# Set theoretic geology in the presence of ω models with a woodin cardinal (Gunter Fuchs)

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fact : if $V = W[g]$, $g \subset \mathbb{P} \in W$ is gen. ; then $W$ may be defined in $V$ (uniformly).

call such $W$s <u>grounds</u> of $V$.

idea / intuition : the canonical objects are those that ex. without having to force.

<u>def</u>. the <u>mantle</u> $\mathbb{M}$ is the intersection of all grounds.
( fuchs – hamkins – reitz )

<u>question</u> : does $\mathbb{M}$ satisfy ZF ?

<u>def</u>. the generic mantle $g\mathbb{M}$ is the intersection of all grounds of set-forcing extensions of $V$.

<u>fact</u>. $g\mathbb{M} \models$ ZF .

quest: $g\mathbb{M} \models AC$ ?

def. the generic HOD is the intersection of all HODs of all set forcing extensions of $V$.

$$= \bigcap_{\alpha} HOD^{V^{Coll(\omega,\alpha)}} = gHOD.$$

fact:

$$\begin{array}{c} HOD \\ \cup \\ gHOD \subset g\mathbb{M} \subset \mathbb{M}, \end{array}$$

$$gHOD \models ZFC.$$

th. if $V = L[x]$, $x$ a set, then $gHOD = g\mathbb{M} = \mathbb{M} \models ZFC$

(in fact, if $V = HOD_{\{x\}}$)

pf: Vopěnka.

<u>th.</u> (fuchs - hamkins - reitz)

there is a class forcing $\mathbb{P}$ s.t. if $G$ is $\mathbb{P}$-gen., then

$$\mathbb{M}^{V[G]} = g\mathbb{M}^{V[G]} = gHOD^{V[G]} = V.$$

prf. idea: use a product of forcings which code every elem of $V$ into the GCH pattern. $\dashv$

what if $V = L[x]$? is $\mathbb{M}$ canonical then?

<u>th.</u> (fuchs - schindler)

supp. $V = L[x]$, $x$ is a set, and there is an inner model with a woodin cardinal. then

$\mathbb{M}$ is a fine str. extender model for an inner model W, then

$\kappa^W$ = least measurable cardinal (if ex.)

$\delta^W$ = least woodin card. ( " ).

We will single out a "minimal" inner model W, fine str., s.t. $\delta^W$ ex.,

$$\mathbb{M} = \bigcap_{\alpha<\infty} W^\alpha = \bigcup_{\alpha<\omega} W^\alpha | \kappa^{W^\alpha}.$$

$\alpha^{th}$ iterate of W by hitting $\kappa^W$ + its images $\alpha$ times

Say that a iteration tree $\mathcal{T}$ on W is <u>short</u> if for all limit $\lambda \leq lh(\mathcal{T})$,

$$L[m(\mathcal{T}\upharpoonright\lambda)] \models \delta(\mathcal{T}\upharpoonright\lambda) \text{ is not woodin.}$$

- assume that W is a proper class ext. model that is short tree it. str and has a woodin cardinal.

$$\mathbb{M} \subset \bigcap_{\alpha<\infty} W^\alpha.$$

show : for any $\alpha$, there is a non-dropping normal tree $\mathcal{T}$ on $W^\alpha$ s.t. $N_\alpha = L[\mathcal{M}(\mathcal{T})]$ is a ground.

w.l.o.g., $x \subset \delta^{W^\alpha}$. do a genericity iteration of $W^\alpha$ to make $x$ generic over the iterate. at limits $\lambda$, if $\mathcal{T}\restriction\lambda$ is short, we continue. if $\mathcal{T}\restriction\lambda$ is not short, then $x$ is gen. / $L[\mathcal{M}(\mathcal{T}\restriction\lambda)]$.

for the other direction, we need a minimality property of $W$:

- $W$ has a woodin cardinal, 1-small
- $W$ is short tree it. able
- if $\gamma < \delta^W$, then any it. tree on $W\|\gamma$ is short.

<u>fact</u>. if there is a inner model with a woodin cardinal, then there is a minimal one.

prf. sketch: let $W$ come from a $K^c$ (or $L[E]$) construction s.t. $\delta^W$ exists. $W$ is short tree iterable.

$$W = W_0 \underset{\mathcal{T}_0}{\rightsquigarrow} L[M(\mathcal{T}_0)] \underset{\mathcal{T}_1}{\rightsquigarrow} L[M(\mathcal{T}_1)] \rightsquigarrow$$

after finitely many steps, we must reach a model $L[M(\mathcal{T}_n)]$ which is minimal.

- if $W, W'$ are minimal, $\kappa = \min\{\kappa^W, \kappa^{W'}\}$.

- if $W$ is minimal, then $(W \text{ is minimal})^{V[g]}$ for all $g$ set generic over $V$.

- if $W$ is minimal, then

$$\bigcup_\alpha W^\alpha|_{\kappa}^{W^\alpha} = \bigcap_\alpha W^\alpha \subset \mathbb{M} .$$

given a ground $W^*$, we have to show that $W^\alpha|\kappa^{W^\alpha} \in W^*$.

$W^*$ has a minimal model $W'$.

so $W'$ is minimal in $V$.

$$(W')^{\kappa^{W^\alpha}} \| \kappa^{W^\alpha} = W^\alpha \| \kappa^{W^\alpha}$$

$$\in W^*$$

<u>def.</u> $x \subset OR$ is <u>solid</u> iff

$x$ can't be added to an inner model

by set-forcing.

i.e., if $x \in L[a][g]$, then $x \in L[a]$.

gen. $/L[a]$

<u>def.</u> the solid core is

$$\mathbb{C} = \bigcup_{x \text{ solid}} L[x].$$

quest: $\mathbb{C} \models ZF$ ?

but:

<u>thm.</u> if $\exists$ an inner model with a woodin cardinal, and $W$ is minimal, then

$$\mathbb{C} = \bigcap_{\alpha} W^{\alpha}.$$