Steve Jackson:

uniform cofinalities and exact partition properties.

$(\lambda \leq \kappa)$:

$\kappa \longrightarrow (\kappa)^\lambda$: for all partitions $\mathcal{P} : (\kappa)^\lambda \longrightarrow \{0,1\}$

$\exists\, H \subset \kappa$, $|H| = \kappa$, $\exists\, h \in \{0,1\}$

$\forall f \in (H)^\lambda \quad \mathcal{P}(f) = h$.

strong partition relation $\kappa \longrightarrow (\kappa)^\kappa$.

weak partition relation $\kappa \longrightarrow (\kappa)^\lambda \ \forall \lambda < \kappa$.

background theory throughout this talk:

ZF + AD.

<u>thm.</u> (Martin) $\omega_1 \longrightarrow (\omega_1)^{\omega_1}$

$\omega_1 = \underset{\sim}{\delta}^1_1$.

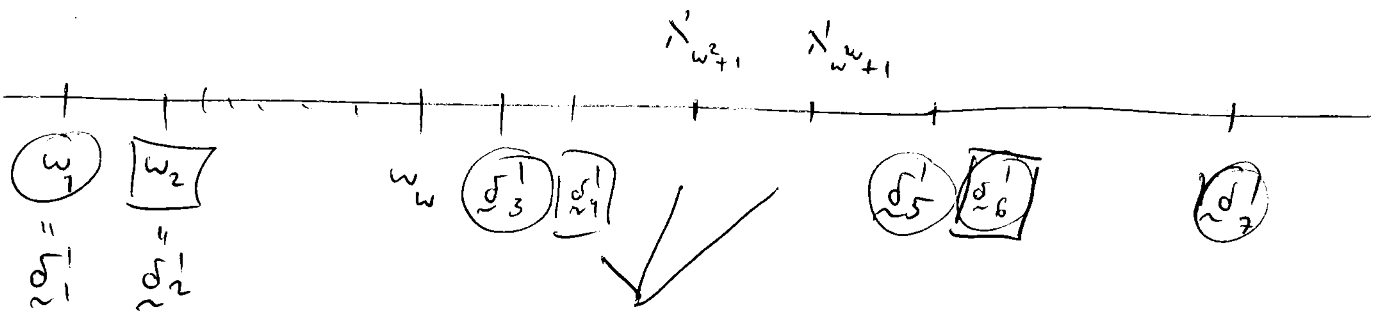
thm. (jackson)

$$\underset{\sim}{\delta}^1_{2n+1} \longrightarrow (\underset{\sim}{\delta}^1_{2n+1})^{\underset{\sim}{\delta}^1_{2n+1}}$$

thm. (paris-martin) $\omega_2 \longrightarrow (\omega_2)^{<\omega_2}$,
but not $\omega_2 \longrightarrow (\omega_2)^{\omega_2}$.

question. for each regular κ, what is the exact partition strength of κ?

we will prove a result which answers this question for all $\kappa < \sup_n \underset{\sim}{\delta}^1_n$.



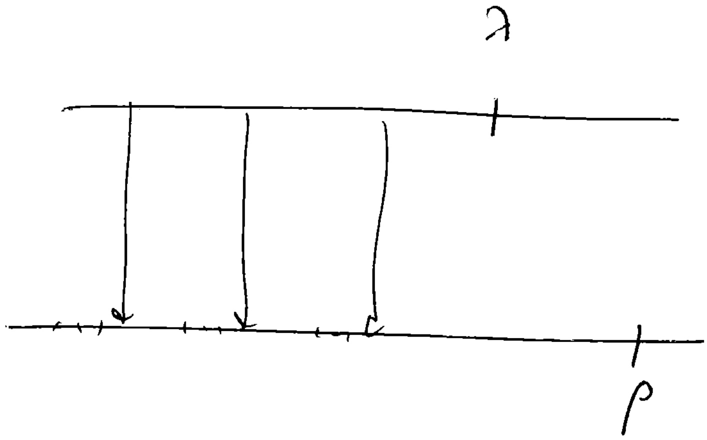
the 3 regulars between $\underset{\sim}{\delta}^1_3$ and $\underset{\sim}{\delta}^1_5$

the $2n$ 2^n-1 regulars between $\underset{\sim}{\delta}^1_{2n-1}$ and $\underset{\sim}{\delta}^1_{2n+1}$.

<u>uniform cofinalities</u>.

example : $f : \lambda \to \rho$ has uniform cofinality ω

iff there is some $f' : \lambda \times \omega \to \rho$ strictly increasing in the 2^{nd} coordinate



$\forall_{\alpha < \lambda} \quad f(\alpha) = \sup_n f'(\alpha, n)$

example: $f(\alpha)$ has unif. cof. ω
for $f : \omega_1 \to \omega_1$: $f(\alpha)$ — " — α "

Let $f : \lambda \to \rho$

$f(\alpha)$ has uniform cofinality $g(\alpha)$ defined in the obvious sense.

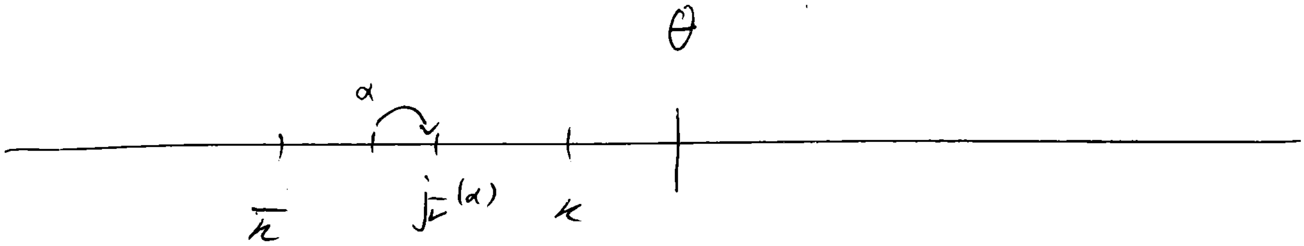
might restrict to some measure on λ.

<u>fact</u>: $\kappa \longrightarrow (\kappa)^{\lambda}$ iff

$$\kappa \xrightarrow[\text{c.u.b.}]{} (\kappa)^{\lambda}$$

for any partition P of functions $f : \lambda \longrightarrow \kappa$ of <u>correct type</u> (e.g. cof. ω + increasing + discontinuous)

then $\exists\, C \subset \kappa$ c.u.b. which is homogeneous for P.



$\bar{\kappa}, \kappa$ regular cardinals

$\kappa \longrightarrow (\kappa)^{\kappa}$

let ν be the $\bar{\kappa}$-complete normal measure on κ.

let $\bar{\nu}$ be the ω-complete normal measure on $\bar{\kappa}$.

[think of $\kappa = \delta^1_3$, $\bar{\kappa} = \omega_1$ or ω_2.]

theorem. let $\kappa \longrightarrow (\kappa)^\kappa$.

let $\omega < \bar{\kappa} < \kappa$ be regular.

let $\iota, \bar{\iota}$ as above.

assume κ is closed under $j_{\bar{\iota}}$,

the embedding ass. with $\bar{\iota}$.

let $\lambda = j_\iota(\kappa)$; let $\theta = [h]_\iota$,

where for $\alpha < \kappa$, $h(\alpha) = j_{\bar{\iota}}(\alpha)$.

then $\lambda \not\longrightarrow (\lambda)^\theta$.

but also $\lambda \longrightarrow (\lambda)^{<\theta}$.

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lem. let $<$ be the lex. order on

(α, β), where $\alpha < \kappa$, $\beta < h(\alpha) =$

$j_{\bar{\iota}}(\alpha)$.

then $\lambda \to (\lambda)^\theta$ iff (*) for every

$g : \theta \longrightarrow (\lambda \setminus \kappa)$ of the correct type then is a $G : \kappa \longrightarrow \kappa$ s.t.
of the correct type
$[G]_\mathcal{U} = g$

prf. (sketch)

assume first (A). given a partition $f : \theta \to \lambda$ of the current type, acc. to $\mathcal{P}'(F) = \mathcal{P}(f)$ where $f = [F]_\mathcal{U}$.

say $C \subset \kappa$ homogeneous for 1 side of $\mathcal{P}'$.

then $D = j_\mathcal{U}(C) \subset \lambda$. this is then a c.u.b. subset of λ.

if $f : \theta \longrightarrow D$ is of the current type.

By $(*)$, let $f = [F]_\iota$.

$\forall^* \alpha \quad \forall \beta < l(\alpha) \quad F(\alpha, \beta) \in C$

(w.l.o.g.) $\quad \forall \alpha, \beta \quad F(\alpha, \beta) \in C$

$$\mathcal{P}(f) = \mathcal{P}'(F) = 1 .$$

Suppose $\lambda \longrightarrow (\lambda)^\theta$ satisfies $(*)$.

partition $g : \theta \longrightarrow \lambda$ of correct type according to whether $\exists G : \kappa \longrightarrow \kappa$ of the correct type with $[G]_\iota = g$.

let $D \subset \lambda$ be c.u.b. and homogeneous for $\mathcal{P}$.

<u>fact</u>. if $D \subset \lambda$ is c.u.b., then $\exists C \subset \kappa$ c.u.b. s.t. $j_\iota(C) \subset D$.

[partition pairs of functions $f_1, f_2 : \kappa \longrightarrow \kappa$ of the correct type,

$f_1(\alpha) < f_2(\alpha) < f_1(\alpha+1)$, acc. to whether $D \cap ([f_1]_\iota, [f_2]_\iota) \neq \emptyset$.

on the homogeneous side, this must hold, and a ~~the~~ c.u.b. $C \subset \kappa$ for P satisfies $j_2(C) = D$.]

have c.u.b. $D \subset \lambda$,

get c.u.b. $C \subset \kappa$, $j_2(C) \subset D$.

D must be homogeneous for the stated side.

[if D hom. for the oth side, let $F: \kappa \to C$ of the correct type, let $f = [F]_2$, so $f: \theta \to j_2(C) \subset D$ is of the correct type, a contradiction.]

let D, C be as above. We then get the results for <u>all</u> $f: \theta \to \lambda$ of the correct type.

[given $g: \theta \to \lambda$ of the correct type, let $g_2: \theta \to \lambda$ be given by

$g_2(\alpha) = g(\alpha)^{th}$ elt. of ~~D~~, $j_\kappa(C) \subset D$.
"sliding up g by ~~D~~ $j_\kappa(C)$."

so g_2 is also of the correct type.

so $\exists\, G_2 : \lambda \longrightarrow \kappa$ of the correct type with $[G_2]_\kappa = g_2$.

define $G : \lambda \longrightarrow \kappa$ by $G_2(\alpha, \beta) = G(\alpha,\beta)^{\underline{th}}$ elt. of C. So $[G]_\kappa = g$, and G is of the correct type.]

note: $(*)$ implies that if $M = \mathrm{ult}(V, j_\kappa)$, $P(\theta) \subset M$.

[any $A \subset \theta$ can be coded by a function ~~for~~ $f : \lambda \longrightarrow \theta$ of the correct type. by $(*)$, $f = [F]_\kappa \in M$. so $A \in M$]

also $P(\theta) \notin M$.

<u>lem</u>. let $\theta = [h]_U$, $h(\alpha) = j_{\bar{E}}(\alpha)$.

let $A = \{\alpha < \theta : cf(\alpha) = \bar{\kappa}\}$.

then $A \notin M$.

pf. $A \in M$, say $A = [F]_U$.

analyze F. consider partitions

$u : \bar{\kappa} \longrightarrow \kappa$ which are increasing,

discont., and $u(\beta)$ has unif. cof. β;

acc. to whether $[u]_{\bar{U}} \in F(\sup(u))$

uniform cofinality then ρ can't have a

hom. set on either side.