i < \omega)$

$$M^\#_{\omega^2} = P_{-1} \longrightarrow P_0 \longrightarrow P_1 \cdots$$
x_0 gen. at $\mathbb{B}^{P_0}_{\delta_0}$ $\qquad x_1 \cdots$

$\dashv$

[simplification: assume Σ is $(2^{\mathfrak{c}})^+$ - u. b. not nec]
<u>thm.</u> in $V^{\mathrm{Col}(\omega, 2^{2^\omega})}$ there is a
Σ - iterate of $M^\#_{\omega_2}$,
say P, and $G \subset \mathrm{Col}(\omega, < \lambda^P_{\omega^2})$
~~s.t.~~ generic over P, ~~then~~
then ~~$\mathbb{R}_G$~~ $\mathbb{R}_G = \mathbb{R}^V$

~~and~~ ~~G~~

s.t. if $(A \in \text{Club})^V \Rightarrow A \in \mathcal{F}_G$.

pf.: let $H \subset \mathrm{Col}(\omega, \mathcal{P}(\mathbb{R}))$ – gen. $/V$.

$\mathcal{P}(\mathbb{R})$ ctbl. in $V[H]$.

γ big enough.

$(X_i : i < \omega)$ $\quad X_i \prec H_\gamma^V$.

$\mathcal{P}(\mathbb{R}) \cap \bigcup X_i = \mathcal{P}(\mathbb{R}) \cap V$

$\sigma_i = \mathbb{R} \cap X_i \in X_{i+1}$ (ctbl. many)

$$\mathcal{P}(\mathbb{R})^V \subset \bigcup \mathcal{P}(\mathbb{R}) \cap X_i =$$

$$\bigcup \sigma_i .$$

get G.

want: if $(A \text{ is club})^V$

$\Rightarrow \quad A \in \mathcal{F}_G$.

$$A = \{ \sigma : \pi''\sigma \subset \sigma \},$$

for $\pi : \mathbb{R}^{<\omega} \longrightarrow \mathbb{R}$.

$\exists i$ s.t. $\pi \in X_i$,

$$\pi''\sigma_i \subset \sigma_i$$

so $A \in \mathcal{F}_G$.

<u>lem.</u> given A stat., then in $V^{G(\omega, \mathbb{R})}$

$\exists\ M^{\#}_{\omega^2} \xrightarrow{\Sigma} P$ and

$G \subset Col(\omega, < \lambda^P_{\omega^2})$ s.t.

$\mathbb{R}_G = \mathbb{R}^V$ and $A \in \mathcal{F}_G$.

Pf.: first shoot a club thru A.

$$\mathbb{P}_A = \{ p : p = (\sigma_i : i < \alpha),$$

$\sigma_i \subset \sigma_j \ \forall i < j$,

cont. at lims,

$\sigma_i \in A \}$.

get $h \subset \mathbb{P}_A$ gen. / V.

apply the uth. in $V[h]$.

get $V[h][H]$.

apply previous lem. $\dashv$

<u>Lem.</u> in $V^{\mathrm{Col}(\omega, P(\mathbb{R}))}$. for any

N_0, N_1 models of cept set thy

$(X_i^0 : i<\omega)$, $X_i^0 \prec N_0$

$(X_i^1 : i<\omega)$, $X_i^1 \prec N_1$

$P(\mathbb{R}) \cap N_0 \subset \bigcup X_i^0$ $\quad \Sigma, M_{\omega_2}^{\#} \in$

$P(\mathbb{R}) \cap N_1 \subset \bigcup X_i^1$ $\quad X_0^0 \cap X_0^1$

$\mathbb{R}^V = \mathbb{R} \cap N_0 = \mathbb{R} \cap N_1$

$L(\mathbb{R}, F_0^*) = L(\mathbb{R}, F_1^*)$.

the tail filters

with $\mathcal{F}$ = the tail filter

generated by some "resolution"

~~$X_i \prec$~~ $(X_i : i < \omega)$,

$X_i \prec H_\gamma$, $\mathbb{R} \cap X_i \in X_{i+1}$

$\mathcal{P}(\mathbb{R}) \cap V = \mathcal{P}(\mathbb{R}) \cap \bigcup_i X_i$.

gen then $L(\mathbb{R}, \mathcal{F})$ is independently

for the choice of the

resolution by another lemma.

th. i) $L(\mathbb{R}, \mu) = L(\mathbb{R}, \mathcal{F})$

ii) . $L(\mathbb{R}, C) = L(\mathbb{R}, \mathcal{F})$

(want to see if $A \in L(\mathbb{R}, \mathcal{F}) \cap \mathcal{F}$

$\Rightarrow \mathcal{F} \in C$)

<u>thm.</u> (trang - rodriguez) $M_{\omega_2}^{\#}$ exists and is $\omega_1 + 1$ iterable.

$\mu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$

$L(\mathbb{R}, \mu) \models$ "AD + μ is $\mathbb{R}$-s.c. measure"

$\Rightarrow L(\mathbb{R}, \mu) = L(\mathbb{R}, C)$.

<u>thm.</u> assume $AD + DC_{\mathbb{R}}$, $L(\mathbb{R}, \mu)$ is unique.

<u>question.</u> $L(\mathbb{R}, \mu) \models$ "$\neg F$ + μ is $\mathbb{R}$-s.c."

can you have different such models?

assume there is no inner model with a woodin, and $\exists \kappa\ o(\kappa) = 2$.

~~then there is a forcing extension~~